



# GREAT BOOKS OF THE WESTERN WORLD

ROBERT MAYNARD HUTCHINS, *EDITOR IN CHIEF*

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## II.

EUCLID

ARCHIMEDES

APOLLONIUS OF PERGA

NICOMACHUS

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THE THIRTEEN BOOKS  
OF EUCLID'S ELEMENTS

THE WORKS OF ARCHIMEDES  
INCLUDING THE METHOD

ON CONIC SECTIONS

BY APOLLONIUS OF PERGA

INTRODUCTION TO  
ARITHMETIC

BY NICOMACHUS OF GERASA



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By NICOMACHUS OF GERASA

*Translated by MARTIN L D OOSE*

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**EUCLID'S ELEMENTS**



## BIOGRAPHICAL NOTE

EUCLID, *fl* c 300 B C.

EUCLID is said to have been younger than the first pupils of Plato but older than Archimedes which would place the time of his flourishing about 300 B C. He probably received his early mathematical education in Athens from the pupils of Plato, since most of the geometers and mathematicians on whom he depended were of that school. Proclus, the Neo-Platonist of the fifth century, asserts that Euclid was of the school of Plato and "intimate with that philosophy." His opinion, however, may have been based only on his view that the treatment of the five regular ("Platonic") solids in Book XIII is the "end of the whole *Elements*."

The only other fact concerning Euclid is that he taught and founded a school at Alexandria in the time of Ptolemy I, who reigned from 306 to 283 B C. The evidence for the place comes from Pappus (fourth century A D), who notes that Apollonius "spent a very long time with the pupils of Euclid at Alexandria, and it was thus that he acquired such a scientific habit of thought." Proclus claims that it was Ptolemy I who asked Euclid if there was no shorter way to geometry than the *Elements* and received as answer "There is no royal road to geometry." The other story about Euclid that has come down from antiquity concerns his answer to a pupil who at the end of his first lesson in geometry asked what he would get by learning such things, whereupon Euclid called his slave and said "Give him a coin since he must needs make gain by what he learns."

Something of Euclid's character would seem to be disclosed in the remark of Pappus regarding Euclid's "scrupulous fairness and his exemplary kindness towards all who advance mathematical science to however small an extent."

problem in conics

Euclid's great work, the thirteen books of the *Elements*, must have become a classic soon after publication. From the time of Archimedes they are constantly referred to and used as a basic text-book. It was recognized in antiquity that Euclid had drawn upon all his predecessors. According to Proclus, he "collected many of the theorems of Eudoxus, perfected many of those of Theaetetus,

visions (of figures), the *Optics* and the *Phenomena*, a treatise on the geometry of the sphere for use in astronomy. His lost *Elements of Music* may have provided the basis for the extant *Sectio Canonis* on the Pythagorean theory of music. Of lost geometrical works all except one belonged to higher geometry.

Since the later Greeks knew nothing about the life of Euclid, the mediaeval

# BIOGRAPHICAL NOTE

translators and editors were left to their own devices. He was usually called *Megarensis*, through confusion with the philosopher Eucleides of Megara, Plato's contemporary. The Arabs found that the name of Euclid, which they

who has not learned the *Elements* of Euclid," thus transferring the inscription over Plato's Academy to all scholastic doors and substituting the *Elements* for geometry.

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# BOOK ONE

## DEFINITIONS

- 1 A *point* is that which has no part
- 2 A *line* is breadthless length
- 3 The *extremities* of a line are points
- 4 A *straight line* is a line which lies evenly with the points on itself
- 5 A *surface* is that which has length and breadth only
- 6 The *extremities* of a surface are lines
- 7 A *plane surface* is a surface which lies evenly with the straight lines on itself
- 8 A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line
- 9 And when the lines containing the angle are straight, the angle is called *rectilineal*
- 10 When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right* and the straight line standing on the other is called a *perpendicular* to that on which it stands
- 11 An *obtuse angle* is an angle greater than a right angle
- 12 An *acute angle* is an angle less than a right angle
- 13 A *boundary* is that which is an extremity of anything

equal to one another,

- 16 And the point is called the *centre* of the circle
- 17 A *diameter* of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle
- 18 A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle
- 19 *Rectilineal figures* are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines

22 Of quadrilateral figures, a *square* is that which is both equilateral and right angled, an *oblong* that which is right-angled but not equilateral, a *rhombus* that which is equilateral but not right angled, and a *rhomboid* that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called *trapezia*.

23 *Parallel straight lines* are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

### POSTULATES

Let the following be postulated

- 1 To draw a straight line from any point to any point
- 2 To produce a finite straight line continuously in a straight line
- 3 To describe a circle with any centre and distance
- 4 That all right angles are equal to one another
- 5 That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles

### COMMON NOTIONS

- 1 Things which are equal to the same thing are also equal to one another
- 2 If equals be added to equals, the wholes are equal
- 3 If equals be subtracted from equals, the remainders are equal
- [7] 4 Things which coincide with one another are equal to one another
- [8] 5 The whole is greater than the part

## BOOK I PROPOSITIONS

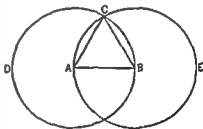
### PROPOSITION 1

*On a given finite straight line to construct an equilateral triangle*

Let  $AB$  be the given finite straight line

Thus it is required to construct an equilateral triangle on the straight line  $AB$

With centre  $A$  and distance  $AB$  let the circle  $BCD$  be described, [Post 3]  
 again, with centre  $B$  and distance  $BA$  let the circle  $ACE$  be described [Post 3]  
 and from the point  $C$ , in which the circles cut one another, to the points  $A, B$  let the straight lines  $CA, CB$  be joined [Post 1]



Now, since the point  $A$  is the centre of the circle  $CDB$ ,

$AC$  is equal to  $AB$

[Def 15]

Again, since the point  $B$  is the centre of the circle  $CAE$ ,

$BC$  is equal to  $BA$

[Def 15]

But  $CA$  was also proved equal to  $AB$ ,

therefore each of the straight lines  $CA, CB$  is equal to  $AB$

And things which are equal to the same thing are also equal to one another,  
 therefore  $CA$  is also equal to  $CB$  [C N 1]

Therefore the three straight lines  $CA, AB, BC$  are equal to one another

Therefore the triangle  $ABC$  is equilateral, and it has been constructed on the given finite straight line  $AB$ .

(Being) what it was required to do

## PROPOSITION 2

To place at a given point (as an extremity) a straight line equal to a given straight line

Let  $A$  be the given point, and  $BC$  the given straight line

Thus it is required to place at the point  $A$  (as an extremity) a straight line equal to the given straight line  $BC$

From the point  $A$  to the point  $B$  let the straight line  $AB$  be joined, [Post 1]  
and on it let the equilateral triangle  $DAB$  be constructed [I 1]

Let the straight lines  $AE$ ,  $BF$  be produced in a straight line with  $DA$ ,  $DB$ , [Post 2]  
with centre  $B$  and distance  $BC$  let the circle  $CGH$  be described, [Post 3]

and again with centre  $D$  and distance  $DG$  let the circle  $GKL$  be described [Post 3]

Then since the point  $B$  is the centre of the circle  $CGH$ ,  
 $BC$  is equal to  $BG$

Again, since the point  $D$  is the centre of the circle  $GKL$ ,  
 $DL$  is equal to  $DG$

And in these  $DA$  is equal to  $DB$ ,

therefore the remainder  $AL$  is equal to the remainder  $BG$  [CN 3]

But  $BC$  was also proved equal to  $BG$ ,

therefore each of the straight lines  $AL$ ,  $BC$  is equal to  $BG$

And things which are equal to the same thing are also equal to one another, [CN 1]

therefore  $AL$  is also equal to  $BC$

Therefore at the given point  $A$  the straight line  $AL$  is placed equal to the given straight line  $BC$

(Being) what it was required to do

## PROPOSITION 3

Given two unequal straight lines, to cut off from the greater a straight line equal to the less

Let  $AB$ ,  $C$  be the two given unequal straight lines and let  $AB$  be the greater of them

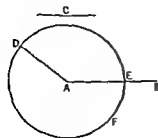
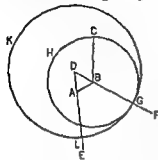
Thus it is required to cut off from  $AB$  the greater a straight line equal to  $C$  the less

At the point  $A$  let  $AD$  be placed equal to the straight line  $C$ , [I 2]

and with centre  $A$  and distance  $AD$  let the circle  $DEF$  be described [Post 3]

Now, since the point  $A$  is the centre of the circle  $DEF$ ,

$AE$  is equal to  $AD$  [De



But  $C$  is also equal to  $AD$

Therefore each of the straight lines  $AE, C$  is equal to  $AD$ ,

so that  $AE$  is also equal to  $C$

[C N 1]

Therefore given the two straight lines  $AB, C$ , from  $AB$  the greater  $AE$  has been cut off equal to  $C$  the less

(Being) what it was required to do

#### PROPOSITION 4

*If two triangles have the two sides equal to two sides respectively and have the angles contained by the equal straight lines equal, they will also have the base equal to the base the triangle will be equal to the triangle and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend*

Let  $ABC, DEF$  be two triangles having the two sides  $AB, AC$  equal to the two sides  $DE, DF$  respectively, namely  $AB$  to  $DE$  and  $AC$  to  $DF$ , and the angle  $BAC$  equal to the angle  $EDF$

I say that the base  $BC$  is also equal to the base  $EF$ , the triangle  $ABC$  will be equal to the triangle  $DEF$ , and the remaining angles will be equal to the rest—those which the equal sides subtend—that and the angle  $ACB$  to the angle  $DFE$

the triangle  $DEF$

and if the point  $A$  be placed on the point  $D$

and the straight line  $AB$  on  $DE$ ,

then the point  $B$  will also coincide with  $E$  because  $AB$  is equal to  $DE$



Again  $AB$  coinciding with  $DE$ ,

the straight line  $AC$  will also coincide with  $DF$ , because the angle  $BAC$  is equal to the angle  $EDF$

hence the point  $C$  will also coincide with the point  $F$ , because  $AC$  is again equal to  $DF$

But  $B$  also coincided with  $E$

hence the base  $BC$  will coincide with the base  $EF$

[For if when  $B$  coincides with  $E$  and  $C$  with  $F$ , the base  $BC$  does not coincide with the base  $EF$  two straight lines will enclose a space which is impossible

Therefore the base  $BC$  will coincide with  $EF$ ] and will be equal to it [C N 4]  
Thus the whole triangle  $ABC$  will coincide with the whole triangle  $DEF$ , and will be equal to it

And the remaining angles will also coincide with the remaining angles and will be equal to them,

the angle  $ABC$  to the angle  $DEF$ ,  
and the angle  $ACB$  to the angle  $DFE$

Therefore etc

(Being) what it was required to prove

#### PROPOSITION 5

*In isosceles triangles the angles at the base are equal to one another and, if the equal straight lines be produced further, the angles under the base will be equal to one another*

Let  $ABC$  be an isosceles triangle having the side  $AB$  equal to the side  $AC$ ,

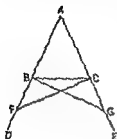
and let the straight lines  $BD$ ,  $CE$  be produced further in a straight line with  $AB$ ,  $AC$  [Post 2]

I say that the angle  $ABC$  is equal to the angle  $ACB$ , and the angle  $CBD$  to the angle  $BCE$

Let  $\equiv$  point  $F$  be taken at random on  $BD$ ,  
from  $AE$  the greater let  $AG$  be cut off equal to  $AF$  the less, [I 3]

and let the straight lines  $FC$ ,  $GB$  be joined [Post 1]

Then, since  $AF$  is equal to  $AG$  and  $AB$  to  $AC$ ,  
the two sides  $FA$ ,  $AC$  are equal to the two sides



and the triangle  $AFC$  is equal to the triangle  $AGB$ ,  
and the remaining angles will be equal to the remaining angles respectively,  
namely those which the equal sides subtend

that is, the angle  $ACF$  to the angle  $ABG$ ,

and the angle  $AFC$  to the angle  $AGB$  [I 4]

And, since the whole  $AF$  is equal to the whole  $AG$ ,

and in these  $AB$  is equal to  $AC$ ,

the remainder  $BF$  is equal to the remainder  $CG$

But  $FC$  was also proved equal to  $GB$ ,

therefore the two sides  $BF$ ,  $FC$  are equal to the two sides  $CG$ ,  $GB$  respectively,

and in these the angle  $CBD$  is equal to the angle  $BCE$ ,  
the remaining angle  $ABC$  is equal to the remaining angle  $ACB$ ,  
and they are at the base of the triangle  $ABC$

But the angle  $FBC$  was also proved equal to the angle  $GCB$ ,  
and they are under the base

Therefore etc

Q.E.D.

#### PROPOSITION 6

If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another

Let  $ABC$  be a triangle having the angle  $ABC$  equal to the angle  $ACB$ ,

I say that the side  $AB$  is also equal to the side  $AC$

For, if  $AB$  is unequal to  $AC$ , one of them is greater

Let  $AB$  be greater, and from  $AB$  the greater let  $DB$  be cut off equal to  $AC$  the less,

let  $DC$  be joined

Then, since  $DB$  is equal to  $AC$ ,

and  $BC$  is common,



the two sides  $DB, BC$  are equal to the two sides  $AC, CB$  respectively,  
 and the angle  $DBC$  is equal to the angle  $ACB$ ,  
 therefore the base  $DC$  is equal to the base  $AB$ ,  
 and the triangle  $DBC$  will be equal to the triangle  $ACB$ ,  
 the less to the greater  
 which is absurd

Therefore  $AB$  is not unequal to  $AC$ ,  
 it is therefore equal to it

Therefore etc

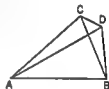
Q E D

### PROPOSITION 7

Given two straight lines constructed on a straight line (from its extremities) and

same extremity with it

For, if possible, given two straight lines  $AC, CB$  constructed on the straight line  $AB$  and meeting at the point  $C$ , let two other straight lines  $AD, DB$  be constructed on the same straight line  $AB$ , on the same side of it, meeting in another point  $D$  and equal to the former two respectively, namely each to that which has the same extremity with it, so that  $CA$  is equal to  $DA$  which has the same extremity  $A$  with it, and  $CB$  to  $DB$  which has the same extremity  $B$  with it, and let  $CD$  be joined



Then, since  $AC$  is equal to  $AD$ ,

[1 5]

Again, since  $CB$  is equal to  $DB$ ,

the angle  $CDB$  is also equal to the angle  $DCB$

But it was also proved much greater than it  
 which is impossible

Therefore etc

Q E D

### PROPOSITION 8

If two triangles have the two sides equal to two sides respectively and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines

Let  $ABC, DEF$  be two triangles having the two sides  $AB, AC$  equal to the two sides  $DE, DF$  respectively, namely  $AB$  to  $DE$ , and  $AC$  to  $DF$ , and let them have the base  $BC$  equal to the base  $EF$ ,

I say that the angle  $BAC$  is also equal to the angle  $EDF$



and if the point  $B$

because  $BC$  is equal to  $EF$

Then,  $BC$  coinciding with  $EF$ ,

$BA, AC$  will also coincide with  $ED, DF$ ,

for, if the base  $BC$  coincides with the base  $EF$ , and the sides  $BA, AC$  do not coincide with  $ED, DF$  but fall beside them as  $EG, GF$ ,

then, given two straight lines constructed on a straight line (from its ex-

But they cannot be so constructed

[1 7]

Therefore it is not possible that if the base  $BC$  be applied to the base  $EF$ , the sides  $BA, AC$  should not coincide with  $ED, DF$ ,

they will therefore coincide,

so that the angle  $BAC$  will also coincide with the angle  $EDF$ , and will be equal to it

If therefore etc

Q E D

### PROPOSITION 9

To bisect a given rectilineal angle



Let the angle  $BAC$  be the given rectilineal angle

Thus it is required to bisect it

Let a point  $D$  be taken at random on  $AB$ ,

let  $AE$  be cut off from  $AC$  equal to  $AD$

[1 3]

let  $DE$  be joined, and on  $DE$  let the equilateral triangle  $DEF$  be constructed,

let  $AF$  be joined

I say that the angle  $BAC$  has been bisected by the straight line  $AF$

For, since  $AD$  is equal to  $AE$ ,

and  $AF$  is common,

the two sides  $DA, AF$  are equal to the two sides  $EA, AF$  respectively

And the base  $DF$  is equal to the base  $EF$ ,

therefore the angle  $DAF$  is equal to the angle  $EAF$

[1 5]

Therefore the given rectilineal angle  $BAC$  has been bisected by the straight line  $AF$

Q E D

### PROPOSITION 10

To bisect a given finite straight line

Let  $AB$  be the given finite straight line

Thus it is required to bisect the finite straight line  $AB$

Let the equilateral triangle  $ABC$  be constructed on it

[1 1]

and let the angle  $ACB$  be bisected by the straight line

$CD$

[1 9]

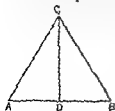
I say that the straight line  $AB$  has been bisected at the point  $D$

For since  $AC$  is equal to  $CB$ ,

and  $CD$  is common,

the two sides  $AC, CD$  are equal to the two sides  $BC, CD$  respectively,

and the angle  $ACD$  is equal to the angle  $BCD$ ,





therefore the base  $AD$  is equal to the base  $BD$  [I 4]

Therefore the given finite straight line  $AB$  has been bisected at  $D$   $\square$   $E F$

### PROPOSITION 11

*To draw a straight line at right angles to a given straight line from a given point on it*

Let  $AB$  be the given straight line, and  $C$  the given point on it

Thus it is required to draw from the point  $C$  a straight line at right angles to the straight line  $AB$

Let a point  $D$  be taken at random on  $AC$ ,

let  $CE$  be made equal to  $CD$ , [I 3]

on  $DE$  let the equilateral triangle  $FDE$  be constructed, [I 1]

and let  $FC$  be joined,

I say that the straight line  $FC$  has been drawn at right angles to the given straight line  $AB$  from  $C$  the given point on it

For, since  $DC$  is equal to  $CE$ ,

and  $CF$  is common,

the two sides  $DC$ ,  $CF$  are equal to the two sides  $EC$ ,  $CF$  respectively,

and the base  $DF$  is equal to the base  $FE$ ,

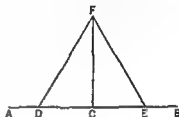
therefore the angle  $DCF$  is equal to the angle  $ECF$ , [I 8]

and they are adjacent angles

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, [Def 10]

therefore each of the angles  $DCF$ ,  $FCE$  is right

Therefore the straight line  $CF$  has been drawn at right angles to the given straight line  $AB$  from the given point  $C$  on it  $\square$   $E F$



### PROPOSITION 12

*To a given infinite straight line, from a given point which is not on it to draw a perpendicular straight line*

Let  $AB$  be the given infinite straight line, and  $C$  the given point which is not on it

thus it is required to draw to the given infinite straight line  $AB$ , from the given point  $C$  which is not on it, a perpendicular straight line

For let a point  $D$  be taken at random on the other side of the straight line  $AB$ , and with centre  $C$  and distance  $CD$  let the circle  $EFG$  be described [Post 3]

let the straight line  $EG$  be bisected at  $H$ ,

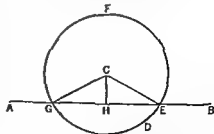
[I 10]

and let the straight lines  $CG$ ,  $CH$ ,  $CE$  be joined [Post 1]

I say that  $CH$  has been drawn perpendicular to the given infinite straight line  $AB$  from the given point  $C$  which is not on it

For, since  $GH$  is equal to  $HE$ ,

and  $HC$  is common,



the two sides  $\underline{GH}$ ,  $\underline{HC}$  are equal to the two sides  $\underline{EH}$ ,  $\underline{HC}$  respectively,

[Def 10]

Therefore  $CH$  has been drawn perpendicular to the given infinite straight line  $AB$  from the given point  $C$  which is not on it

Q E F

### PROPOSITION 13

*If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles*

For let any straight line  $AB$  set up on the straight line  $CD$  make the angles  $CBA$ ,  $ABD$ ,

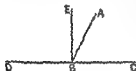
I say that the angles  $CBA$ ,  $ABD$  are either two right angles or equal to two right angles

Now, if the angle  $CBA$  is equal to the angle  $ABD$ , they are two right angles

[Def 10]

But, if not, let  $BE$  be drawn from the point  $B$  at right angles to  $CD$ ,

[I 11]



therefore the angles  $CBA$ ,  $EBD$  are equal to the three angles  $CBA$ ,  $ABE$ ,  $EBD$

[C A 2]

Again, since the angle  $DBA$  is equal to the two angles  $DBE$ ,  $EBA$ ,

let the angle  $ABC$  be added to each,

therefore the angles  $DBA$ ,  $ABC$  are equal to the three angles  $DBE$ ,  $EBA$ ,  $ABC$

[C N 2]

But the angles  $CBE$ ,  $EBD$  were also proved equal to the same three angles, and things which are equal to the same thing are also equal to one another

[C N 1]

therefore the angles  $CBE$ ,  $EBD$  are also equal to the angles  $DBA$ ,  $ABC$

But the angles  $CBE$ ,  $EBD$  are two right angles

therefore the angles  $DBA$ ,  $ABC$  are also equal to two right angles

Therefore etc

Q E D

### PROPOSITION 14

*If with any straight line, and at a point on it two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another*

For with any straight line  $AB$ , and at the point  $B$  on it let the two straight lines  $BC$ ,  $BD$  not lying on the same side make the adjacent angles  $ABC$ ,  $ABD$  equal to two right angles,

I say that  $BD$  is in a straight line with  $CB$

For, if  $BD$  is not in a straight line with  $BC$ , let  $BE$  be in a straight line with  $CB$

CR

Then, since the straight line  $AB$  stands on the straight line  $CBE$ ,  
the angles  $ABC$ ,  $ABE$  are equal to two right angles [I 13]

But the angles  $ABC$ ,  $ABD$  are also equal to two  
right angles,  
therefore the angles  $CBA$ ,  $ABE$  are equal to the  
angles  $CBA$ ,  $ABD$  [Post 4 and CN 1]

Let the angle  $CBA$  be subtracted from each,  
therefore the remaining angle  $ABE$  is equal to the remaining angle  $ABD$ ,  
[CN 3]

the less to the greater which is impossible

Therefore  $BE$  is not in a straight line with  $CB$

Similarly we can prove that neither any other straight line except  $BD$

Therefore  $CB$  is in a straight line with  $BD$

Therefore etc

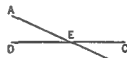
Q E D

### PROPOSITION 15

*If two straight lines cut one another, they make the vertical angles equal to one another*

For let the straight lines  $AB$ ,  $CD$  cut one another at the point  $E$ ,

I say that the angle  $AEC$  is equal to the angle  $DEB$ ,  
and the angle  $CEB$  to the angle  $AED$



lin

angles  $AED$ ,  $DEB$ ,

the angles  $AED$ ,  $DEB$  are equal to two right angles [I 13]

But the angles  $CEA$ ,  $AED$  were also proved equal to two right angles,  
therefore the angles  $CEA$ ,  $AED$  are equal to the angles  $AED$ ,  $DEB$

[Post 4 and CN 1]

Let the angle  $AED$  be subtracted from each,  
therefore the remaining angle  $CEA$  is equal to the remaining angle  $DEB$

[CN 3]

Similarly it can be proved that the angles  $CEB$ ,  $DEA$  are also equal

Therefore etc

Q E D

[PORISM From this it is manifest that, if two straight lines cut one another, they will make the angles at the point of section equal to four right angles]

### PROPOSITION 16

*In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles*

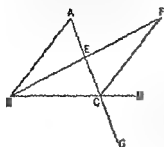
Let  $ABC$  be a triangle and let one side of it  $BC$  be produced to  $D$ ,  
[I 13] is greater than either of the interior and

and let  $BE$  be joined and produced in a  
straight line to  $F$ ,

let  $EF$  be made equal to  $BE$ , [I 3]

let  $FC$  be joined [Post 1], and let  $AC$  be drawn through to  $G$  [Post 2]

Then, since  $AE$  is equal to  $EC$ , and  $BE$  to  $EF$ ,  
 the two sides  $AE$ ,  $EB$  are equal to the two sides  $CE$ ,  $EF$  respectively;  
 and the angle  $AEB$  is equal to the angle  $FEC$ ,  
 for they are vertical angles [1. 15]



Therefore the base  $AB$  is equal to the base  $FC$ ,  
 and the triangle  $ABE$  is equal to the triangle  $FEC$ ,  
 and the remaining angles are equal to the remain-  
 ing angles respectively, namely, those which the  
 equal sides subtend; [1. 4]

therefore the angle  $BAE$  is equal to the angle  $ECF$   
 But the angle  $ECD$  is greater than the angle  
 $ECF$ ; [C.N. 3]

angle  $ACD$

$\angle E D$

Therefore etc.

### PROPOSITION 17

In any triangle two angles taken together in any manner are less than two right angles

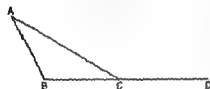
Let  $ABC$  be a triangle;

I say that two angles of the triangle  $ABC$  taken together in any manner are  
 less than two right angles

For let  $BC$  be produced to  $D$  [Post. 2,

Then, since the angle  $ACD$  is an exterior  
 angle of the triangle  $ABC$ ,  
 it is greater than the interior and opposite  
 angle  $ABC$  [1. 16]

Let the angle  $ACB$  be added to each,



### PROPOSITION 18

In any triangle the greater side subtends the greater angle

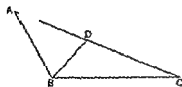
$\angle B$ ;

$AB$ , let  $AD$   
 and let  $BD$  be

joined

Then, since the angle  $ADB$  is an exterior  
 angle of the triangle  $BCD$ ,  
 it is greater than the interior and opposite  
 angle  $DCB$  [1. 16]

But the angle  $ADB$  is equal to the angle  $ABD$ ,  
 since the side  $AB$  is equal to  $AD$ ;



therefore the angle  $ABD$  is also greater than the angle  $ACB$ ,  
therefore the angle  $ABC$  is much greater than the angle  $ACB$

Therefore etc

Q E D

PROPOSITION

In any triangle

I. The angle  $ABC$  is greater than the angle  $BCA$ ;  
I. The side  $AC$  is greater than the side  $AB$

F. The side  $AC$  is either equal to  $AB$  or less

Now  $AC$  is not equal to  $AB$ ,  
for then the angle  $ABC$  would also have been equal to the angle  $ACB$ , [1 5]

but it is not,

therefore  $AC$  is not equal to  $AB$

Neither is  $AC$  less than  $AB$ ,  
for then the angle  $ABC$  would also have been less than the angle  $ACB$ , [1 18]

but it is not,

therefore  $AC$  is not less than  $AB$

And it was proved that it is not equal either

Therefore  $AC$  is greater than  $AB$

Therefore etc

Q E D

# PROPOSITION 20

In any triangle two sides taken together in any manner are greater than the remaining one

For let  $ABC$  be a triangle

I say that in the triangle  $ABC$  two sides taken together in any manner are greater than the remaining one, namely

$BA, AC$  greater than  $BC$ ,

$AB, BC$  greater than  $AC$ ,

$BC, CA$  greater than  $AB$

For let  $BA$  be drawn through to the point  $D$ , let  $DA$  be made equal to  $CA$ , and let  $DC$  be joined

Then since  $DA$  is equal to  $CA$ ,

the angle  $ADC$  is also equal to the angle  $ACD$  [1 5]

therefore the angle  $BCD$  is greater than the angle  $ADC$

[C N 5]

And, since  $DCB$  is a triangle having the angle  $BCD$  greater than the angle  $BDC$ ,

and the greater angle is subtended by the greater side, [1 19]

therefore  $DB$  is greater than  $BC$

But  $DA$  is equal to  $AC$ ,

therefore  $BA, AC$  are greater than  $BC$

Similarly we can prove that  $AB, BC$  are also greater than  $CA$ , and  $BC, CA$  than  $AB$

Therefore etc

Q E D

# PROPOSITION 21

If on one of the sides of a triangle, from its extremities, there be constructed two



straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle

On  $BC$ , one of the sides of the triangle  $ABC$ , from its extremities  $B, C$ , let the two straight lines  $BD, DC$  be constructed meeting within the triangle;

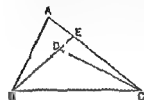
I say that  $BD, DC$  are less than the remaining two sides of the triangle  $BA, AC$ , but contain an angle  $BDC$  greater than the angle  $BAC$ .

For let  $BD$  be drawn through to  $E$

Then, since in any triangle two sides are greater  
[I 20]  
the two sides  $AB$ ,

Let  $EC$  be added to each,

therefore  $BA, AC$  are greater than  $BE, EC$



therefore  $CE, EB$  are greater than  $CD, DB$

But  $BA, AC$  were proved greater than  $BE, EC$ ,

therefore  $BA, AC$  are much greater than  $BD, DC$

Again, since in any triangle the exterior angle is greater than the interior and opposite angle,  
[I 16]

therefore, in the triangle  $CDE$ ,

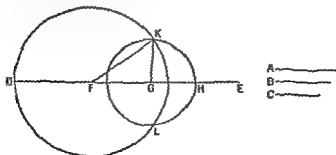
Therefore etc

Q E D

### PROPOSITION 22

Out of three straight lines, which are equal to three given straight lines to construct a triangle thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one  
[I 20]

Let the three given straight lines be  $A, B, C$ , and of these let two taken together in any manner be greater than the remaining one,



namely

$A, B$  greater than  $C$ ,

and

$A, C$  greater than  $B$ ,

$B, C$  greater than  $A$ ;

and let  $DF$  be made equal to  $A$ ,  $FG$  equal to  $B$ , and  $GH$  equal to  $C$  [1 3]

With centre  $F$  and distance  $FD$  let the circle  $DKL$  be described,  
again, with centre  $G$  and distance  $GH$  let the circle  $KLH$  be described,  
and let  $KF$ ,  $KG$  be joined,

I say that the triangle  $KFG$  has been constructed out of three straight lines equal to  $A$ ,  $B$ ,  $C$

For, since the point  $F$  is the centre of the circle  $DKL$ ,  
 $FD$  is equal to  $FK$

But  $FD$  is equal to  $A$ ,  
therefore  $KF$  is also equal to  $A$

Again, since the point  $G$  is the centre of the circle  $LKH$ ,  
 $GH$  is equal to  $GK$

But  $GH$  is equal to  $C$ ,  
therefore  $KG$  is also equal to  $C$

And  $FG$  is also equal to  $B$ ,  
therefore the three straight lines  $KF$ ,  $FG$ ,  $GK$  are equal to the three straight lines  $A$ ,  $B$ ,  $C$

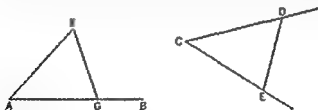
Therefore out of the three straight lines  $KF$ ,  $FG$ ,  $GK$ , which are equal to the three given straight lines  $A$ ,  $B$ ,  $C$ , the triangle  $KFG$  has been constructed  
Q E F

### PROPOSITION 23

On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle

Let  $AB$  be the given straight line  $A$  the point on it, and the angle  $DCE$  the given rectilineal angle,

thus it is required to construct on the given straight line  $AB$ , and at the point  $A$  on it, a rectilineal angle equal to the given rectilineal angle  $DCE$



On the straight lines  $CD$ ,  $CE$  respectively let the points  $D$ ,  $E$  be taken at random,

let  $DE$  be joined,

tively,

and the base  $DE$  is equal to the base  $FG$ ,  
the angle  $DCE$  is equal to the angle  $FAG$

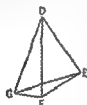
[1 8]

Therefore on the given straight line  $AB$ , and at the point  $A$  on it, the rectilinear angle  $FAG$  has been constructed equal to the given rectilinear angle  $DCE$  Q E F

## PROPOSITION 24

Let  $ABC$ ,  $DEF$  be two triangles having the two sides  $AB$ ,  $AC$  equal to the two sides  $DE$ ,  $DF$  respectively, namely  $AB$  to  $DE$ , and  $AC$  to  $DF$ , and let the angle at  $A$  be greater than the angle at  $D$ ,

I say that the base  $BC$  is also greater than the base  $EF$



For, since the angle  $BAC$  is greater than the angle  $EDF$ , let there be constructed, on the straight line  $DE$ , and at the point  $D$  on it, the angle  $EDG$  equal to the angle  $BAC$ ; [1 23]

let  $DG$  be made equal to either of the two straight lines  $AC$ ,  $DF$ , and let  $EG$ ,  $FG$  be joined

Then, since  $AB$  is equal to  $DE$ , and  $AC$  to  $DG$ ,

Again, since  $DF$  is equal to  $DG$ ,

4)

$EGF$ ,

and the greater angle is subtended by the greater side, [1 19]  
the side  $EG$  is also greater than  $EF$

But  $EG$  is equal to  $BC$

Therefore  $BC$  is also greater than  $EF$

Therefore etc

Q E D

## PROPOSITION 25

If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other

equal to the  
, and let the

:

Now the angle  $BAC$  is not equal to the angle  $EDF$ , for then the base  $BC$  would also have been equal to the base  $EF$ , [1 4]

but it is not,

therefore the angle  $BAC$  is not equal to the angle  $EDF$ .



Neither again  $\equiv$  the angle  $BAC$  less than the angle  $EDF$ , for then the base  $BC$  would also have been less than the base  $EF$ , [1. 24]



but it is not, therefore the angle  $BAC$  is not less than the angle  $EDF$ .  
But it was proved that it is not equal either,  
therefore the angle  $BAC$  is greater than the angle  $EDF$ .  
Therefore etc Q E D

## PROPOSITION 26

remaining sides and the remaining angle to the remaining angle

t  
l  
e

sides respectively, namely  $AB$  to  $DE$  and  $AC$  to  $DF$ , and the remaining angle to the remaining angle, namely the angle  $BAC$  to the angle  $EDF$



For, if  $AB$  is unequal to  $DE$ , one of them is greater.  
Let  $AB$  be greater and let  $BG$  be made equal to  $DE$ , and let  $GC$  be joined.  
Then, since  $BG$  is equal to  $DE$  and  $BC$  to  $EF$ ,  
the two sides  $GB$ ,  $BC$  are equal to the two sides  $DE$ ,  $EF$  respectively,  
and the angle  $GBC$  is equal to the angle  $DEF$ ,  
therefore the base  $GC$  is equal to the base  $DF$ ,  
and the triangle  $GBC$  is equal to the triangle  $DEF$ ,  
and the remaining angles will be equal to the remaining angles, namely those  
which the equal sides subtend, [1. 4]

But the ar

Therefore  $AB$  is not unequal to  $DE$   
and is therefore equal to it  
But  $BC$  is al-o equal to  $EF$ ,

therefore the two sides  $AB$ ,  $BC$  are equal to the two sides  $DE$ ,  $EF$  respectively,

and the angle  $ABC$  is equal to the angle  $DEF$ ,  
therefore the base  $AC$  is equal to the base  $DF$ ,

namely  $AC$  to  $DF$  and  $BC$  to  $EF$ , and further the remaining angle  $BAC$  is equal to the remaining angle  $EDF$

For, if  $BC$  is unequal to  $EF$ , one of them is greater

Let  $BC$  be greater, if possible, and let  $BH$  be made equal to  $EF$ , let  $AH$  be joined

Then, since  $BH$  is equal to  $EF$ , and  $AB$  to  $DE$ ,  
the two sides  $AB$ ,  $BH$  are equal to the two sides  $DE$ ,  $EF$  respectively, and they contain equal angles,

therefore the base  $AH$  is equal to the base  $DF$ ,

and the triangle  $ABH$  is equal to the triangle  $DEF$ ,

and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend, [1 4]

and opposite angle  $BCA$

which is impossible

[1 16]

Therefore  $BC$  is not unequal to  $EF$ ,

and is therefore equal to it

But  $AB$  is also equal to  $DE$ ,

therefore the two sides  $AB$ ,  $BC$  are equal to the two sides  $DE$ ,  $EF$  respectively, and they contain equal angles,

therefore the base  $AC$  is equal to the base  $DF$ ,

the triangle  $ABC$  equal to the triangle  $DEF$ ,

and the remaining angle  $BAC$  equal to the remaining angle  $EDF$  [1 4]

Therefore etc

Q E D

### PROPOSITION 27

If two straight lines falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another

For let the straight line  $EF$  falling on the two straight lines  $AB$ ,  $CD$  make the alternate angles  $AEF$ ,  $EFD$  equal to one another,

I say that  $AB$  is parallel to  $CD$

For if not,  $AB$ ,  $CD$  when produced will meet either in the direction of  $B$ ,  $D$  or towards  $A$ ,  $C$

Let them be produced and meet, in the direction of  $B$ ,  $D$ , at  $G$

Then in the triangle  $GEF$ ,

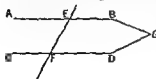
the exterior angle  $AEF$  is equal to the interior and opposite angle  $EFG$

which is impossible

[1 16]

Therefore  $AB$ ,  $CD$  when produced will not meet in the direction of  $B$ ,  $D$

Similarly it can be proved that neither will they meet towards  $A$ ,  $C$



But straight lines which do not meet in either direction are parallel;

[Def 23]

therefore  $AB$  is parallel to  $CD$

Therefore etc

Q E D

### PROPOSITION 28

*If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another*

For let the straight line  $EF$  fall on the two straight lines  $AB$ ,  $CD$  make

For, since the angle  $EGB$  is equal to the angle  $GHD$ ,

while the angle  $EGB$  is equal to the angle  $AGH$ ,

[I 15]

the angle  $AGH$  is also equal to the angle  $GHD$ ,

and they are alternate,

therefore  $AB$  is parallel to  $CD$

[I 27]

Again, since the angles  $BGH$ ,  $GHD$  are equal to two right angles, and the angles  $AGH$ ,  $BGH$  are also equal to two right angles,

[I 13]

the angles  $AGH$ ,  $BGH$  are equal to the angles  $BGH$ ,  $GHD$

Let the angle  $BGH$  be subtracted from each,

therefore the remaining angle  $AGH$  is equal to the remaining angle  $GHD$ ,

and they are alternate,

therefore  $AB$  is parallel to  $CD$

[I 27]

Therefore etc

Q E D }

### PROPOSITION 29

*A straight line falling on parallel straight lines makes the alternate angles equal to one another the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles*

For let the straight line  $EF$  fall on the parallel straight lines  $AB$ ,  $CD$ ,

I say that it makes the alternate angles  $AGH$ ,  $GHD$  equal, the exterior angle  $EGB$  equal to the interior and opposite angle  $GHD$  and the interior angles on the same side, namely  $BGH$ ,  $GHD$  equal to two right angles

For, if the angle  $AGH$  is unequal to the angle  $GHD$ , one of them is greater

Let the angle  $AGH$  be greater

Let the angle  $BGH$  be added to each,

therefore the angles  $AGH$ ,  $BGH$  are greater than the angles  $BGH$ ,  $GHD$

But the angles  $AGH$ ,  $BGH$  are equal to two right angles,

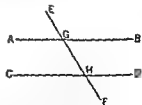
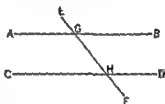
[I 13]

therefore the angles  $BGH$ ,  $GHD$  are less than two right angles

But straight lines produced indefinitely from angles less than two right angles meet,

[Post 5]

therefore  $AB$ ,  $CD$ , if produced indefinitely, will meet,



but they do not  $\Rightarrow$   $\angle$  between the two lines is  $0^\circ$  or  $180^\circ$ .  
Therefore the angle  $\angle$

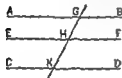
Again, the angle  $AGH$  is equal to the angle  $EGB$ , [15]

D [CN 2]  
[I 13]

therefore the angles  $BGH$ ,  $GHD$  are also equal to two right angles  
Therefore etc Q.E.D.

### PROPOSITION 30

**Straight lines parallel to the same straight line are also parallel to one another**  
 Let each of the straight lines  $AB, CD$  be parallel to  $EF$ , I say that  $AB$  is also parallel to  $CD$



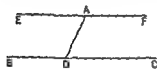
But the

Therefore  $AB$  is parallel to  $CD$  and they are alternate

### PROPOSITION 31

*Through a given point to draw a straight line parallel to a given straight line*

Let  $A$  be the given point, and  $BC$  the given straight line  
thus it is required to draw through the point  $A$  a straight line parallel to the  
straight line  $BC$



Let a point  $D$  be taken at random on  $BC$ , and let  $AD$  be joined, on the straight line  $DA$ , and at the point  $A$  on it, let the angle  $DAE$  be constructed equal to the angle  $ADC$  [I 23], and let the straight line  $AF$  be produced in a straight line with  $EA$

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### PROPOSITION 32

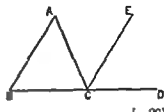
*In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles*

Let  $ABC$  be a triangle, and let one side of it  $BC$  be produced to  $D$ ,  
 I say that the exterior angle  $ACD$  is equal to the two interior and opposite angles  $CAB$ ,  $ABC$ , and the three interior angles of the triangle  $ABC$ ,  $BCA$ ,  $CAB$  are equal to two right angles

For let  $CE$  be drawn through the point  $C$  parallel to the straight line  $AB$  [I 31]

Then, since  $AB$  is parallel to  $CE$ ,  
 and  $AC$  has fallen upon them,  
 the alternate angles  $BAC$ ,  $ACE$  are equal to one another [I 29]

Again, since  $AB$  is parallel to  $CE$ ,  
 and the straight line  $BD$  has fallen upon them,  
 the exterior angle  $ECD$  is equal to the interior



$BAC$ ,  $ABC$

Let the angle  $ACB$  be added to each,  
 therefore the angles  $ACD$ ,  $ACB$  are equal to the three angles  $ABC$ ,  $BCA$ ,  $CAB$

But the angles  $ACD$ ,  $ACB$  are equal to two right angles, [I 13]  
 therefore the angles  $ABC$ ,  $BCA$ ,  $CAB$  are also equal to two right angles

Therefore etc

Q E D

### PROPOSITION 33

The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel

Let  $AB$  and  $CD$  be equal and parallel straight lines, and let them be joined

I say

Let  $BC$  be joined

Then since  $AB$  is parallel to  $CD$ , and  $BC$  has fallen upon them  
 the alternate angles  $ABC$ ,  $BCD$  are equal to one another [I 29]

And, since  $AB$  is equal to  $CD$ ,

and  $BC$  is common

the two sides  $AB$ ,  $BC$  are equal to the two sides  $DC$ ,  $CB$ ,  
 and the angle  $ABC$  is equal to the angle  $BCD$ ,

therefore the base  $AC$  is equal to the base  $BD$ ,

and the triangle  $ABC$  is equal to the triangle  $DCB$ ,

and the remaining angles will be equal to the remaining angles respectively,  
 namely those which the equal sides subtend [I 4]

therefore the angle  $ACB$  is equal to the angle  $CBD$

And since the straight line  $BC$  falling on the two straight lines  $AC$ ,  $BD$  has made the alternate angles equal to one another,

$AC$  is parallel to  $BD$

[I 27]

And it was also proved equal to it

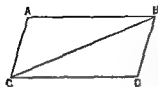
Therefore etc

Q E D



## PROPOSITION 34

*In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas*



For, since  $AB$  is parallel to  $CD$ ,  
and the straight line  $BC$  has fallen upon them,  
the alternate angles  $ABC$ ,  $BCD$  are equal to one another [I 29]

Again, since  $AC$  is parallel to  $BD$  and  $BC$  has fallen upon them,

therefore they will also have the remaining sides equal to the remaining sides respectively, and the remaining angle to the remaining angle, [I 26]  
therefore the side  $AB$  is equal to  $CD$ ,

and  $AC$  to  $BD$ ,

angle  $CDB$

to one another

I say, next, that the diameter also bisects the areas

For, since  $AB$  is equal to  $CD$ ,

and  $BC$  is common,

the two sides  $AB$ ,  $BC$  are equal to the two sides  $DC$ ,  $CB$  respectively,

and the angle  $ABC$  is equal to the angle  $BCD$ ,

therefore the base  $AC$  is also equal to  $BD$

and the triangle  $ABC$  is equal to the triangle  $DCB$  [I 4]

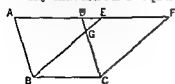
Therefore the diameter  $BC$  bisects the parallelogram  $ACDB$  Q E D

## PROPOSITION 35

*Parallelograms which are on the same base and in the same parallels are equal to one another*

Let  $ABCD$ ,  $EBCF$  be parallelograms on the same base  $BC$  and in the same parallels  $AF$ ,  $BC$ ,

I say that  $ABCD$  is equal to the parallelogram  $EBCF$



For since  $ABCD$  is a parallelogram,

$AD$  is equal to  $BC$

[I 34]

For the same reason also

$EF$  is equal to  $BC$ ,

so that  $AD$  is also equal to  $EF$ ,

[C.N 1]

and  $DE$  is common,

therefore the whole  $AE$  is equal to the whole  $DF$  [C V 2]

But  $AB$  is also equal to  $DC$ , [I 34]

therefore the two sides  $EA, AB$  are equal to the two sides  $FD, DC$  respectively,  
and the angle  $FDC$  is equal to the angle  $EAB$ ,

the exterior to the interior, [I 29]

therefore the base  $FB$  is equal to the base  $EC$ ,

and the triangle  $EAB$  will be equal to the triangle  $FDC$  [I 4]

Let  $DGE$  be subtracted from each,

therefore the trapezium  $ABGD$  which remains is equal to the trapezium  $EGCF$   
which remains [C A 3]

Let the triangle  $GBC$  be added to each,

therefore the whole parallelogram  $ABCD$  is equal to the whole parallelogram  
 $EBCF$  [C A 2]

Therefore etc

Q E D

### PROPOSITION 36

*Parallelograms which are on equal bases and in the same parallels are equal to one another*

Let  $ABCD, EFGH$  be parallelograms which are on equal bases  $BC, FG$  and  
in the same parallels  $AH, BG$ ,

I say that the parallelogram  $ABCD$  is equal to  $EFGH$

For let  $BE, CH$  be joined

Then, since  $BC$  is equal to  $FG$ ,

while

$FG$  is equal to  $EH$ ,

$BC$  is also equal to  $EH$  [C A 1]

But they are also parallel

And  $EB, HC$  join them,

but straight lines joining equal and parallel straight lines (at the extremities  
which are) in the same directions (respectively) are equal and parallel [I 33]

Therefore  $EBCH$  is a parallelogram

[I 34]

And it is equal to  $ABCD$ ,

for it has the same base  $BC$  with it, and is in the same parallels  $BC, AH$  with  
it [I 35]

For the same reason also  $EFGH$  is equal to the same  $EBCH$ , [I 35]

so that the parallelogram  $ABCD$  is also equal to  $EFGH$  [C N 1]

Therefore etc

Q E D

### PROPOSITION 37

*Triangles which are on the same base and in the same parallels are equal to one another*

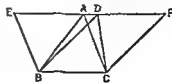
Let  $ABC, DBC$  be triangles on the same base  $BC$  and in the same parallels  
 $AD, BC$ ,

I say that the triangle  $ABC$  is equal to the  
triangle  $DBC$

Let  $AD$  be produced in both directions to  
 $E, F$ ,

through  $B$  let  $BE$  be drawn parallel to  $CA$ ,

[I 31]



and through  $C$  let  $CF$  be drawn parallel to  $BD$

[I 31]

Then each of the figures  $EBCA$ ,  $DBCF$  is a parallelogram, and they are equal,

for they are on the same base  $BC$  and in the same parallels  $BC$ ,  $EF$  [I 35]

Moreover the triangle  $ABC$  is half of the parallelogram  $EBCA$ , for the diameter  $AB$  bisects it

[I 34]

And the triangle  $DBC$  is half of the parallelogram  $DBCF$ , for the diameter  $DC$  bisects it

[I 34]

[But the halves of equal things are equal to one another]

Therefore the triangle  $ABC$  is equal to the triangle  $DBC$

Therefore etc

Q E D

## PROPOSITION 38

*Triangles which are on equal bases and in the same parallels are equal to one another*

Let  $ABC$ ,  $DEF$  be triangles on equal bases  $BC$ ,  $EF$  and in the same parallels  $BC$ ,  $AD$ ,

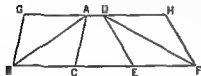
I say that the triangle  $ABC$  is equal to the triangle  $DEF$

For let  $AD$  be produced in both directions to  $G$ ,  $H$ ,

through  $B$  let  $BG$  be drawn parallel to  $CA$ ,

[I 31]

and through  $F$  let  $FH$  be drawn parallel to  $DE$



Then each of the figures  $GBCA$ ,  $DEFH$  is a parallelogram,

and  $GBCA$  is equal to  $DEFH$ ,

for they are on equal bases  $BC$ ,  $EF$  and in the same parallels  $BF$ ,  $GH$

[I 36]

Moreover the triangle  $ABC$  is half of the parallelogram  $GBCA$ , for the diameter  $AB$  bisects it

[I 34]

And the triangle  $FED$  is half of the parallelogram  $DEFH$ , for the diameter  $DF$  bisects it

[I 34]

[But the halves of equal things are equal to one another]

Therefore the triangle  $ABC$  is equal to the triangle  $DEF$

Therefore etc

Q E D

## PROPOSITION 39

*Equal triangles which are on the same base and on the same side are also in the same parallels*

Let  $ABC$ ,  $DBC$  be equal triangles which are on the same base  $BC$  and on the same side of it

[I say that they are also in the same parallels]

And [For] let  $AD$  be joined, I say that  $AD$  is parallel to  $BC$

For, if not, let  $AE$  be drawn through the point  $A$  parallel to the straight line  $BC$ ,

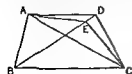
[I 31]

and let  $EC$  be joined

Therefore the triangle  $ABC$  is equal to the triangle  $EBC$ ,

for it is on the same base  $BC$  with it and in the same parallels

[I 37]



But  $ABC$  is equal to  $DBC$ ,



therefore  $DBC$  is also equal to  $EBC$ , [C.N. 1]

the greater to the less which is impossible

Therefore  $AE$  is not parallel to  $BC$

Similarly we can prove that neither is any other straight line except  $AD$ ;  
therefore  $AD$  is parallel to  $BC$ .

Therefore etc

Q E D

#### PROPOSITION 40

*Equal triangles which are on equal bases and on the same side are also in the same parallels*

Let  $ABC$ ,  $CDE$  be equal triangles on equal bases  $BC$ ,  $CE$  and on the same side

I say that they are also in the same parallels

For let  $AD$  be joined,

I say that  $AD$  is parallel to  $BE$

For, if not, let  $AF$  be drawn through  $A$  parallel to  $BE$

[I. 31] and let  $FE$  be joined

Therefore the triangle  $ABC$  is equal to the triangle  $FCE$ ,

for they are on equal bases  $BC$ ,  $CE$  and in the same parallels  $BE$ ,  $AF$  [I. 38]

But the triangle  $ABC$  is equal to the triangle  $DCE$ ,

therefore the triangle  $DCE$  is also equal to the triangle  $FCE$ , [C.N. 1]

the greater to the less which is impossible

Therefore  $AF$  is not parallel to  $BE$

Similarly we can prove that neither is any other straight line except  $AD$ ,  
therefore  $AD$  is parallel to  $BE$

Therefore etc

Q E D

#### PROPOSITION 41

*If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle*

For let the parallelogram  $ABCD$  have the same base  $BC$  with the triangle  $EBC$  and let it be in the same parallels  $BC$ ,  $AE$ ,

I say that the parallelogram  $ABCD$  is double of the triangle  $BEC$

For let  $AC$  be joined

Then the triangle  $ABC$  is equal to the triangle  $EBC$ ,  
for it is on the same base  $BC$  with it and in the same parallels  $BC$ ,  $AE$  [I. 37]

But the parallelogram  $ABCD$  is double of the triangle  $ABC$ ,

for the diameter  $AC$  bisects it, [I. 34]

so that the parallelogram  $ABCD$  is also double of the triangle  $EBC$

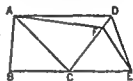
Therefore etc

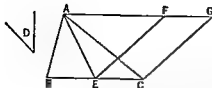
Q E D

#### PROPOSITION 42

*To construct in a given rectilineal angle a parallelogram equal to a given triangle*

Let  $ABC$  be the given triangle, and  $D$  the given rectilineal angle,  
thus it is required to construct in the rectilineal angle  $D$  a parallelogram equal





to the triangle  $ABC$

Let  $BC$  be bisected at  $E$ , and let  $AE$  be joined,  
on the straight line  $EC$ , and at the point  
 $E$  on it, let the angle  $CEF$  be constructed  
equal to the angle  $D$ , [I 23]

through  $A$  let  $AG$  be drawn parallel to  $EC$ , and [I 31]  
through  $C$  let  $CG$  be drawn parallel to  $EF$

Then  $FECG$  is a parallelogram

And, since  $BE$  is equal to  $EC$ ,

therefore the parallelogram  $FECG$  is equal to the triangle  $ABC$

#### PROPOSITION 43

*In any parallelogram the complements of the parallelograms about the diameter are equal to one another*

Let  $ABCD$  be a parallelogram, and  $AC$  its diameter,  
and about  $AC$  let  $EH$ ,  $FG$  be parallelograms and  $BK$ ,  $KD$  the so-called complements,

I say that the complement  $BK$  is equal to the complement  $KD$

For, since  $ABCD$  is a parallelogram, and  $AC$  its diameter,

the triangle  $ABC$  is equal to the triangle  $ACD$  [I 34]

Again, since  $EH$  is a parallelogram, and  $AK$  is  
its diameter,

the triangle  $AEK$  is equal to the triangle  $AHK$

For the same reason

the triangle  $KFC$  is also equal to  $AGC$

Now, since the triangle  $AEK$  is equal to the  
triangle  $AHK$ ,

and  $KFC$  to  $AGC$ ,

the triangle  $AEK$  together with  $KGC$  is equal to the triangle  $AHK$  together  
with  $KFC$  [C.N 2]

And the whole triangle  $ABC$  is also equal to the whole  $ADC$ ,

therefore the complement  $BK$  which remains is equal to the complement  $KD$   
which remains [C.N 3]

Therefore etc

Q E D

#### PROPOSITION 44

*To a given straight line to apply in a given rectilineal angle a parallelogram equal to a given triangle*

Let  $AB$  be the given straight line,  $C$  the given triangle and  $D$  the given rectilineal angle,

Let  $HB$  be joined  
 Then, since the straight line  $HF$  falls upon the parallels  $AH$ ,  $FF$ , the angles  $AHF$ ,  $HFE$  are equal to two right angles [I 29]

Therefore the angles  $BHG$ ,  $GFE$  are less than two right angles, and straight lines produced indefinitely from angles less than two right angles meet, [Post 5]

therefore  $HB$ ,  $FE$ , when produced, will meet

Let them be produced and meet at  $K$ , through the point  $K$  let  $KL$  be drawn parallel to either  $EA$  or  $FH$ , [I 31]

and let  $HA$ ,  $GB$  be produced to the points  $L$ ,  $M$

Then  $HLKF$  is a parallelogram,  $HK$  is its diameter, and  $AG$ ,  $ME$  are parallelograms, and  $LB$ ,  $BF$  the so-called complements about  $HK$ ,

therefore  $LB$  is equal to  $BF$ . [I 43]

But  $BF$  is equal to the triangle  $C$

therefore  $LB$  is also equal to  $C$  [C.N. 1]

And, since the angle  $GBE$  is equal to the angle  $ABM$ , [I 15]

while the angle  $GBE$  is equal to  $D$ ,

the angle  $ABM$  is also equal to the angle  $D$

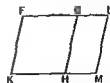
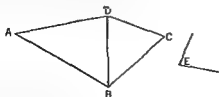
Therefore the parallelogram  $LB$  equal to the given triangle  $C$  has been applied to the given straight line  $AB$ , in the angle  $ABM$  which is equal to  $D$ .

Q.E.F.

#### PROPOSITION 45

To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure

Let  $ABCD$  be the given rectilineal figure and  $E$  the given rectilineal angle, thus it is required to construct, in the given angle  $E$ , a parallelogram equal to the rectilineal figure  $ABCD$



Let  $DB$  be joined and let the parallelogram  $FH$  be constructed equal to the triangle  $ABD$ , in the angle  $HKF$  which is equal to  $E$ , [I 42]  
 let the parallelogram  $GM$  equal to the triangle  $DBC$  be applied to the straight

line  $GH$ , in the angle  $GHM$  which is equal to  $E$

[I 44]

 $GHM$ ,

[CN 1]

therefore the angles  $FAH$ ,  $KHG$  are equal to the angles  $KHG$ ,  $GHM$

[I 29]

right angles,

therefore  $KH$  is in a straight line with  $HM$

[I 14]

And, since the straight line  $HG$  falls upon the parallels  $KM$ ,  $FG$ , the alternate angles  $MHG$ ,  $HGF$  are equal to one another

[I 29]

Let the angle  $HGL$  be added to each,

therefore the angles  $MHG$ ,  $HGL$  are equal to the angles  $HGF$ ,  $HGL$

[CN 2]

But the angles  $MHG$ ,  $HGL$  are equal to two right angles,

[I 29]

[CN 1]

[I 14]

[I 34]

and  $HG$  to  $ML$  also,

$KF$  is also equal and parallel to  $ML$ ,

[CN 1, I 30]

and the straight lines  $KM$ ,  $FL$  join them (at their extremities), therefore  $KM$ ,  $FL$  are also equal and parallel

[I 33]

Therefore  $KFLM$  is a parallelogram

And, since the triangle  $ABD$  is equal to the parallelogram  $FH$ ,

and  $DBC$  to  $GM$ ,

Q E D

## PROPOSITION 46

On a given straight line to describe a square

Let  $AB$  be the given straight line, thus it is required to describe a square on the straight line  $AB$

Let  $AC$  be drawn at right angles to the straight line  $AB$  from the point  $A$  on it [I 11] and let  $AD$  be made equal to  $AB$ ,

through the point  $D$  let  $DE$  be drawn parallel to  $AB$ , and through the point  $B$  let  $BE$  be drawn parallel to  $AD$

[I 31]

Therefore  $ADEB$  is a parallelogram,

therefore  $AB$  is equal to  $DE$ , and  $AD$  to  $BE$

[I 31]

and  $AD$  is equal to  $AB$

therefore  $BE$  is equal to  $AB$ , and  $AD$  to  $BE$

[I 31]

I say next that it is also right angled

For, since the straight line  $AD$  falls upon the parallels  $AB$ ,  $DE$ , the angles  $BAD$ ,  $ADE$  are equal to two right angles

But the angle  $BAD$  is right

therefore the angle  $ADE$  is also right

And in parallelogrammic areas the opposite sides and angles are equal to one another, [I 34]

therefore each of the opposite angles  $ABE$ ,  $BED$  is also right

Therefore  $ADEB$  is right-angled

And it was also proved equilateral

Therefore it is a square and it is described on the straight line  $AB$   $Q E T$

#### PROPOSITION 47

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle

Let  $ABC$  be a right-angled triangle having the angle  $BAC$  right,

I say that the square on  $BC$  is equal to the squares on  $BA$ ,  $AC$

For let there be described on  $BC$  the square  $BDEC$  and on  $BA$ ,  $AC$  the squares  $GBHC$  [I 46]

through  $A$  let  $AL$  be drawn parallel to either  $BD$  or  $CE$  and let  $AD$ ,  $FC$  be joined

Then since each of the angles  $BAC$ ,  $BAG$  is right it follows that with a straight line  $BA$  and at the point  $A$  on it the two straight lines  $AC$ ,  $AG$  not lying on the same side make the adjacent angles equal to two right angles

therefore  $CA$  is in a straight line with  $AG$  [I 14]

For the same reason

$BA$  is also in a straight line with  $AH$

And since the angle  $DBC$  is equal to the angle  $FBA$  for each is right

let the angle  $ABC$  be added to each

therefore the whole angle  $DBA$  is equal to the whole angle  $FBC$  [CN 2]

And since  $DB$  is equal to  $BC$  and  $FB$  to  $BA$

the two sides  $AB$ ,  $BD$  are equal to the two sides  $FB$ ,  $BC$  respectively,

and the angle  $ABD$  is equal to the angle  $FBC$

therefore the base  $AD$  is equal to the base  $FC$ ,

and the triangle  $ABD$  is equal to the triangle  $FBC$  [I 4]

Now the parallelogram  $BL$  is double of the triangle  $ABD$  for they have the same base  $BD$  and are in the same parallels  $BD$ ,  $AL$  [I 41]

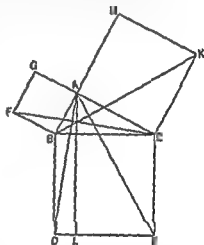
And the square  $GB$  is double of the triangle  $FBC$

for they again have the same base  $FB$  and are in the same parallels  $FB$ ,  $GC$  [I 41]

[But the doubles of equals are equal to one another]

Therefore the parallelogram  $BL$  is also equal to the square  $GB$

Similarly if  $AE$ ,  $BK$  be joined



and the squares  $GB, HC$  on  $BA, AC$

Therefore the square on the side  $BC$  is equal to the squares on the sides  $BA, AC$

Therefore etc

Q E D

### PROPOSITION 48

*If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right*

For in the triangle  $ABC$  let the square on one side  $BC$  be equal to the squares on the sides  $BA, AC$ ,

I say that the angle  $BAC$  is right

For let  $AD$  be drawn from the point  $A$  at right angles to the straight line  $BC$ , let  $AD$  be made equal to  $BA$ , and let  $DC$  be joined

Since  $DA$  is equal to  $AB$ ,

the square on  $DA$  is also equal to the square on  $AB$

Let the square on  $AC$  be added to each,  
therefore the squares on  $DA, AC$  are equal to the squares on  $BA, AC$

But the square on  $DC$  is equal to the squares on  $DA, AC$ , for the angle  $DAC$  is right, [I 47]  
and the square on  $BC$  is equal to the squares on  $BA, AC$ , for this is the hypothesis,

therefore the square on  $DC$  is equal to the square on  $BC$ ,  
so that the side  $DC$  is also equal to  $BC$

And, since  $DA$  is equal to  $AB$ ,

and  $AC$  is common,

the two sides  $DA, AC$  are equal to the two sides  $BA, AC$ ,

and the base  $DC$  is equal to the base  $BC$ ,

therefore the angle  $DAC$  is equal to the angle  $BAC$  [I 8]

But the angle  $DAC$  is right,

therefore the angle  $BAC$  is also right

Therefore etc

Q E D.



## BOOK TWO

### DEFINITIONS

1 Any rectangular parallelogram is said to be *contained* by the two straight lines containing the right angle

2 And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a *gnomon*

### BOOK II PROPOSITIONS

#### PROPOSITION 1

*If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments*

Let  $A, BC$  be two straight lines, and let  $BC$  be cut at random at the points  $D, E$ ,

I say that the rectangle contained by  $A, BC$  is equal to the rectangle contained by  $A, BD$  that contained by  $A, DE$  and that contained by  $A, EC$

For let  $BF$  be drawn from  $B$  at right angles to  $BC$ ,

[1 11]

let  $BG$  be made equal to  $A$ ,

[1 3]

through  $G$  let  $GH$  be drawn parallel to  $BC$

[1 31]

and through  $D, E, C$  let  $DK, EL, CH$  be drawn parallel to  $BG$

Then  $BH$  is equal to  $BK, DL, EH$

Now  $BH$  is the rectangle  $A, BC$  for it is contained by  $GB, BC$  and  $BG$  is equal to  $A$ ,

$BK$  is the rectangle  $A, BD$  for it is contained by  $GB, BD$  and  $BG$  is equal to  $A$ ,

and  $DL$  is the rectangle  $A, DE$  for  $DK$  that is  $BG$  is equal to  $A$  [1 34]

Similarly also  $EH$  is the rectangle  $A, EC$

Therefore the rectangle  $A, BC$  is equal to the rectangle  $A, BD$ , the rectangle  $A, DE$  and the rectangle  $A, EC$

Therefore etc

Q E D

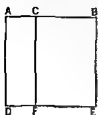
#### PROPOSITION 2

*If a straight line be cut at random the rectangle contained by the whole and both of the segments is equal to the square on the whole*

For let the straight line  $AB$  be cut at random at the point  $C$

I say that the rectangle contained by  $AB, BC$  together with the rectangle contained by  $BA, AC$  is equal to the square on  $AB$

For let the square  $ADEB$  be described on  $AB$  [I 46], and let  $CF$  be drawn through  $C$  parallel to either  $AD$  or  $BE$  [I 31]



Then  $AE$  is equal to  $AF$ ,  $CE$

Now  $AE$  is the square on  $AB$ ,

$AF$  is the rectangle contained by  $BA$ ,  $AC$ , for it is contained by  $DA$ ,  $AC$ , and  $AD$  is equal to  $AB$ ,

and  $CE$  is the rectangle  $AB$ ,  $BC$ , for  $BE$  is equal to  $AB$

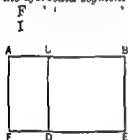
Therefore the rectangle  $BA$ ,  $AC$  together with the rectangle  $AB$ ,  $BC$  is equal to the square on  $AB$

Therefore etc.

Q E D

### PROPOSITION 3

If a straight line be cut at random, the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the segments and the square on the aforesaid segment



the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the segments and the square on the aforesaid segment

For let the square  $CDEB$  be described on  $CB$ , [I 46]

let  $ED$  be drawn through to  $F$ , and through  $A$  let  $AF$  be drawn parallel to either  $CD$  or  $BE$  [I 31]

Then  $AE$  is equal to  $AD$ ,  $CE$

Now  $AE$  is the rectangle contained by  $AB$ ,  $BC$ , for it is contained by  $AB$ ,  $BE$ , and  $BE$  is equal to  $BC$ ,

$AD$  is the rectangle  $AC$ ,  $CB$ , for  $DC$  is equal to  $CB$ ;

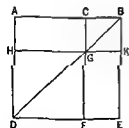
and  $DB$  is the square on  $CB$

Therefore the rectangle contained by  $AB$ ,  $BC$  is equal to the rectangle contained by  $AC$ ,  $CB$  together with the square on  $BC$

Therefore etc

Q E D

### PROPOSITION 4



rectangle contained by  $AC$ ,  $CB$

For let the square  $ADEB$  be described on  $AB$ , [I 46]

let  $BD$  be joined,

through  $C$  let  $CF$  be drawn parallel to either  $AD$  or  $EB$ , and through  $G$  let  $HK$  be drawn parallel to either  $AB$  or  $DE$  [I 31]

Then, since  $CF$  is parallel to  $AD$ , and  $BD$  has fallen on them,

the exterior angle  $CGB$  is equal to the interior and opposite angle  $ADB$  [I 29]

But the angle  $ADB$  is equal to the angle  $ABD$ ,

since the side  $BA$  is also equal to  $AD$ ,

[I 51]



therefore the angle  $CGB$  is also equal to the angle  $GBC$ ,

so that the side  $BC$  is also equal to the side  $CG$

[I 6]

But  $CB$  is equal to  $GK$ , and  $CG$  to  $KB$ ,

[I 34]

therefore  $GK$  is also equal to  $KB$ ,

therefore  $CGKB$  is equilateral

I say next that it is also right-angled

For, since  $CG$  is parallel to  $BK$ ,

the angles  $KBC$ ,  $GCB$  are equal to two right angles

[I 29]

But the angle  $KBC$  is right,

therefore the angle  $BCG$  is also right,

so that the opposite angles  $CGK$ ,  $GKB$  are also right

[I 34]

Therefore  $CGKB$  is right-angled,

and it was also proved equilateral,

therefore it is a square,

and it is described on  $CB$

For the same reason

$HF$  is also a square,

and it is described on  $HG$ , that is  $AC$

[I 34]

Therefore the squares  $HF$ ,  $KC$  are the squares on  $AC$ ,  $CB$

Now, since  $AG$  is equal to  $GE$

and  $AG$  is the rectangle  $AC$ ,  $CB$ , for  $GC$  is equal to  $CB$ ,

therefore  $GE$  is also equal to the rectangle  $AC$ ,  $CB$

Therefore  $AG$ ,  $GE$  are equal to twice the rectangle  $AC$ ,  $CB$

But the squares  $HF$ ,  $KC$  are also the squares on  $AC$ ,  $CB$ , therefore the four areas  $HF$ ,  $KC$ ,  $AG$ ,  $GE$  are equal to the squares on  $AC$ ,  $CB$  and twice the rectangle contained by  $AC$ ,  $CB$

But  $HF$ ,  $KC$ ,  $AG$ ,  $GE$  are the whole  $ADFB$ ,

which is the square on  $AB$

Therefore the square on  $AB$  is equal to the squares on  $AC$ ,  $CB$  and twice the rectangle contained by  $AC$ ,  $CB$

Therefore etc

Q E D

### PROPOSITION 5

*If a straight line be cut into equal and unequal segments the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half*

For let a straight line  $AB$  be cut into equal segments at  $C$  and into unequal segments at  $D$ ,

I say that the rectangle contained by  $AD$ ,  $DB$  together with the square on  $CD$  is equal to the square on  $CB$

For let the square  $CEFB$  be described on  $CB$ ,

[I 46]

and let  $BE$  be joined,  
through  $D$  let  $DG$  be drawn parallel to  
either  $CE$  or  $BF$ ,

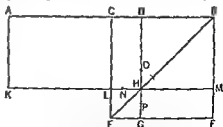
through  $H$  again let  $KM$  be drawn parallel to either  $AB$  or  $EF$ ,

and again through  $A$  let  $AK$  be drawn parallel to either  $CL$  or  $BF$

[I 31]

Then, since the complement  $CH$  is equal to the complement  $HF$ ,

[I 43]



let  $DM$  be added to each,  
therefore the whole  $CM$  is equal to the whole  $DF$ .

But  $CM$  is equal to  $AL$ ,

since  $AC$  is also equal to  $CB$ ,  
therefore  $AL$  is also equal to  $DF$ .

[I 36]

Let  $CH$  be added to each,

But

Let  $LG$ , which is equal to the square on  $CD$ , be added to each,  
therefore the gnomon  $NOP$  and  $LG$  are equal to the rectangle contained by  $AD$ ,  $DB$  and the square on  $CD$

But the gnomon  $NOP$  and  $LG$  are the whole square  $CEFB$ , which is described on  $CB$ ,  
therefore the rectangle contained by  $AD$ ,  $DB$  together with the square on  $CD$  is equal to the square on  $CB$

Therefore etc.

Q E D

## PROPOSITION 6

If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line

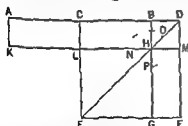
$CB$  is equal to the square on  $CD$

For let the square  $CEFD$  be described on  $CD$ ,  
and let  $DE$  be joined;

[I 46]

through the point  $B$  let  $BG$  be drawn parallel to either  $EC$  or  $DF$ ,  
through the point  $H$  let  $KM$  be drawn parallel to either  $AB$  or  $EF$ ,  
and further through  $A$  let  $AK$  be drawn parallel to either  $CL$  or  $DM$

[I 31]



Then, since  $AC$  is equal to  $CB$ ,

$AL$  is also equal to  $CH$

[I 36]

But  $CH$  is equal to  $HF$

[I 43]

Therefore  $AL$  is also equal to  $HF$

Let  $CM$  be added to each,

therefore the whole  $AM$  is equal to the gnomon  $NOP$

But  $AM$  is the rectangle  $AD$ ,  $DB$ ,

for  $DM$  is equal to  $DB$ ,

therefore the gnomon  $NOP$  is also equal to the rectangle  $AD$ ,  $DB$

Let  $LG$ , which is equal to the square on  $BC$ , be added to each,

therefore the rectangle contained by  $AD$ ,  $DB$  together with the square on  $CB$

is equal to the square on  $CD$ ,

therefore the rectangle contained by  $AD$ ,  $DB$  together with the square on  $CB$  is equal to the square on  $CD$

Therefore etc.

Q E D



therefore  $AD$  is also equal to  $GR$ ,

therefore the four areas  $DK, CK, GR, RN$  are equal to one another

Therefore the four are quadruple of  $CK$

Again, since  $CB$  is equal to  $BD$ ,

while  $BD$  is equal to  $BF$ , that is  $CG$ ,

and  $CB$  is equal to  $GK$ , that is  $GQ$ ,

therefore  $CG$  is also equal to  $GQ$

And, since  $CG$  is equal to  $GQ$ , and  $QR$  to  $RP$ ,

$AG$  is also equal to  $MQ$ , and  $QL$  to  $RF$  [I 36]

But  $MQ$  is equal to  $QL$ , for they are complements of the parallelogram  $ML$ , [I 43]

therefore  $AG$  is also equal to  $RF$ ,

therefore the four areas  $AG, MQ, QL, RF$  are equal to one another

Therefore the four are quadruple of  $AG$

But the four areas  $CK, KD, GR, RN$  were proved to be quadruple of  $CK$ , therefore the eight areas, which contain the gnomon  $STU$ , are quadruple of  $AK$

Now, since  $AK$  is the rectangle  $AB, BD$ , for  $BK$  is equal to  $BD$ ,

with the square on  $AC$  is

are  $AEFD$ ,

therefore four times the rectangle  $AB, BD$  together with the square on  $AC$  is equal to the square on  $AD$

But  $BD$  is equal to  $BC$ ,

therefore four times the rectangle contained by  $AB, BC$  together with the square on  $AC$  is equal to the square on  $AD$ , that is to the square described on  $AB$  and  $BC$  as on one straight line

Therefore etc

Q E D

### PROPOSITION 9

If a straight line be cut into equal and unequal segments the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section

For let a straight line  $AB$  be cut into equal segments at  $C$  and into unequal segments at  $D$ ,

I say that the squares on  $AD, DB$  are double of the squares on  $AC, CD$

For let  $CE$  be drawn from  $C$  at right angles to  $AB$ , and let it be made equal to either  $AC$  or  $CB$ ,

let  $EA, EB$  be joined

let  $L$

Then, since  $AC$  is equal to  $CF$

the angle  $ECF$  is also equal to the angle  $AEC$

And since the angle at  $C$  is right

the remaining angles  $EAC, AEC$  are equal to one right angle [I 32]

And they are equal,

therefore each of the angles  $CEA, CAE$  is half a right angle.

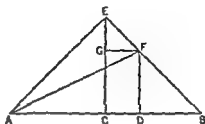
For the same reason

each of the angles  $CFB, EBC$  is also half a right angle,

therefore the whole angle  $AEB$  is right

And since the angle  $GEF$  is half a right angle

and the angle  $EGF$  is right for it is equal to the interior



[I 29]

[I 32]

so that the side  $FG$  is also equal to  $GF$

[I 6]

Again since the angle at  $B$  is half a right angle

and the angle  $FDB$  is right for it is again equal to the interior and opposite angle  $ECB$ ,

[I 29]

the remaining angle  $BFD$  is half a right angle,

[I 32]

therefore the angle at  $B$  is equal to the angle  $DFB$ ,

so that the side  $FD$  is also equal to the side  $DB$

[I 6]

Now since  $AC$  is equal to  $CE$

the square on  $AC$  is also equal to the square on  $CE$ ,

therefore the squares on  $AC, CE$  are double of the square on  $AC$

But the square on  $EA$  is equal to the squares on  $AC, CE$ , for the angle  $ACE$

is right

[I 47]

therefore the square on  $EA$  is double of the square on  $AC$

Again since  $EG$  is equal to  $GF$

the square on  $EG$  is also equal to the square on  $GF$ ,

therefore the squares on  $EG, GF$  are double of the square on  $GF$

But the square on  $EF$  is equal to the squares on  $EG, GF$ ,

therefore the square on  $EF$  is double of the square on  $GF$

But  $GF$  is equal to  $CD$

[I 34]

therefore the square on  $EF$  is double of the square on  $CD$

But the square on  $EA$  is also double of the square on  $AC$

therefore the squares on  $AE, EF$  are double of the squares on  $AC, CD$

And the square on  $AF$  is equal to the squares on  $AE, EF$  for the angle  $AEF$

is right

[I 47]

therefore the square on  $AF$  is double of the squares on  $AC, CD$

But the squares on  $AD, DF$  are equal to the square on  $AF$  for the angle at  $D$  is right

[I 47]

therefore the squares on  $AD, DF$  are double of the squares on  $AC, CD$

And  $DF$  is equal to  $DB$

therefore the squares on  $AD, DB$  are double of the squares on  $AC, CD$

Therefore etc

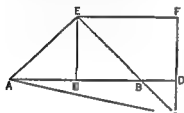
Q E D

## PROPOSITION 10

line

For let a straight line  $AB$  be bisected at  $C$ , and let a straight line  $BD$  be added to it in a straight line,

I say that the squares on  $AD$ ,  $DB$  are double of the squares on  $AC$ ,  $CD$



For let  $CE$  be drawn from the point  $C$  at right angles to  $AB$  [I 11], and let it be made equal to either  $AC$  or  $CB$  [I 3],

let  $EA$ ,  $EB$  be joined,

through  $E$  let  $EF$  be drawn parallel to  $AD$ , and through  $D$  let  $FD$  be drawn parallel to  $CE$  [I 31], on the

[I 29]

But straight lines produced from angles less than two right angles meet, [I Post 5]

therefore  $EB$ ,  $FD$ , if produced in the direction  $B$ ,  $D$ , will meet

Let them be produced and meet at  $G$ ,

and let  $AG$  be joined

Then, since  $AC$  is equal to  $CE$ ,

the angle  $\angle AEC$  is a right angle [I 5]

therefore each of

[I 32]

For the same reason

each of the angles  $\angle CEB$ ,  $\angle EBC$  is also half a right angle,

[I 15]

But the angle  $\angle BDG$  is also right,

for it is equal to the angle  $\angle DCE$ , they being alternate, [I 29]

[I 32]

[I 6]

Again, since the angle  $\angle LGF$  is half a right angle,

and the angle at  $F$  is right for it is equal to the opposite angle, the angle at  $C$ ,

[I 34]

the remaining angle  $\angle FLG$  is half a right angle, [I 32]

therefore the angle  $\angle EGF$  is equal to the angle  $\angle FEG$ ,

so that the side  $GF$  is also equal to the side  $EF$  [I 6]

Now, since the square on  $EC$  is equal to the square on  $CA$ ,

the squares on  $EC$ ,  $CA$  are double of the square on  $CA$ .

But the square on  $EA$  is equal to the squares on  $EC$ ,  $CA$ ,

[I 47]

therefore the square on  $EA$  is double of the square on  $AC$  [C.N. 1]  
 Again, since  $FG$  is equal to  $EF$ ,

the square on  $FG$  is also equal to the square on  $FE$ ,

therefore the squares on  $GF, FE$  are double of the square on  $EF$   
 But the square on  $EG$  is equal to the squares on  $GF, FE$ , [1 47]

therefore the square on  $EG$  is double of the square on  $EF$   
 And  $EF$  is equal to  $CD$ , [1 34]

therefore the square on  $EG$  is double of the square on  $CD$

But the square on  $EA$  was also proved double of the square on  $AC$ ;  
 therefore the squares on  $AE, EG$  are double of the squares on  $AC, CD$

And the square on  $AG$  is equal to the squares on  $AE, EG$ , [1 47]

therefore the square on  $AG$  is double of the squares on  $AC, CD$

But the squares on  $AD, DG$  are equal to the square on  $AG$ , [1 47]

therefore the squares on  $AD, DG$  are double of the squares on  $AC, CD$

And  $DG$  is equal to  $DB$ ,

therefore the squares on  $AD, DB$  are double of the squares on  $AC, CD$

Therefore etc

Q E D

### PROPOSITION 11

*To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment*

Let  $AB$  be the given straight line,  
 thus it is required to cut  $AB$  so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment

For let the square  $ABDC$  be described on  $AB$ , [1 46]

let  $AC$  be bisected at the point  $E$  and let  $BF$  be joined,  
 let  $CA$  be drawn through to  $F$ , and let  $EF$  be made equal to  $BE$ ,

let the square  $FHI$  be described on  $AF$ , and let  $GH$  be drawn through to  $K$

I say that  $AB$  has been cut at  $H$  so as to make the rectangle contained by  $AB, BH$  equal to the square on  $AH$

For, since the straight line  $AC$  has been bisected at  $E$ , and  $FA$  is added to it  
 the rectangle contained by  $CF, FA$  together with the square on  $AE$  is equal to the square on  $EF$  [11 6]

But  $EF$  is equal to  $EB$ ,  
 therefore the rectangle  $CF, FA$  together with the square on  $AE$  is equal to the square on  $EB$

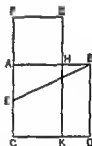
But the squares on  $BA, AE$  are equal to the square on  $EB$ , for the angle at  $A$  is right [1 47]

therefore the rectangle  $CF, FA$  together with the square on  $AE$  is equal to the squares on  $BA, AE$

Let the square on  $AE$  be subtracted from each,  
 therefore the rectangle  $CF, FA$  which remains is equal to the square on  $AB$

Now the rectangle  $CF, FA$  is  $FK$ , for  $AF$  is equal to  $FG$ ,  
 and the square on  $AB$  is  $AD$ ,  
 therefore  $FK$  is equal to  $AD$

Let  $AK$  be subtracted from each,



therefore  $FH$  which remains  $\equiv$  equal to  $HD$

And  $HD \equiv$  the rectangle  $AB, BH$ , for  $AB$  is equal to  $BD$ ,  
and  $FH$  is the square on  $AH$ ,

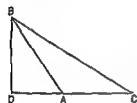
therefore the rectangle contained by  $AB, BH \equiv$  equal to the square on  $HA$   
therefore the given straight line  $AB$  has been cut at  $H$  so as to make the rectangle contained by  $AB, BH$  equal to the square on  $HA$  Q E F

## PROPOSITION 12

*In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle*

Let  $ABC$  be an obtuse-angled triangle having the angle  $BAC$  obtuse, and let  $BD$  be drawn from the point  $B$  perpendicular to  $CA$  produced,

I say



dom at the point  $A$ ,

the square on  $DC$  is equal to the squares on  $CA, AD$   
and twice the rectangle contained by  $CA, AD$  [I 4]

Let the square on  $DB$  be added to each,  
therefore the squares on  $CD, DB$  are equal to the  
squares on  $CA, AD, DB$  and twice the rectangle  $CA, AD$

But the square on  $CB$  is equal to the squares on  $CD, DB$ , for the angle at  $D$   
is right, [I 47]

and the square on  $AB$  is equal to the squares on  $AD, DB$ , [I 47]  
therefore the square on  $CB$  is equal to the squares on  $CA, AB$  and twice the  
rectangle contained by  $CA, AD$ ,  
so that the square on  $CB$  is greater than the squares on  $CA, AB$  by twice the  
rectangle contained by  $CA, AD$

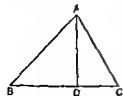
Therefore etc

Q E D

## PROPOSITION 13

*In acute angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle*

Let  $ABC$



on  $CB, BA$  by twice the rectangle contained by  $CB, BD$

For since the straight line  $CB$  has been cut at random at  $D$ ,  
the squares on  $CB, BD$  are equal to twice the rectangle contained by  $CB, BD$  and the square on  $DC$  [I 37]



Let the square on  $DA$  be added to each,  
therefore the squares on  $CB$ ,  $BD$ ,  $DA$  are equal to twice the rectangle contained by  $CB$ ,  $BD$  and the squares on  $AD$ ,  $DC$

But the square on  $AB$  is equal to the squares on  $BD$ ,  $DA$ , for the angle at  $D$  is right, [1 47]

and the square on  $AC$  is equal to the squares on  $AD$ ,  $DC$ ,  
therefore the squares on  $CB$ ,  $BA$  are equal to the square on  $AC$  and twice the rectangle  $CB$ ,  $BD$

so that the square on  $AC$  alone is less than the squares on  $CB$ ,  $BA$  by twice the rectangle contained by  $CB$ ,  $BD$

Therefore etc

Q E D

### PROPOSITION 14

To construct a square equal to a given rectilinear figure

I let  $A$  be the given rectilinear figure,  
thus it is required to construct a square equal to the rectilinear figure  $A$

For let there be constructed  
the rectangular parallelogram  $BD$   
equal to the rectilinear figure  $A$

[1 45]

Then if  $BE$  is equal to  $ED$ ,  
that which was enjoined will  
have been done, for a square  $BD$

has been constructed equal to the rectilinear figure  $A$

But if not one of the straight lines  $BE$ ,  $ED$  is greater

Let  $BE$  be greater, and let it be produced to  $F$ ,

let  $EF$  be made equal to  $ED$  and let  $BF$  be bisected at  $G$

With centre  $G$  and distance one of the straight lines  $GB$ ,  $GF$  let the semicircle  $BHF$  be described let  $DE$  be produced to  $H$  and let  $GH$  be joined

Then since the straight line  $BF$  has been cut into equal segments at  $G$  and into unequal segments at  $E$

the rectangle contained by  $BE$ ,  $EF$  together with the square on  $EG$  is equal to the square on  $GF$  [11 5]

But  $GF$  is equal to  $GH$

therefore the rectangle  $BE$ ,  $EF$  together with the square on  $GE$  is equal to the square on  $GH$

But the squares on  $HE$ ,  $EG$  are equal to the square on  $GH$  [1 47]

therefore the rectangle  $BE$ ,  $EF$  together with the square on  $GE$  is equal to the squares on  $HE$ ,  $EG$

Let the square on  $GE$  be subtracted from each

therefore the rectangle contained by  $BE$ ,  $EF$  which remains is equal to the square on  $EH$

But the rectangle  $BE$ ,  $EF$  is  $BD$  for  $EF$  is equal to  $ED$

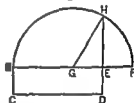
therefore the parallelogram  $BD$  is equal to the square on  $HE$

And  $BD$  is equal to the rectilinear figure  $A$

Therefore the rectilinear figure  $A$  is also equal to the square which can be described on  $EH$

Therefore a square namely that which can be described on  $EH$ , has been constructed equal to the given rectilinear figure  $A$

Q E D



## BOOK THREE

### DEFINITIONS

1 *Equal circles* are those the diameters of which are equal, or the radii of which are equal

2 A straight line is said to *touch a circle* which, meeting the circle and being produced does not cut the circle

3 *Circles* are said to *touch one another* which, meeting one another, do not cut one another

4 In a circle straight lines are said to be *equally distant from the centre* when the perpendiculars drawn to them from the centre are equal

5 And that straight line is said to be at a *greater distance* on which the greater perpendicular falls

6 A *segment of a circle* is the figure contained by a straight line and a circumference of a circle

7 An *angle of a segment* is that contained by a straight line and a circumference of a circle

straight lines so joined

8 And, when the straight lines containing the angle cut off a circumference,

the circumference cut off by them

9 *Similar segments of circles* are those which admit equal angles, or in which the angles are equal to one another

## BOOK III PROPOSITIONS

### PROPOSITION 1

*To find the centre of a given circle*

Let  $ABC$  be the given circle,

thus it is required to find the centre of the circle  $ABC$

Let a straight line  $AB$  be drawn through it at random, and let it be bisected at the point  $D$ ,

from  $D$  let  $DC$  be drawn at right angles to  $AB$  and let it be drawn through to  $E$ , let  $CE$  be bisected at  $F$ ,

I say that  $F$  is the centre of the circle  $ABC$

For suppose it is not, but if possible, let  $G$  be the centre,

and let  $GI$ ,  $GD$ ,  $GB$  be joined

Then since  $AD$  is equal to  $DB$  and  $DG$  is common

the two sides  $AD$   $DG$  are equal to the two sides  $BD$ ,  $DG$  respectively,  
and the base  $GI$  is equal to the base  $GB$ , for they are radii,

therefore the angle  $ADG$  is equal to the angle  $GDB$

[I 8]

But when a straight line set up on a straight line makes the adjacent angles equal to one another each of the equal angles is right,

[I Def 10]

therefore the angle  $GDB$  is right

But the angle  $FDB$  is also right,

therefore the angle  $FDB$  is equal to the angle  $GDB$ ,  
the greater to the less which is impossible

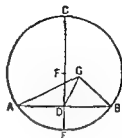
Therefore  $G$  is not the centre of the circle  $ABC$

Similarly we can prove that neither is any other point except  $F$

Therefore the point  $F$  is the centre of the circle  $ABC$

PROPOSITION 1 From this it is manifest that if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line

Q E D



### PROPOSITION 2

If on the circumference of a circle two points be taken at random the straight line joining the points will fall within the circle

Let  $ABC$  be a circle and let two points  $A$ ,  $B$  be taken at random on its circumference,

I say that the straight line joined from  $A$  to  $B$  will fall within the circle

For suppose it does not but if possible let it fall outside as  $AEB$ ,  
let the centre of the circle  $ABC$  be taken [I 1] and let it be  $D$ , let  $DA$ ,  $DB$  be joined and let  $DFE$  be drawn through

Then since  $DA$  is equal to  $DB$ ,  
the angle  $DAE$  is also equal to the angle  $DBE$  [I 5]  
And since one side  $AEB$  of the triangle  $DAE$  is produced

the angle  $DEB$  is greater than the angle  $DAE$  [I 16]

But the angle  $DAE$  is equal to the angle  $DBE$ ,  
therefore the angle  $DEB$  is greater than the angle  $DBE$

And the greater angle is subtended by the greater side,

[I 19]

therefore  $DB$  is greater than  $DE$

But  $DB$  is equal to  $DF$

therefore  $DF$  is greater than  $DE$ ,

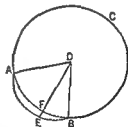
the less than the greater which is impossible

Therefore the straight line joined from  $A$  to  $B$  will not fall outside the circle

Similarly we can prove that neither will it fall on the circumference itself,  
therefore it will fall within

Therefore etc

Q E D



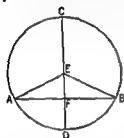
## PROPOSITION 3

*If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles, and if it cut it at right angles, it also bisects it*

Let  $ABC$  be a circle, and in it let a straight line  $CD$  through the centre bisect a straight line  $AB$  not through the centre at the point  $F$ ,

I say that it also cuts it at right angles

For let the centre of the circle  $ABC$  be taken, and let it be  $E$ , let  $EA$ ,  $EB$  be joined



Then, since  $AF$  is equal to  $FB$ , and  $FE$  is common,  
two sides are equal to two sides,  
and the base  $EA$  is equal to the base  $EB$ ,  
therefore the angle  $AFE$   
is equal to the angle  $BFE$  [I 8]

But, when a straight line set up on a straight line  
makes the adjacent angles equal to one another, each  
of the equal angles is right, [I Def 10]

therefore each of the angles  $AFE$ ,  $BFE$  is right

Therefore  $CD$ , which is through the centre, and  
bisects  $AB$  which is not through the centre, also cuts it at right angles

Again, let  $CD$  cut  $AB$  at right angles,

I say that it also bisects it that is, that  $AF$  is equal to  $FB$

For, with the same construction,

since  $EA$  is equal to  $EB$ ,

the angle  $EAF$  is also equal to the angle  $EBF$  [I 5]

But the right angle  $AFE$  is equal to the right angle  $BFE$ , therefore  $EAF$ ,  
 $EBF$  are two triangles having two angles equal to two angles and one side  
equal to one side, namely  $EF$ , which is common to them and subtends one of  
the equal angles,

therefore they will also have the remaining sides equal to the remaining sides,  
[I 26]

therefore  $AF$  is equal to  $FB$

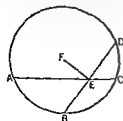
Therefore etc

Q E D

## PROPOSITION 4

*If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another*

Let  $ABCD$  be a circle, and in it let the two straight lines  $AC$ ,  $BD$ , which are



bisects a straight line  $AC$  not through the centre,  
it also cuts it at right angles, [III 3]  
therefore the angle  $FEA$  is right

Again, since  $\equiv$  straight line  $FE$  bisects a straight line  $BD$ ,  
it also cuts it at right angles,  
therefore the angle  $FEB$  is right

[III 3]

But the angle  $FEA$  was also proved right,  
therefore the angle  $FEA \equiv$  equal to the angle  $FEB$ , the less to the greater  
which is impossible

Therefore  $AC, BD$  do not bisect one another

Therefore etc

Q E D

## PROPOSITION 5

*If two circles cut one another, they will not have the same centre*

For let the circles  $ABC, CDG$  cut one another at the points  $B, C$ ,

I say that they will not have the same centre

For, if possible, let it be  $E$ , let  $EC$  be joined, and let  $EFG$  be drawn through at random

Then, since the point  $E$  is the centre of the circle  $ABC$ ,

$EC \equiv$  equal to  $EF$  [I Def 15]

Again, since the point  $E$  is the centre of the circle  $CDG$ ,

$EC \equiv$  equal to  $EG$

But  $EC$  was proved equal to  $EF$  also,  
therefore  $EF$  is also equal to  $EG$ , the less to the greater which is impossible

Therefore the point  $E$  is not the centre of the circles  $ABC, CDG$

Therefore etc

Q E D

## PROPOSITION 6

*If two circles touch one another, they will not have the same centre*

For let the two circles  $ABC, CDE$  touch one another at the point  $C$ ,

I say that they will not have the same centre

For, if possible let it be  $F$ , let  $FC$  be joined, and let  $FEB$  be drawn through at random

Then, since the point  $F$  is the centre of the circle  $ABC$ ,

$FC$  is equal to  $FB$

Again, since the point  $F$  is the centre of the circle  $CDE$ ,

$FC \equiv$  equal to  $FE$

But  $FC$  was proved equal to  $FB$ ,  
therefore  $FE$  is also equal to  $FB$ , the less to the greater which is impossible

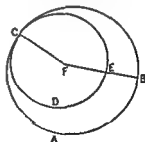
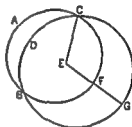
Therefore  $F$  is not the centre of the circles  $ABC, CDE$

Therefore etc

Q E D

## PROPOSITION 7

*If on the diameter of a circle a point be taken which is not the centre of the circle, and from the point straight lines fall upon the circle that will be greatest on which the centre is, the remainder of the same diameter will be least, and of the rest the*



nearer to the straight line through the  
and only two on each  
side of the line

more remote,  
, one on each

Let  $ABCD$  be a circle, let  $AD$  be a diameter of it, on  $AD$  let a point  $F$  be taken which is not the centre of the circle, let  $E$  be the centre of the circle, and from  $F$  let straight lines  $FB$ ,  $FC$ ,  $FG$  fall upon the circle  $ABCD$ , I say that  $FA$  is greatest,  $FD$  is least, and of the rest  $FB$  is greater than  $FC$ , and  $FC$  than  $FG$

For let  $BE$ ,  $CE$ ,  $GE$  be joined

Then, since in any triangle two sides are greater than the remaining one,

[1 20]

$EB$ ,  $EF$  are greater than  $BF$

But  $AE$  is equal to  $BE$ ,

therefore  $AF$  is greater than  $BF$

Again, since  $BE$  is equal to  $CE$ , and  $FE$  is common,

the two sides  $BE$ ,  $EF$  are equal to the two sides  $CE$ ,  $EF$

But the angle  $BEF$  is also greater than the angle  $CEF$ ,

therefore the base  $BF$  is greater than the base  $CF$

[1 24]

For the same reason

$CF$  is also greater than  $FG$

Again, since  $GF$ ,  $FE$  are greater than  $EG$ ,

and  $EG$  is equal to  $ED$ ,

$GF$ ,  $FE$  are greater than  $ED$

Let  $EF$  be subtracted from each,

therefore the remainder  $GF$  is greater than the remainder  $FD$

Therefore  $FA$  is greatest,  $FD$  is least, and  $FB$  is greater than  $FC$ , and  $FC$  than  $FG$

I say also that from the point  $F$  only two equal straight lines can be drawn to the circle  $ABCD$ , one on each side of the straight line  $AD$

For on the straight line  $AD$  let a point  $H$  be taken between  $A$  and  $F$ , and let  $HE$  be joined, and let  $EF$  be subtracted from each,

construct

Then, since  $HE$  is common,  
and  $EF$  is common,  
the two sides  $GE$ ,  $EF$  are equal to the two sides  $HE$ ,  $EF$ ,

from the point  $F$

For, if possible let  $FK$  so fall

Then, since  $FK$  is equal to  $FG$ , and  $FH$  to  $FG$ ,

$FK$  is also equal to  $FH$ ,

the nearer to the straight line through the centre being thus equal to the more remote which is impossible

Therefore another straight line equal to  $GF$  will not fall from the point  $F$  upon the circle,

therefore only one straight line will so fall

Therefore etc

Q. E. D.

### PROPOSITION 8

If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others are drawn at random, then of the straight lines which fall on the concave circumference, that through the centre is greatest, while of the rest the nearer to that through the centre is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote, and only two equal straight lines will fall on the circle from the point, one on each side of the least

Let  $ABC$  be a circle, and let a point  $D$  be taken outside  $ABC$ , let there be drawn through from it straight lines  $DA, DE, DF, DC$ , and let  $DA$  be through the centre,

I say that of the straight lines falling on the concave circumference  $AEFC$ , the straight line  $DA$  through the centre is greatest,

while  $DE$  is greater than  $DF$  and  $DF$  than  $DC$ ,  
but, of the straight lines falling on the convex circumference  $HLKG$ , the straight line  $DG$  between the point and the diameter  $AG$  is least, and the nearer to the least  $DG$  is always less than the more remote, namely  $DK$  than  $DL$  and  $DL$  than  $DH$

For let the centre of the circle  $ABC$  be taken (in 1)  
and let it be  $M$ , let  $ME, MF, MC, MA, ML, MH$  be joined

Then since  $AM$  is equal to  $EM$ , let  $MD$  be added to each

therefore  $AD$  is equal to  $FM + MD$

and  $ME + MD$  is equal to  $FM + MD$

therefore  $AD$  is equal to  $FM + MD$

and  $MD$  is common

therefore  $EM + MD$  are equal to  $FM + MD$ ,

and the angle  $EMD$  is greater than the angle  $FMD$ ,

therefore the base  $ED$  is greater than the base  $FD$

(I 24)

Similarly we can prove that  $FD$  is greater than  $CD$ , therefore  $DA$  is greatest, while  $DE$  is greater than  $DF$ , and  $DF$  than  $DC$

Next since  $MA, MD$  are greater than  $MD$

(I 20)

and  $MG$  is equal to  $MA$ ,

therefore the remainder  $AD$  is greater than the remainder  $GD$ ,

so that  $GD$  is less than  $KD$

And since on  $MD$  one of the sides of the triangle  $MLD$ , two straight lines  $MA, AD$  were constructed meeting within the triangle

therefore  $MA + AD$  are less than  $ML + LD$ ,

(I 21)

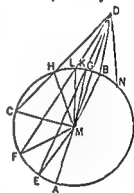
and  $MA$  is equal to  $ML$ ,

therefore the remainder  $AD$  is less than the remainder  $LD$

Similarly we can prove that  $DL$  is also less than  $DH$ ,

therefore  $DG$  is least, while  $DK$  is less than  $DL$  and  $DL$  than  $DH$

I say also that only two equal straight lines will fall from the point  $D$  on the



circle, one on each side of the least  $DG$

On the straight line  $MD$ , and at the point  $M$  on it, let the angle  $DMB$  be constructed equal to the angle  $KMD$ , and let  $DB$  be joined

Then, since  $MA$  is equal to  $MB$ ,

and  $MD$  is common

the two sides  $KM$ ,  $MD$  are equal to the two sides  $BM$ ,  $MD$  respectively,

and the angle  $KMD$  is equal to the angle  $BMD$ ,

therefore the base  $DK$  is equal to the base  $DB$  [I 4]

I say that no other straight line equal to the straight line  $DK$  will fall on the circle from the point  $D$

For, if possible, let a straight line so fall, and let it be  $DN$

Then, since  $DK$  is equal to  $DN$ ,

while  $DK$  is equal to  $DB$ ,

$DB$  is also equal to  $DN$ ,

that is the nearer to the least  $DG$  equal to the more remote which was proved impossible

Therefore no more than two equal straight lines will fall on the circle  $ABC$  from the point  $D$ , one on each side of  $DG$  the least

Therefore etc

Q E D

### PROPOSITION 9

If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle the point taken is the centre of the circle

Let  $ABC$  be a circle and  $D$  a point within it, and from  $D$  let more than two equal straight lines, namely  $DA$ ,  $DB$ ,  $DC$ , fall on the circle  $ABC$ ,

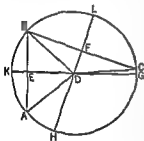
I say that the point  $D$  is the centre of the circle  $ABC$

For let  $AB$ ,  $BC$  be joined and bisected at the points  $E$ ,  $F$ , and let  $ED$ ,  $FD$  be joined and drawn through to the points  $G$ ,  $K$ ,  $H$ ,  $L$

Then, since  $AE$  is equal to  $EB$  and  $ED$  is common,

the two sides  $AE$ ,  $ED$  are equal to the two sides  $BE$ ,  $ED$ ,

and the base  $DA$  is equal to the base  $DB$ ,



an

[III 1 Por]

the centre of the circle is on  $GK$

For the same reason

the centre of the circle  $ABC$  is also on  $HL$

And the straight lines  $GK$ ,  $HL$  have no other point common but the point  $D$ ,

therefore the point  $D$  is the centre of the circle  $ABC$

Therefore etc

Q E



## PROPOSITION 10

*A circle does not cut a circle at more points than two*

For, if possible let the circle  $ABC$  cut the circle  $DEF$  at more points than two namely  $B, C, F, H$ ,

let  $BH, BG$  be joined and bisected at the points  $K, L$ ,  
and from  $K, L$  let  $AC, LM$  be drawn at right angles to  $BH, BG$  and carried through to the points  $A, E$

Then since in the circle  $ABC$  a straight line  $AC$  cuts a straight line  $BH$  into two equal parts and at right angles

the centre of the circle  $ABC$  is on  $AC$

[III 1, Por.]

Again, since in the same circle  $ABC$  a straight line  $AO$  cuts a straight line  $BG$  into two equal parts and at right angles,

the centre of the circle  $ABC$  is on  $NO$

But it was also proved to be on  $AC$ , and the straight lines  $AC, AO$  meet at no point except at  $P$ ,

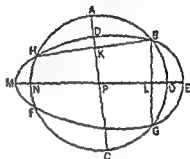
therefore the point  $P$  is the centre of the circle  $ABC$

Similarly we can prove that  $P$  is also the centre of the circle  $DEF$ ,  
therefore the two circles  $ABC, DEF$  which cut one another have the same centre  $P$  which is impossible

[III 5]

Therefore etc

Q E D



## PROPOSITION 11

*If two circles touch one another internally and their centres be taken, the straight line joining their centres if it be also produced will fall on the point of contact of the circles*

For let the two circles  $ABC, ADE$  touch one another internally at the point  $A$ , and let the centre  $F$  of the circle  $ABC$  and the centre  $G$  of  $ADE$ , be taken,

I say that the straight line joined from  $G$  to  $F$  and produced will fall on  $A$

For suppose it does not but if possible let it fall as  $FGH$  and let  $AF, AG$  be joined

Then, since  $AG, GF$  are greater than  $FA$  that is, than  $FH$

let  $FG$  be subtracted from each,  
therefore the remainder  $AG$  is greater than the remainder  $GH$

But  $AG$  is equal to  $GD$ ,

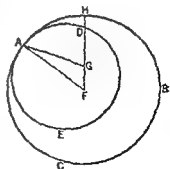
therefore  $GD$  is also greater than  $GH$

the less than the greater which is impossible

Therefore the straight line joined from  $F$  to  $G$  will not fall outside

therefore it will fall at  $A$  on the point of contact

Therefore etc



Q E D

## PROPOSITION 12

*If two circles touch one another externally, the straight line joining their centres will pass through the point of contact*

For let the two circles  $ABC$ ,  $ADE$  touch one another externally at the point  $A$ , and let the centre  $F$  of  $ABC$ , and the centre  $G$  of  $ADE$ , be taken,

I say that the straight line joined from  $F$  to  $G$  will pass through the point of contact at  $A$

For suppose it does not, but, if possible, let it pass as  $FCDG$ , and let  $AF$ ,  $AG$  be joined

Then, since the point  $F$  is the centre of the circle  $ABC$ ,

$FA$  is equal to  $FC$

Again, since the point  $G$  is the centre of the circle  $ADE$ ,

$GA$  is equal to  $GD$

But  $FA$  was also proved equal to  $FC$ , therefore  $FA$ ,  $AG$  are equal to  $FC$ ,  $GD$ , so that the whole  $FG$  is greater than  $FA$ ,  $AG$ ,

but it is also less [I 20] which is impossible

Therefore the straight line joined from  $F$  to  $G$  will not fail to pass through the point of contact at  $A$ ,

therefore it will pass through it

Therefore etc

Q E D

## PROPOSITION 13

*A circle does not touch a circle at more points than one, whether it touch it internally or externally*

For, if possible, let the circle  $ABDC$  touch the circle  $EBFD$  first internally, at more points than one, namely  $D$ ,  $B$

Let the centre  $G$  of the circle  $ABDC$ , and the centre  $H$  of  $EBFD$ , be taken

Therefore the straight line joined from  $G$  to  $H$  will fall on  $B$ ,  $D$  [III 11]

Let it so fall as  $BGHD$

Then, since the point  $G$  is the centre of the circle  $ABDC$ ,

$BG$  is equal to  $GD$ ,

therefore  $BG$  is greater than  $HD$ ,

therefore  $BH$  is much greater than  $HD$

Again, since the point  $H$  is the centre of the circle  $EBFD$ ,

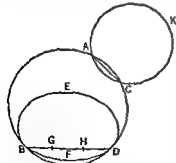
$BH$  is equal to  $HD$ ,

but it was also proved much greater than it which is impossible

Therefore a circle does not touch a circle internally at more points than one

I say further that neither does it so touch it externally

For, if possible, let the circle  $ACK$  touch the circle  $ABDC$  at more points than one, namely  $A$ ,  $C$ ,



and let  $AC$  be joined

Then since on the circumference of each of the circles  $ABDC$ ,  $ACK$  two points  $A$ ,  $C$  have been taken at random, the straight line joining the points will fall within each circle, [III 2]  
but it fell within the circle  $ABDC$  and outside  $ACK$  [III Def 3] which is absurd

Therefore a circle does not touch a circle externally at more points than one  
And it was proved that neither does it so touch it internally  
Therefore etc Q E D

#### PROPOSITION 14

*In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another*

Let  $ABDC$  be a circle, and let  $AB$ ,  $CD$  be equal straight lines in it,

I say that  $AB$ ,  $CD$  are equally distant from the centre

For let the centre of the circle  $ABDC$  be taken [III 1] and let it be  $E$ , from  $E$  let  $EF$ ,  $EG$  be drawn perpendicular to  $AB$ ,  $CD$ , and let  $AF$ ,  $EC$  be joined  
Then since  $AB$  is equal to  $CD$ ,  $EF$  is equal to  $EG$ , and  $AE$  is equal to  $CE$ ,  
th

therefore  $AB$  is double of  $AF$

For the same reason

$CD$  is also double of  $CG$ ,

and  $AB$  is equal to  $CD$ ,

therefore  $AF$  is also equal to  $CG$

And, since  $AE$  is equal to  $EC$ ,

the square on  $AE$  is also equal to the square on  $EC$

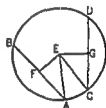
But the squares on  $AF$ ,  $FE$  are equal to the square on  $AE$  for the angle at  $F$  is right,

and the squares on  $EG$ ,  $GC$  are equal to the square on  $EC$ , for the angle at  $G$  is right, [I 47]

therefore the squares on  $AF$ ,  $FE$  are equal to the squares on  $CG$ ,  $GE$ ,  
of which the square on  $AF$  is equal to the square on  $CG$ , for  $AF$  is equal to  $CG$ ,

therefore the square on  $FE$  which remains is equal to the square on  $EG$ ,

therefore  $EF$  is equal to  $EG$



the centre  
[III Def 4],

Next let the straight lines  $AB$ ,  $CD$  be equally distant from the centre, that is, let  $EF$  be equal to  $EG$

I say that  $AB$  is also equal to  $CD$

For, with the same construction, we can prove similarly, that  $AB$  is double of  $AF$ , and  $CD$  of  $CG$

And, since  $AE$  is equal to  $CE$ ,

the square on  $AE$  is equal to the square on  $CE$

But the squares on  $EF$ ,  $FA$  are equal to the square on  $AE$ , and the squares on  $EG$ ,  $GC$  equal to the square on  $CE$  [I 47]

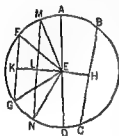
Therefore the squares on  $EF$ ,  $FA$  are equal to the squares on  $EG$ ,  $GC$ ,  
of which the square on  $EF$  is equal to the square on  $EG$ , for  $EF$  is equal to  $EG$ ,

therefore the square on  $AF$  which remains is equal to the square on  $CG$ ,  
 therefore  $AF$  is equal to  $CG$   
 And  $AB$  is double of  $AF$ , and  $CD$  double of  $CG$ ,  
 therefore  $AB$  is equal to  $CD$   
 Therefore etc Q E D

## PROPOSITION 15

*Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote*

Let  $ABCD$  be a circle, let  $AD$  be its diameter and  $E$  the centre, and let  $BC$  be nearer to the diameter  $AD$ , and  $FG$  more remote,  
 I say that  $AD$  is greatest and  $BC$  greater than  $FG$   
 For from the centre  $E$  let  $EH$ ,  $EK$  be drawn perpendicular to  $BC$ ,  $FG$   
 Then, since  $BC$  is nearer to the centre and  $FG$  more remote  $EK$  is greater than  $EH$  [III Def 5]  
 Let  $EL$  be made equal to  $EH$ , through  $L$  let  $LM$  be drawn at right angles to  $EK$  and carried through to  $N$ , and let  $ME$ ,  $EN$ ,  $FE$ ,  $EG$  be joined



Then, since  $EH$  is equal to  $EL$ ,

$BC$  is also equal to  $MN$

[III 14]

Again, since  $AE$  is equal to  $EM$ , and  $ED$  to  $EN$ ,

$AD$  is equal to  $ME$ ,  $EN$ .

But  $ME$ ,  $EN$  are greater than  $MN$ ,

[I 20]

and  $MN$  is equal to  $BC$ ,

therefore  $AD$  is greater than  $BC$

And, since the two sides  $ME$ ,  $EN$  are equal to the two sides  $FE$ ,  $EG$ ,

and the angle  $MEN$  greater than the angle  $FEG$ ,

therefore the base  $MN$  is greater than the base  $FG$

[I 24]

But  $MN$  was proved equal to  $BC$

Therefore the diameter  $AD$  is greatest and  $BC$  greater than  $FG$

Therefore etc Q E D

## PROPOSITION 16

*The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed, further the angle of the semi-circle is greater, and the remaining angle less, than any acute rectilineal angle*

Let  $ABC$  be a circle about  $D$  as centre and  $AB$  as diameter,

I say that the straight line drawn from  $A$  at right angles to  $AB$  from its extremity will fall outside the circle

For suppose it does not but, if possible, let it fall within as  $CA$ , and let  $DC$  be joined

Since  $DA$  is equal to  $DC$ ,

the angle  $DAC$  is also equal to the angle  $ACD$

[I 5]

But the angle  $DAC$  is right,

Therefore the straight line drawn from the point *A* at right angles to *BA* will not fall within the circle

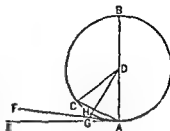
Similarly we can prove that neither will it fall on the circumference,

therefore it will fall outside

Let it fall as *AE*,

I say next that into the space between the straight line *AE* and the circumference *CHA* another straight line cannot be interposed

For, if possible, let another straight line be so interposed, as *FA*, and let *DG* be drawn from the point *D* perpendicular to *FA*



Then, since the angle *AGD* is right,

and the angle *DAG* is less than a right angle,

*AD* is greater than *DG*

[I 19]

But *DA* is equal to *DH*;

therefore *DH* is greater than *DG*, the less than the greater which is impossible

Therefore another straight line cannot be interposed into the space between the straight line and the circumference

line will be interposed such as will make an angle contained by straight lines

is greater than the angle contained by the straight line *BA* and the circumference *CHA*

to the diameter of a circle from its extremity touches the circle Q E D

### PROPOSITION 17

From a given point to draw a straight line touching a given circle

Let *A* be the given point, and *BCD* the given circle, thus it is required to draw from the point *A* a straight line touching the circle *BCD*

For let the centre *E* of the circle be taken, [III 1]  
let *AE* be joined, and with centre *E* and distance *EA* let the circle *AFG* be described,

from *D* let *DF* be drawn at right angles to *EA*,  
and let *EF*, *AB* be joined,

I say that  $AB$  has been drawn from the point  $A$  touching the circle  $BCD$

For, since  $E$  is the centre of the circles  $BCD$ ,  $AFG$ ,

$EA$  is equal to  $EF$ , and  $ED$  to  $EB$ ,

therefore the two sides  $AE$ ,  $EB$  are equal to the two sides  $FE$ ,  $ED$ ,

and they contain a common angle, the angle at  $E$ ,

therefore the base  $DF$  is equal to the base  $AB$ ,

and the triangle  $DEF$  is equal to the triangle  $BEA$ ,

and the remaining angles to the remaining angles,

[I 4]

therefore the angle  $EDF$  is equal to the angle  $EBA$

But the angle  $EDF$  is right,

therefore the angle  $EBA$  is also right

Now  $EB$  is a radius,

and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle,

[III 16, Por 1]

therefore  $AB$  touches the circle  $BCD$

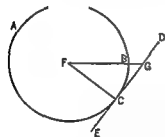
Therefore from the given point  $A$  the straight line  $AB$  has been drawn touching the circle  $BCD$

Q E F

### PROPOSITION 18

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent

F I



For, if not, let  $FG$  be drawn from  $F$  perpendicular to  $DE$

Then the angle  $FGC$  is right

or side,

[I 19]

therefore  $FC$  is greater than  $FG$

But  $FC$  is equal to  $FB$ ,

therefore  $FB$  is also greater than  $FG$ ,

the less than the greater which is impossible

Therefore  $FG$  is not perpendicular to  $DE$

Similarly we can prove that neither is any other straight line except  $FC$ , therefore  $FC$  is perpendicular to  $DE$

Therefore etc

Q E D

### PROPOSITION 19

If a straight line touch a circle and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn

and let  $CF$  be joined

Since a straight line  $DE$  touches the circle  $ABC$ , and  $FC$  has been joined from the centre to the point of contact,

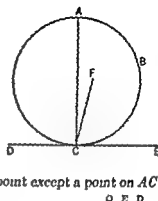
$FC$  is perpendicular to  $DE$ , [iii 18]

the angle  $ACE$ ,

the less to the greater which is impossible

Therefore  $F$  is not the centre of the circle  $ABC$

Similarly we can prove that neither is any other point except a point on  $AC$   
Therefore etc



### PROPOSITION 20

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base

Let  $ABC$  be a circle, let the angle  $BEC$  be an angle at its centre, and the angle  $BAC$  an angle at the circumference, and let them have the same circumference  $BC$  as base,

I say that the angle  $BEC$  is double of the angle  $BAC$

For let  $AE$  be joined and drawn through to  $F$

Then, since  $EA$  is equal to  $EB$ ,

the angle  $EAB$  is also equal to the angle  $EBA$ , [i 5]  
therefore the angles  $EAB$ ,  $EBA$  are double of the angle  $EAB$

But the angle  $BEF$  is equal to the angles  $EAB$ ,  $EBA$ , [i 32]  
therefore the angle  $BEF$  is also double of the angle  $EAB$

For the same reason

the angle  $FEC$  is also double of the angle  $EAC$

Therefore the whole angle  $BEC$  is double of the angle  $BAC$

angle

Similarly it can be proved that if the angle  $BAC$  is obtuse, the angle  $BEC$  is double of it

Therefore etc

Q E D

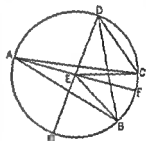
### PROPOSITION 21

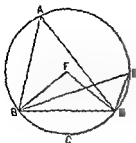
In a circle the angles in the same segment are equal to one another

Let  $ABCD$  be a circle and let the angles  $BAD$ ,  $BED$  be angles in the same segment  $BAED$ ,

I say that the angles  $BAD$ ,  $BED$  are equal to one another

For let the centre of the circle  $ABCD$  be taken, and let it be  $F$ , let  $BF$ ,  $FD$  be joined





Now since the angle  $BFD$  is at the centre  
and the angle  $BAD$  at the circumference  
and they have the same circumference  $BCD$  as base  
therefore the angle  $BFD$  is double of the angle  
 $BAD$  [III 20]

For the same reason  
the angle  $BFD$  is also double of the angle  $BED$   
therefore the angle  $BAD$  is equal to the angle  $BED$   
Therefore etc Q E D

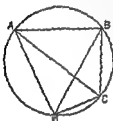
## PROPOSITION 22

The opposite angles of quadrilaterals in circles are equal to two right angles

Let  $ABCD$  be a circle and let  $ABCD$  be a quadrilateral in it  
I say that the opposite angles are equal to two right angles

Let  $AC$   $BD$  be joined

Then since in any triangle the three angles are equal  
to two right angles [I 32]  
the three angles  $CAB$   $ABC$   $BCA$  of the triangle  $ABC$



and the angle  $ACB$  is equal to the angle  $ADB$  for they

Let  
therefore the angles  $ABC$   $BAC$   $ACB$  are equal to the angles  $ABC$   $ADC$   
But  $\angle ABC$  is common to both triangles  $ABC$  and  $ADC$   
Therefore the angles  $BAC$  and  $ADC$  are equal  
Similarly the angles  $ACB$  and  $ADB$  are equal  
Therefore the opposite angles of the quadrilateral  $ABCD$  are equal to two right angles  
Therefore etc Q E D

## PROPOSITION 23

On the same straight line there cannot be constructed two similar and unequal

two similar and unequal  
be constructed on the

same side

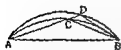
let  $ACD$  be drawn through and let  $CB$   $DB$  be joined

Then since the segment  $ACB$  is similar to the segment  $ADB$ ,

and similar segments of circles are those which admit equal angles [III Def 11]  
the angle  $ACB$  is equal to the angle  $ADB$ , the exterior to the interior which is impossible [I 16]

Therefore etc

Q E D





## PROPOSITION 21

*Similar segments of circles on equal straight lines are equal to one another.*

For let  $AEB$ ,  $CFD$  be similar segments of circles on equal straight lines  $AB$ ,  $CD$ ,



F

COINCIDE WITH  $CFD$ ,

it will either fall within it, or outside it;

or it will fall awry, as  $CGD$ , and a circle cuts a circle at more points than two; which is impossible [III. 10]

Therefore, if the straight line  $AB$  be applied to  $CD$ , the segment  $AEB$  will not fail to coincide with  $CFD$  also;

therefore it will coincide with it and will be equal to it

Therefore etc

Q E D.

## PROPOSITION 23

*Given a segment of a circle, to describe the complete circle of which it is a segment.*

Let  $ABC$  be the given segment of a circle, thus it is required to describe the complete circle belonging to the segment  $ABC$ , that is, of which it is a segment

For let  $AC$  be bisected at  $D$ , let  $DB$  be drawn from the point  $D$  at right angles to  $AC$ , and let  $AB$  be joined,

the angle  $ABD$  is then greater than, equal to, or less than the angle  $BAD$ .

First let it be greater,

and on the straight line  $BA$ , and at the point  $A$  on it, let the angle  $BAE$  be constructed equal to the angle  $ABD$ , let  $DB$  be drawn through to  $E$ , and let  $EC$  be joined

Then, since the angle  $ABE$  is equal to the angle  $BAE$ ,

the straight line  $EB$  is also equal to  $EA$  [I. 6]

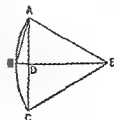
And, since  $AD$  is equal to  $DC$ , and  $DE$  is common, the two sides  $AD$ ,  $DE$  are equal to the two sides  $CD$ ,  $DE$  respectively,

and the angle  $ADE$  is equal to the angle  $CDE$ , for each is right;

therefore the base  $AE$  is equal to the base  $CE$

But  $AE$  was proved equal to  $BE$ ,

therefore  $BE$  is also equal to  $CE$ ,



therefore the three straight lines  $AE, EB, EC$  are equal to one another

Therefore the circle drawn with centre  $E$  and distance one of the straight lines  $AE, EB, EC$  will also pass through the remaining points and will have been completed

[III 9]



$AD$  being equal to each of the two  $BD, DC$ , the three straight lines  $DA, DB, DC$  will be equal to one another,

$D$  will be the centre of the completed circle, and  $ABC$  will clearly be a semicircle

But, if the angle  $ABD$  be less than the angle  $BAD$ , and if we construct, on the straight line  $BA$  and at the point  $A$  on it, an angle equal to the angle  $ABD$ , the centre will fall on  $DB$  within the segment  $ABC$ , and the segment  $ABC$  will clearly be greater than a semicircle



Therefore, given a segment of a circle, the complete circle has been described

Q E F

## PROPOSITION 26

*In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences*

Let  $ABC, DEF$  be equal circles, and in them let there be equal angles, namely at the centres the angles  $BGC, EHF$ , and at the circumferences the angles  $BAC, EDF$ ,

I say that the circumference  $BKC$  is equal to the circumference  $ELF$

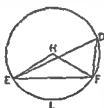
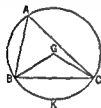
For let  $BC, EF$  be joined

Now, since the circles  $ABC, DEF$  are equal,

the radii are equal

Thus the two straight lines  $BG, GC$  are equal to the two straight lines  $EH, HF$ , and the angle at  $G$  is equal to the angle at  $H$ ,

therefore the base  $BC$  is equal to the base  $EF$  [I 4]



Q, [III Def 11]

But similar segments of circles on equal straight lines are equal to one another, [III 24]

therefore the segment  $BAC$  is equal to  $EDF$

But the whole circle  $ABC$  is also equal to the whole circle  $DEF$ ; therefore the circumference  $BKC$  which remains is equal to the circumference  $ELF$

Therefore etc

Q E D

## PROPOSITION 27

*In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences*

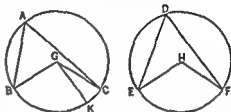
For in equal circles  $ABC$ ,  $DEF$ , on equal circumferences  $BC$ ,  $EF$ , let the angles  $BGC$ ,  $EHF$  stand at the centres  $G$ ,  $H$ , and the angles  $BAC$ ,  $EDF$  at the circumferences;

I say that the angle  $BGC$  is equal to the angle  $EHF$ ,  
and the angle  $BAC$  is equal to the angle  $EDF$

For, if the angle  $BGC$  is unequal to the angle  $EHF$ ,

one of them is greater

Let the angle  $BGC$  be greater and on



Now equal angles stand on equal circumferences, when they are at the centres,  
therefore the circumference  $BK$  is equal to the circumference  $EF$  [III 26]

But  $EF$  is equal to  $BC$ ;  
therefore  $BK$  is also equal to  $BC$ , the less to the greater which is impossible  
Therefore the angle  $BGC$  is not unequal to the angle  $EHF$ ,  
therefore it is equal to it

And the angle at  $A$  is half of the angle  $BGC$ ,  
and the angle at  $D$  half of the angle  $EHF$ , [III 20]  
therefore the angle at  $A$  is also equal to the angle at  $D$

Therefore etc

Q. E. D.

## PROPOSITION 28

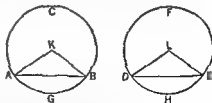
*In equal circles equal straight lines cut off equal circumferences, the greater equal to the greater and the less to the less*

Let  $ABC$ ,  $DEF$  be equal circles and in the circles let  $AB$ ,  $DE$  be equal straight lines cutting off  $ACB$ ,  $DFE$  as greater circumferences and  $AGB$ ,  $DHE$  as lesser,

I say that the greater circumference  $ACB$  is equal to the greater circumference  $DFE$ , and the less circumference  $AGB$  to  $DHE$

For let the centres  $K$ ,  $L$  of the circles be taken, and let  $AK$ ,  $KB$ ,  $DL$ ,  $LE$  be joined

Now since the circles are equal,  
the radii are also equal,  
therefore the two sides  $AK$ ,  $KB$  are equal  
to the two sides  $DL$ ,  $LE$ ,



and the base  $AB$  is equal to the base  $DE$ ,  
therefore the angle  $AKB$  is equal to the angle  $DLE$  [I 8]

But equal angles stand on equal circumferences, when they are at the centres, [III 26]

therefore the circumference  $AGB$  is equal to  $DHE$

And the whole circle  $ABC$  is also equal to the whole circle  $DEF$ ,

therefore the circumference  $ACB$  which remains is also equal to the circumference  $DFE$  which remains

Therefore etc

Q E D

### PROPOSITION 29

*In equal circles equal circumferences are subtended by equal straight lines*

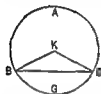
Let  $ABC$ ,  $DEF$  be equal circles, and in them let equal circumferences  $BGC$ ,  $EHF$  be cut off, and let the straight lines  $BC$ ,  $EF$  be joined,

I say that  $BC$  is equal to  $EF$

For let the centres of the circles be taken, and let them be  $K$ ,  $L$ , let  $BK$ ,  $KC$ ,  $EL$ ,  $LF$  be joined

Now, since the circumference  $BGC$  is equal to the circumference  $EHF$ ,

the angle  $BKC$  is also equal to the angle  $ELF$ . [III 27]



And, since the circles  $ABC$ ,  $DEF$  are equal

the radii are also equal,

therefore the two sides  $BK$ ,  $KC$  are equal to the two sides  $EL$ ,  $LF$ , and they contain equal angles,

therefore the base  $BC$  is equal to the base  $EF$  [I 4]

Therefore etc

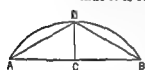
Q E D

### PROPOSITION 30

*To bisect a given circumference*

Let  $ADB$  be the given circumference,

thus it is required to bisect the circumference  $ADB$



Let  $AB$  be joined and bisected at  $C$ , from the point  $C$  let  $CD$  be drawn at right angles to the straight line  $AB$  and let  $AD$ ,  $DB$  be joined

Then, since  $AC$  is equal to  $CB$ , and  $CD$  is common

the two sides  $AC$ ,  $CD$  are equal to the two sides  $BC$ ,  $CD$ ,

and the angle  $ACD$  is equal to the angle  $BCD$ , for each is right,

therefore the base  $AD$  is equal to the base  $DB$  [I 4]

But equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less, [III 28]

and each of the circumferences  $AD$ ,  $DB$  is less than a semicircle,

therefore the circumference  $AD$  is equal to the circumference  $DB$

Therefore the given circumference has been bisected at the point  $D$

Q E F

### PROPOSITION 31

*In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle, and further the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle*

Let  $ABCD$  be a circle let  $BC$  be its diameter, and  $E$  its centre, and  $AC$ ,  $AD$ ,  $DC$  be joined,



the point  $B$  let there be drawn across, in the circle  $ABCD$  a straight line  $BD$  cutting it,

I say that the angles which  $BD$  makes with the tangent  $EF$  will be equal to the angles in the alternate segments of the circle, that is, that the angle  $FBD$  is equal to the angle constructed in the segment  $BAD$ , and the angle  $EBD$  is equal to the angle constructed in the segment  $DCB$

For let  $BA$  be drawn from  $B$  at right angles to  $EF$ ,

let a point  $C$  be taken at random on the circumference  $BD$ ,

and let  $AD$ ,  $DC$ ,  $CB$  be joined

Then, since a straight line  $EF$  touches the circle  $ABCD$  at  $B$ ,

and  $BA$  has been drawn from the point of contact at right angles to the tangent, the centre of the circle  $ABCD$  is on  $BA$  [III 19]

Therefore  $BA$  is a diameter of the circle  $ABCD$ ,

therefore the angle  $ADB$ , being an angle in a semicircle, is right [III 31]

Therefore the remaining angles  $BAD$ ,  $ABD$  are equal to one right angle [I 32]

But the angle  $ABF$  is also right,

therefore the angles  $BAD$ ,  $ABD$

are equal to the angle  $BAD$  in the alternate segment of the circle

Next, since  $ABCD$  is a quadrilateral in a circle,

its opposite angles are equal to two right angles [III 22]

Therefore the angle  $DCB$  of the circle

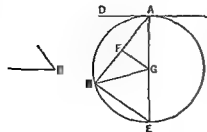
is equal to the angle  $ABD$

Q E D

### PROPOSITION 33

On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilineal angle

Let  $AB$  be the given straight line and the angle at  $C$  the given rectilineal angle,



thus it is required to describe on the given straight line  $AB$  a segment of a circle admitting an angle equal to the angle at  $C$

The angle at  $C$  is then acute or right, or obtuse

First, let it be acute and as in the first figure on the straight line  $AB$ , and at the point  $A$ , let the angle  $BAD$  be constructed equal to the angle at  $C$ .

therefore the angle  $BAD$  is also acute

Let  $AE$  be drawn at right angles to  $DA$ , let  $AB$  be bisected at  $F$ , let  $FG$  be drawn from the point  $F$  at right angles to  $AB$ , and let  $GB$  be joined

Then, since  $AF$  is equal to  $FB$

and  $FG$  is common,

the two sides  $AF$ ,  $FG$  are equal to the two sides  $BF$ ,  $FG$ ,

and the angle  $AFG$  is equal to the angle  $BFG$ ,

therefore the base  $AG$  is equal to the base  $BG$  [I 4]

Therefore the circle described with centre  $G$  and distance  $GA$  will pass through  $B$  also

Let it be drawn and let it be  $ABE$ ,

let  $EB$  be joined

Now since  $AD$  is drawn from  $A$ , the extremity of the diameter  $AE$ , at right angles to  $AE$ ,

therefore  $AD$  touches the circle  $ABE$  [III 16 Por]

Since then a straight line  $AD$  touches the circle  $ABE$ , and from the point of contact at  $A$  a straight line  $AB$  is drawn across in the circle  $ABE$ ,

the angle  $DAB$  is equal to the angle  $AEB$  in the alternate segment of the circle [III 32]

at  $C$

Next let the angle at  $C$  be right, and let it be again required to describe on  $AB$  a segment of a circle admitting an angle equal to the right angle at  $C$

Let the angle  $BAD$  be constructed equal to the right angle at  $C$ , as is the case in the second figure

let  $AB$  be bisected at  $F$  and with centre  $F$  and distance either  $FA$  or  $FB$  let the circle  $AEB$  be described

Therefore the straight line  $AD$  touches the circle  $ABE$  because the angle at  $A$  is right [III 16 Por]

And the angle  $BAD$  is equal to the angle in the segment  $AEB$  for the latter too is itself a right angle being an angle in a semicircle [III 31]

But the angle  $BAD$  is also equal to the angle at  $C$

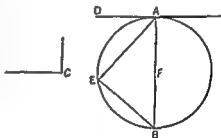
Therefore the angle  $AEB$  is also equal to the angle at  $C$

Therefore again the segment  $AEB$  of a circle has been described on  $AB$  admitting an angle equal to the angle at  $C$

Next let the angle at  $C$  be obtuse

and on the straight line  $AB$  and at the point  $A$  let the angle  $BAD$  be con-

circle has  
the angle



Then, since  $AF$  is again equal to  $FB$ ,  
and  $FG$  is common.

the two sides  $AF, FG$  are equal to  
the two sides  $BF, FG$ ,  
and the angle  $AFG$  is equal to the  
angle  $BFG$ ;  
therefore the base  $AG$  is equal to  
the base  $BG$  (I. 4).

Therefore the circle described with centre  $G$  and distance  $GA$  will pass through  $B$  also, let it so pass, as  $AEB$

Now, since  $AD$  is drawn at right angles to the diameter  $AE$  from its extremity,

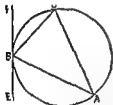
$AD$  touches the circle  $AEB$  [III 16, Por 1]

And  $AB$  has been drawn across from the point of contact at  $A$ , therefore the angle  $BAD$  is equal to the angle constructed in the alternate segment  $AHB$  of the circle. [III 32]

### PROPOSITION 34

*From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle*

Let  $ABC$  be the given circle, and the angle at  $D$  the given rectilineal angle, thus it is required to cut off from the circle  $ABC$  a segment admitting an angle



Then, since a straight line  $EF$  touches the circle  $ABC$ , and  $BC$  has been drawn across from the point of contact at  $B$ , the angle  $FBC$  is equal to the angle constructed in the alternate segment  $BAC$  [III 32]

But the angle  $FBC$  is equal to the angle at  $D$ ,

### PROPOSITION 35

the point  $E$ .



I say that the rectangle contained by  $AE$ ,  $EC$  is equal to the rectangle contained by  $DE$ ,  $EB$

If now  $AC$ ,  $BD$  are through the centre, so that  $E$  is the centre of the circle  $ABCD$ ,

it is manifest that,  $AE$ ,  $EC$ ,  $DE$ ,  $EB$  being equal the rectangle contained by  $AE$ ,  $EC$  is also equal to the rectangle contained by  $DE$ ,  $EB$

Next let  $AC$ ,  $DB$  not be through the centre, let the centre of  $ABCD$  be taken, and let it be  $F$ , from  $F$  let  $FG$ ,  $FH$  be drawn perpendicular to the straight lines  $AC$ ,  $DB$ , and let  $FB$ ,  $FC$ ,  $FE$  be joined

Then, since a straight line  $GF$  through the centre cuts a straight line  $AC$  not through the centre at right angles,

it also bisects it,

[III 3]

therefore  $AG$  is equal to  $GC$

Since then, the straight line  $AC$  has been cut into equal parts at  $G$  and into unequal parts at  $E$ , the rectangle contained by  $AE$ ,  $EC$  together with the square on  $EG$  is equal to the square on  $GC$ , [II 5]

Let the square on  $GF$  be added, therefore the rectangle  $AE$ ,  $EC$  together with the squares on  $GF$ ,  $GF$  is equal to the squares on  $CG$ ,  $GF$

But the square on  $FE$  is equal to the squares on  $EG$ ,  $GF$ , and the square on  $FC$  is equal to the squares on  $CG$ ,  $GF$ , [I 47]

therefore the rectangle  $AE$ ,  $EC$  together with the square on  $FE$  is equal to the square on  $FC$

And  $FC$  is equal to  $FB$ , therefore the rectangle  $AE$ ,  $EC$  together with the square on  $FE$  is equal to the square on  $FB$

For the same reason also the rectangle  $DE$ ,  $EB$  together with the square on  $FE$  is equal to the square on  $FB$

But the rectangle  $AE$ ,  $EC$  together with the square on  $FE$  was also proved equal to the square on  $FB$ ,

therefore the rectangle  $AE$ ,  $EC$  together with the square on  $FE$  is equal to the rectangle  $DE$ ,  $EB$  together with the square on  $FE$

Let the square on  $FE$  be subtracted from each, therefore the rectangle contained by  $AE$ ,  $EC$  which remains is equal to the rectangle contained by  $DE$ ,  $EB$

Therefore etc

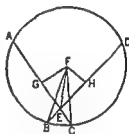
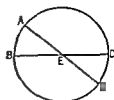
Q E D

### PROPOSITION 36

If a point be taken outside a circle and from it there fall on the circle two straight lines and a tangent

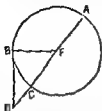
square on the tangent

For let a point  $D$  be taken outside the circle  $ABC$ , and from  $D$  let the two



straight lines  $DCA$ ,  $DB$  fall on the circle  $ABC$ , let  $DCA$  cut the circle  $ABC$  and let  $DB$  touch it,

I say that the rectangle contained by  $AD$ ,  $DC$  is equal to the square on  $DB$



Then  $DCA$  is either through the centre or not through the centre

First let it be through the centre, and let  $F$  be the centre of the circle  $ABC$ , let  $FB$  be joined,

therefore the angle  $FBD$  is right [III 18]

And, since  $AC$  has been bisected at  $F$ , and  $CD$  is added to it,

the rectangle  $AD$ ,  $DC$  together with the square on  $FC$  is equal to the square on  $FD$  [II 6]

But  $FC$  is equal to  $FB$ ,  
therefore the rectangle  $AD$ ,  $DC$  together with the square on  $FB$  is equal to the square on  $FD$

And the squares on  $FB$ ,  $BD$  are equal to the square on  $FD$ , [I 47]  
therefore the rectangle  $AD$ ,  $DC$  together with the square on  $FB$  is equal to the squares on  $FB$ ,  $BD$

Let the square on  $FB$  be subtracted from each,  
therefore the rectangle  $AD$ ,  $DC$  which remains is equal to the square on the tangent  $DB$

Again, let  $DCA$  not be through the centre of the circle  $ABC$ ,  
let the centre  $E$  be taken, and from  $E$ , let  $EF$  be drawn perpendicular to  $AC$ ,  
let  $EB$ ,  $EC$ ,  $ED$  be joined

Then the angle  $EBD$  is right [III 18]

And since a straight line  $EF$  through the centre cuts a straight line  $AC$  not through the centre at right angles,

it also bisects it, [III 3]

therefore  $AF$  is equal to  $FC$

Now, since the straight line  $AC$  has been bisected at the point  $F$ , and  $CD$  is added to it  
the rectangle contained by  $AD$ ,  $DC$  together with the square on  $FC$  is equal to the square on  $FD$  [II 6]

Let the square on  $FE$  be added to each,  
therefore the rectangle  $AD$ ,  $DC$  together with the squares on  $CF$ ,  $FE$  is equal to the squares on  $FD$ ,  $FE$

But the square on  $EC$  is equal to the squares on  $CF$ ,  $FE$  for the angle  $EFC$  is right, [I 47]

and the square on  $ED$  is equal to the squares on  $DF$ ,  $FE$ ,  
therefore the rectangle  $AD$ ,  $DC$  together with the square on  $EC$  is equal to the square on  $ED$

And  $EC$  is equal to  $EB$ ,  
therefore the rectangle  $AD$ ,  $DC$  together with the square on  $EB$  is equal to the square on  $ED$

But the squares on  $EB$ ,  $BD$  are equal to the square on  $ED$  for the angle  $EBD$  is right, [I 47]

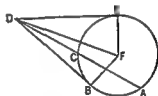
therefore the rectangle  $AD, DC$  together with the square on  $EB$  is equal to the squares on  $EB, BD$

Let the square on  $EB$  be subtracted from each,  
therefore the rectangle  $AD, DC$  which remains is equal to the square on  $DB$   
Therefore etc Q E D

### PROPOSITION 37

*If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle*

For let a point  $D$  be taken outside the circle  $ABC$ , from  $D$  let the two straight lines  $DCA, DB$  fall on the circle  $ABC$ , let  $DCA$  cut the circle and  $DB$  fall on it, and let the rectangle  $AD, DC$  be equal to the square on  $DB$



I say that  $DB$  touches the circle  $ABC$   
For let  $DF$  be drawn touching  $ABC$ , let the centre of the circle  $ABC$  be taken, and let it be  $F$ , let  $FE, FB, FD$  be joined

Thus the angle  $FED$  is right [III 18]

Now since  $DE$  touches the circle  $ABC$ , and  $DCA$  cuts it, the rectangle  $AD, DC$  is equal to the square on  $DE$  [III 36]

But the rectangle  $AD, DC$  was also equal to the square on  $DB$ ,  
therefore the square on  $DE$  is equal to the square on  $DB$ ,  
therefore  $DE$  is equal to  $DB$

And  $FE$  is equal to  $FB$ ,  
therefore the two sides  $DE, EF$  are equal to the two sides  $DB, BF$ ,  
and  $FD$  is the common base of the triangles,  
therefore the angle  $DEF$  is equal to the angle  $DBF$  [I 8]

But the angle  $DEF$  is right,  
therefore the angle  $DBF$  is also right

And  $FB$  produced is a diameter,  
and the straight line drawn at right angles to the diameter of a circle, from its extremity touches the circle, [III 16, Por]

therefore  $DB$  touches the circle

Similarly this can be proved to be the case even if the centre be on  $AC$   
Therefore etc Q E D

## BOOK FOUR

### DEFINITIONS

1 A rectilineal figure is said to be *inscribed in a rectilineal figure* when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed

2 Similarly a figure is said to be *circumscribed about a figure* when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed

6 A circle is said to be *circumscribed about a figure* when the circumference of the circle passes through each angle of the figure about which it is circumscribed

7 A straight line is said to be *fitted into a circle* when its extremities are on the circumference of the circle

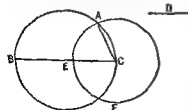
### BOOK IV PROPOSITIONS

#### PROPOSITION I

*Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle*

Let  $ABC$  be the given circle, and  $D$  the given straight line not greater than the diameter of the circle,

thus it is required to fit into the circle  $ABC$  a straight line equal to the straight line  $D$



Let a diameter  $BC$  of the circle  $ABC$  be drawn

Then, if  $BC$  is equal to  $D$ , that which was enquired will have been done, for  $BC$  has been fitted into the circle  $ABC$  equal to the straight line  $D$

But, if  $BC$  is greater than  $D$ , let  $CE$  be made equal to  $D$ , and with centre

$C$  and distance  $CE$  let the circle  $EAF$  be described,

let  $CA$  be joined

Then, since the point  $C$  is the centre of the circle  $EAF$ ,

$CA$  is equal to  $CE$

But  $CE$  is equal to  $D$ ,

therefore  $D$  is also equal to  $CA$

Therefore into the given circle  $ABC$  there has been fitted  $CA$  equal to the given straight line  $D$  Q E F

### PROPOSITION 2

*In a given circle to inscribe a triangle equiangular with a given triangle*

Let  $ABC$  be the given circle, and  $DEF$  the given triangle, thus it is required to inscribe in the circle  $ABC$  a triangle equiangular with the triangle  $DEF$

Let  $GH$  be drawn touching the circle  $ABC$  at  $A$  [III 16, Por], on the straight line  $AH$ , and at the point  $A$  on it, let the angle  $HAC$  be constructed equal to the angle  $DEF$ ,

and on the straight line  $AG$ , and at the point  $A$  on it, let the angle  $GAB$  be constructed equal to the angle  $DFE$ , [I 23]

let  $BC$  be joined

Then since a straight line  $AH$  touches the circle  $ABC$ , and from the point of contact at  $A$  the straight line  $AC$  is drawn across in the circle,

therefore the angle  $HAC$  is equal to the angle  $ABC$  in the alternate segment of the circle [III 32]

But the angle  $HAC$  is equal to the angle  $DEF$ ,

therefore the angle  $ABC$  is also equal to the angle  $DEF$

For the same reason

the angle  $ACB$  is also equal to the angle  $DFE$ ,

therefore the remaining angle  $BAC$  is also equal to the remaining angle  $EDF$  [I 32]

Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle Q E F

### PROPOSITION 3

*At a given point to draw a straight line tangent to a given circle*

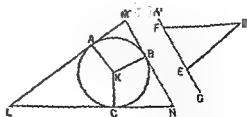
with the triangle  $DEF$

Let  $EF$  be produced in both directions to the points  $G$   $H$

let the centre  $K$  of the circle  $ABC$  be taken [III 1] and let the straight line  $AK$  be drawn across at random,

on the straight line  $AK$  and at the point  $K$  on it let the angle  $BKA$  be constructed equal to the angle  $DEG$ ,

and the angle  $BKC$  equal to the angle  $DFH$ , [I 23]



and through the points  $A, B, C$  let  $LAM, MBN, NCL$  be drawn touching the circle  $ABC$  [III 16, Por.]

Now, since  $LM, MN, NL$  touch the circle  $ABC$  at the points  $A, B, C$ , and  $KA, KB, KC$  have been joined from the centre  $K$  to the points  $A, B, C$ , therefore the angles at the points  $A, B, C$  are right [III 18]

And, since the four angles of the quadrilateral  $AMBK$  are equal to four right

3]

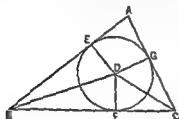
therefore the angles  $AKB, AMB$  are equal to the angles  $DEG, DEF$ , of which the angle  $AKB$  is equal to the angle  $DEG$ , therefore the angle  $AMB$  which remains is equal to the angle  $DEF$  which remains

Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle Q E F

## PROPOSITION 4

In a given triangle to inscribe a circle

Let  $ABC$  be the given triangle, thus it is required to inscribe a circle in the triangle  $ABC$



Let the angles  $ABC, ACB$  be bisected by the straight lines  $BD, CD$  [I 9], and let these meet one another at the point  $D$ , from  $D$  let  $DE, DF, DG$  be drawn perpendicular to the straight lines  $AB, BC, CA$

Now, since the angle  $ABD$  is equal to the angle  $CBD$ , and the right angle  $BED$  is also equal to the right angle  $BFD$ ,

$EBD, FBD$  are two triangles having two angles equal to two angles and one side equal to one side, namely that subtending one of the equal angles, which is  $BD$  common to the triangles,

therefore they will also have the remaining sides equal to the remaining sides, [I 26]

therefore  $DE$  is equal to  $DF$

For the same reason

$DG$  is also equal to  $DF$

Therefore the three straight lines  $DE, DF, DG$  are equal to one another, therefore the circle described with centre  $D$  and distance one of the straight lines  $DE, DF, DG$  will pass also through the remaining points and will touch the straight lines  $AB, BC, CA$ , because the angles at the points  $E, F, G$  are right

For, if it cuts them, the straight line drawn at right angles to the

the circle from its extremity will be found to fall within the circle which was proved absurd [iii 16]

therefore the circle described with centre  $D$  and distance one of the straight lines  $DE$   $DF$   $DG$  will not cut the straight lines  $AB$   $BC$   $CA$ , therefore it will touch them and will be the circle inscribed in the triangle  $ABC$  [iv Def 5]

Let it be inscribed as  $FGE$

Therefore in the given triangle  $ABC$  the circle  $EFG$  has been inscribed

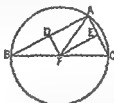
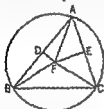
Q E F

### PROPOSITION 5

About a given triangle to circumscribe a circle

Let  $ABC$  be the given triangle

thus it is required to circumscribe a circle about the given triangle  $ABC$



Let the straight lines  $AB$   $AC$  be bisected at the points  $D$   $E$  [i 10] and from the points  $D$   $E$  let  $DF$   $EF$  be drawn at right angles to  $AB$ ,  $AC$ , they will then meet within the triangle  $ABC$ , or on the straight line  $BC$  or outside  $BC$

First let them meet within at  $F$ , and let  $FB$   $FC$   $FA$  be joined

Then since  $AD$  is equal to  $DB$ ,

and  $DF$  is common and at right angles

therefore the base  $AF$  is equal to the base  $FB$

[i 4]

Similarly we can prove that

$CF$  is also equal to  $AF$ ,

so that  $FB$  is also equal to  $FC$ ,

therefore the three straight lines  $FA$   $FB$   $FC$  are equal to one another

Therefore the circle described with centre  $F$  and distance one of the straight lines  $FA$   $FB$   $FC$  will pass also through the remaining points and the circle will have been circumscribed about the triangle  $ABC$

Let it be circumscribed as  $ABC$

as is the case in the se-

is the centre of the circle

the case in the

[i 4]

Similarly we can prove that

$CF$  is also equal to  $AF$

so that  $BF$  is also equal to  $FC$

therefore the circle described with centre  $F$  and distance one of the straight lines  $FA, FB, FC$  will pass also through the remaining points and will have been circumscribed about the triangle  $ABC$

Therefore about the given triangle a circle has been circumscribed

Q E F

And it is manifest that, when the centre of the circle falls within the triangle the angle  $BAC$ , being in a segment greater than the semicircle, is less than a right angle,  
when the centre falls on the straight line  $BC$ , the angle  $BAC$ , being in a semicircle is right,  
and when the centre of the circle falls outside the triangle, the angle  $BAC$ , being in a segment less than the semicircle, is greater than a right angle [III 31]

### PROPOSITION 6

In a given circle to inscribe a square

Let  $ABCD$  be the given circle,

thus it is required to inscribe a square in the circle  $ABCD$

Let two diameters  $AC, BD$  of the circle  $ABCD$  be drawn at right angles to one another, and let  $AB, BC, CD, DA$  be joined

Then, since  $BE$  is equal to  $ED$ , for  $E$  is the centre, and  $EA$  is common and at right angles

therefore the base  $AB$  is equal to the base  $AD$  [I 4]

For the same reason

each of the straight lines  $BC, CD$  is also equal to each of the straight lines  $AB, AD$ ,

therefore the quadrilateral  $ABCD$  is equilateral

I say next that it is also right-angled

For, since the straight line  $BD$  is a diameter of the circle  $ABCD$ ,

therefore  $BAD$  is a semicircle,

therefore the angle  $BAD$  is right

[III 31]

For the same reason

each of the angles  $ABC, BCD, CDA$  is also right,

therefore the quadrilateral  $ABCD$  is right-angled

But it was also proved equilateral,

therefore it is a square,

[I Def 22]

and it has been inscribed in the circle  $ABCD$

Therefore in the given circle the square  $ABCD$  has been inscribed Q E F

### PROPOSITION 7

About a given circle to circumscribe a square

Let  $ABCD$  be the given circle

thus it is required to circumscribe a square about the circle  $ABCD$

I say that it is possible to circumscribe a square about the circle  $ABCD$

one

tol

Then since  $FG$  touches the circle  $ABCD$ ,

and  $EA$  has been joined from the centre  $E$  to the point of contact at  $A$ ,

therefore the angles at  $A$  are right

[III 18]



For the same reason

For the same reason

$AC$  is also parallel to  $FK$ ,

so that  $GH$  is also parallel to  $FK$  [I 30]

Similarly we can prove that

each of the straight lines  $GF$ ,  $HK$  is parallel to  $BED$

Therefore  $GK$ ,  $GC$ ,  $AK$ ,  $FB$ ,  $BK$  are parallelograms,  
therefore  $GF$  is equal to  $HK$ , and  $GH$  to  $FK$  [I 34]

And, since  $AC$  is equal to  $BD$ ,

and  $AC$  is also equal to each of the straight lines  $GH$ ,  $FK$ ,

while  $BD$  is equal to each of the straight lines  $GF$ ,  $HK$ , [I 34]

therefore the quadrilateral  $FGHK$  is equilateral

I say next that it is also right-angled

For, since  $GBEA$  is a parallelogram,

and the angle  $AEB$  is right,

therefore the angle  $AGB$  is also right [I 34]

Similarly we can prove that

the angles at  $H$ ,  $K$ ,  $F$  are also right

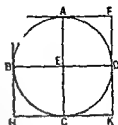
Therefore  $FGHA$  is right-angled

But it was also proved equilateral;

therefore it is a square,

and it has been circumscribed about the circle  $ABCD$

Therefore about the given circle a square has been circumscribed Q.E.D.



### PROPOSITION 8

In a given square to inscribe a circle

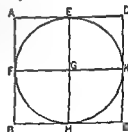
Let  $ABCD$  be the given square,

thus it is required to inscribe a circle in the given square  $ABCD$

Let the straight lines  $AD$ ,  $AB$  be bisected at the points  $E$ ,  $F$  respectively, [I 10]

through  $E$  let  $EH$  be drawn parallel to either  $AB$  or  $CD$ , and through  $F$  let  $FK$  be drawn parallel to either  $AD$  or  $BC$ , [I 31]

therefore each of the figures  $AK$ ,  $KB$ ,  $AH$ ,  $HD$ ,  $AG$ ,  $GC$ ,  $BG$ ,  $GD$  is a parallelogram, and their opposite sides are evidently equal [I 34]



Now, since  $AD$  is equal to  $AB$ ,

and  $AE$  is half of  $AD$ , and  $AF$  half of  $AB$ ,

therefore  $AE$  is equal to  $AF$ ,

so that the opposite sides are also equal,

therefore  $FG$  is equal to  $GE$

Similarly we can prove that each of the straight lines  $GH$ ,  $GK$  is equal to each of the straight lines  $FG$ ,  $GE$ ,

therefore the four straight lines  $GE$ ,  $GF$ ,  $GH$ ,  $GK$  are equal to one another

lin

$E, F, H, K$  are right

For, if the circle cuts  $AB, BC, CD, DA$ , the straight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle which was proved absurd, [III 16]

therefore the circle described with centre  $G$  and distance one of the straight lines  $GE, GF, GH, GK$  will not cut the straight lines  $AB, BC, CD, DA$

Therefore it will touch them, and will have been inscribed in the square  $ABCD$

Therefore in the given square a circle has been inscribed

Q E F

## PROPOSITION II

About a given square to circumscribe a circle

Let  $ABCD$  be the given square,

thus it is required to circumscribe a circle about the square  $ABCD$

For let  $AC, BD$  be joined, and let them cut one another at  $E$

Then, since  $DA$  = equal to  $AB$ , and  $AC$  is common,

therefore the two sides  $DA, AC$  are equal to the two sides  $BA, AC$ ,

and the base  $DC$  is equal to the base  $BC$ ,

therefore the angle  $DAC$  is equal to the angle  $BAC$

[I 8]

Therefore the angle  $DAB$  is bisected by  $AC$

Similarly we can prove that each of the angles  $ABC, BCD, CDA$  is bisected by the straight lines  $AC, DB$

Now, since the angle  $DAB$  = equal to the angle  $ABC$ ,

and the angle  $EAB$  is half the angle  $DAB$ ,

and the angle  $EBA$  half the angle  $ABC$ ,

therefore the angle  $EAB$  is also equal to the angle  $EBA$ ,

so that the side  $EA$  = also equal to  $EB$

[I 6]

Similarly we can prove that each of the straight lines  $EA, EB$  = equal to each of the straight lines  $EC, ED$

Therefore the four straight lines  $EA, EB, EC, ED$  are equal to one another

Therefore the circle described with centre  $E$  and distance one of the straight lines  $EA, EB, EC, ED$  will pass also through the remaining points,

and it will have been circumscribed about the square  $ABCD$

Let it be circumscribed as  $ABCD$

Therefore about the given square a circle has been circumscribed

Q E F

## PROPOSITION 10

To construct an isosceles triangle having each of the angles at the base double of the remaining one

Let any straight line  $AB$  be set out, and let it be cut at the point  $C$  so that the rectangle contained by  $AB, BC$  is equal to the square on  $CA$ , [II 11]

with centre  $A$  and distance  $AB$  let the circle  $BDE$  be described, and let there be fitted in the circle  $BDE$  the straight line  $BD$  equal to the straight line  $AC$  which = not greater than the diameter of the circle  $BDE$



the triangle  $FGH$ , so that the angle  $CAD$  is equal to the angle at  $F$  and the angles at  $G, H$  respectively equal to the angles  $ACD, CDA$ , [iv 2]  
therefore each of the angles  $ACD, CDA$  is also double of the angle  $CAD$



Now let the angles  $ACD, CDA$  be bisected respectively by the straight lines  $CE, DB$  [i 9] and let  $AB, BC, DE, EA$  be joined

Then, since each of the angles  $ACD, CDA$  is double of the angle  $CAD$ , and they have been bisected by the straight lines  $CE, DB$ ,

therefore the five angles  $DAC, ACE, ECD, CDB, BDA$  are equal to one another

But equal angles stand on equal circumferences, [iii 26]  
therefore the five circumferences  $AB, BC, CD, DE, EA$  are equal to one another

But equal circumferences are subtended by equal straight lines, [iii 29]  
therefore the five straight lines  $AB, BC, CD, DE, EA$  are equal to one another,  
therefore the pentagon  $ABCDE$  is equilateral

I say next that it is also equiangular

For, since the circumference  $AB$  is equal to the circumference  $DE$ , let  $BCD$  be added to each,  
therefore the whole circumference  $ABCD$  is equal to the whole circumference  $EDCB$

And the angle  $AED$  stands on the circumference  $ABCD$ , and the angle  $BAE$  on the circumference  $EDCB$ ,

therefore the angle  $BAE$  is also equal to the angle  $AED$  [iii 27]

For the same reason  
each of the angles  $ABC, BCD, CDE$  is also equal to each of the angles  $BAE, AED$ ,

therefore the pentagon  $ABCDE$  is equiangular

But it was also proved equilateral,  
therefore in the given circle an equilateral and equiangular pentagon has been inscribed

Q E F

## PROPOSITION 12

About a given circle to circumscribe an equilateral and equiangular pentagon

Let  $ABCDE$  be the given circle,  
thus it is required to circumscribe an equilateral and equiangular pentagon about the circle  $ABCDE$

Let  $A, B, C, D, E$  be conceived to be the angular points of the inscribed pentagon, so that the circumferences  $AB, BC, CD, DE, EA$  are equal, [iv 11]  
through  $A, B, C, D, E$  let  $GH, HK, KL, LM, MG$  be drawn touching the circle, [iii 16, Por]

let the centre  $F$  of the circle  $ABCDE$  be taken [iii 1], and let  $FB, FK, FC, FL, FD$  be joined

Then, since the straight line  $KL$  touches the circle  $ABCDE$  at  $C$ ,  
and  $FC$  has been joined from the centre  $F$  to the point of contact at  $C$ ,  
therefore  $FC$  is perpendicular to  $KL$ , [iii 18]

therefore each of the angles at  $C$  is right

For the same reason

$\therefore \angle BCF = \angle FCK = \angle KCL = \angle LCF = \angle FCL = \angle FCB$

$\therefore$

$FC, CK$

[1. 47]

For the same reason

the square on  $FK$  is also equal to the squares on  $FB, BK$ ,

so that the squares on  $FC, CK$  are equal to the squares on  $FB, BK$ ,

of which the square on  $FC$  is equal to the square on  $FB$ ,

therefore the square on  $CK$  which remains is equal to the square on  $BK$

Therefore  $BK$  is equal to  $CK$

And, since  $FB$  is equal to  $FC$ ,

and  $FK$  common,

the two sides  $BF, FK$  are equal to the two sides  $CF, FK$ ; and the base  $BK$  equal to the base  $CK$ ,

therefore the angle  $BFK$  is equal to the angle  $KFC$ ,

[1. 8]

and the angle  $BKF$  to the angle  $FKC$

Therefore the angle  $BFC$  is double of the angle  $KFC$ ,

and the angle  $BKC$  of the angle  $FKC$

For the same reason

the angle  $CFD$  is also double of the angle  $CFL$ ,

and the angle  $DLC$  of the angle  $FLC$

Now, since the circumference  $BC$  is equal to  $CD$ ,

the angle  $BFC$  is also equal to the angle  $CFD$

[III. 27]

And the angle  $BFC$  is double of the angle  $KFC$ , and the angle  $DFC$  of the angle  $LFC$ ,

therefore the angle  $KFC$  is also equal to the angle  $LFC$

But the angle  $FCK$  is also equal to the angle  $FCL$ ,

therefore  $FKC = FLC$  = two angles each equal to  $KFC$  and  $LFC$

and the remaining angle to the remaining angle,

[1. 26]

therefore the straight line  $KC$  is equal to  $CL$ ,

and the angle  $FKC$  to the angle  $FLC$ ,

And, since  $KC$  is equal to  $CL$ ,

therefore  $KL$  is double of  $KC$ .

For the same reason it can be proved that

$HK$  is also double of  $BK$

And  $BK$  is equal to  $KC$ ,

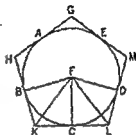
therefore  $HK$  is also equal to  $KL$

Similarly each of the straight lines  $HG, GM, ML$  can also be proved equal to each of the straight lines  $HK, KL$ ,

therefore the pentagon  $GHKLM$  is equilateral

I say next that it is also equiangular

For, since the angle  $FKC$  is equal to the angle  $FLC$ ,



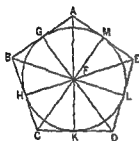
Simile  
each of  
therefore the five angles  $GHK, HKL, KLM, LMG, MGH$  are equal to one another

Therefore the pentagon  $GHKLM$  is equiangular

And it was also proved equilateral, and it has been circumscribed about the circle  $ABCDE$  Q E F

### PROPOSITION 13

In a given pentagon, which is equilateral and equiangular, to inscribe a circle



the two sides  $BC, CF$  are equal to the two sides  $DC, CF$ ,

and the angle  $BCF$  is equal to the angle  $DCF$ ,  
therefore the base  $BF$  is equal to the base  $DF$ ,

subtend

[14]

Therefore the angle  $CBF$  is equal to the angle  $CDF$

And since the angle  $CDE$  is double of the angle  $CDF$ ,

therefore the angle  $ABF$  is equal to the angle  $FBC$ ,

therefore the angle  $ABC$  has been bisected by the straight line  $BF$

Similarly it can be proved that

the angles  $BAE, AED$  have also been bisected by the straight lines  $FA, FE$  respectively

Now let  $FG, FH, FK, FL, FM$  be drawn from the point  $F$  perpendicular to the straight lines  $AB, BC, CD, DE, EA$

$FHC, FKC$  are two triangles having two angles equal to two angles and one side equal to one side, namely  $FC$  which is common to them and subtends one of the equal angles,

therefore they will also have the remaining sides equal to the remaining sides; [16]

therefore the perpendicular  $FH$  is equal to the perpendicular  $FK$

Similarly it can be proved that

each of the straight lines  $FL$ ,  $FM$ ,  $FG$  is also equal to each of the straight lines  $FH$ ,  $FK$ ,

therefore the five straight lines  $FG$ ,  $FH$ ,  $FK$ ,  $FL$ ,  $FM$  are equal to one another

Therefore the circle described with centre  $F$  and distance one of the straight

For, if it does not touch them, but cuts them,

it will result that the straight line drawn at right angles to the diameter of the circle from its extremity falls within the circle which was proved absurd

[III 16]

Therefore the circle described with centre  $F$  and distance one of the straight lines  $FG$ ,  $FH$ ,  $FK$ ,  $FL$ ,  $FM$  will not cut the straight lines  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$ ; therefore it will touch them

Let it be described, as  $GHLIM$

Therefore in the given pentagon, which is equilateral and equiangular, a circle has been inscribed

Q E F

#### PROPOSITION 14

About a given pentagon, which is equilateral and equiangular, to circumscribe a circle

Let  $ABCDE$  be the given pentagon, which is equilateral and equiangular, thus it is required to circumscribe a circle about the pentagon  $ABCDE$

Let the angles  $BCD$ ,  $CDE$  be bisected by the straight lines  $CF$ ,  $DF$  respectively, and from the point  $F$ , at which the straight lines meet, let the straight lines  $FB$ ,  $FA$ ,  $FE$  be joined to the points  $B$ ,  $A$ ,  $E$

Then in manner similar to the preceding it can be proved that the angles  $CBA$ ,  $BAE$ ,  $AED$  have also been bisected by the straight lines  $FB$ ,  $FA$ ,  $FE$  respectively

Now, s11



[I 6]

Similarly it can be proved that

each of the straight lines  $FB$ ,  $FA$ ,  $FE$  is also equal to each of the straight lines  $FC$ ,  $FD$ ,

therefore the five straight lines  $FA$ ,  $FB$ ,  $FC$ ,  $FD$ ,  $FE$  are equal to one another

Therefore the circle described with centre  $F$  and distance one of the straight lines  $FA$ ,  $FB$ ,  $FC$ ,  $FD$ ,  $FE$  will pass also through the remaining points, and will have been circumscribed

Let it be circumscribed and let it be  $ABCDE$

Therefore about the given pentagon, which is equilateral and equiangular, a circle has been circumscribed

Q E F

## PROPOSITION 15

In a given circle to inscribe an equilateral and equiangular hexagon

Let  $ABCDEF$  be the given circle,  
thus it is required to inscribe an equilateral and equiangular hexagon in the circle  $ABCDEF$

Let the diameter  $AD$  of the circle  $ABCDEF$  be drawn,

let the centre  $G$  of the circle be taken and with centre  $D$  and distance  $DG$  let the circle  $EGCH$  be described, let  $EG$ ,  $CG$  be joined and carried through to the points  $B$ ,  $F$ ,

and let  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EF$ ,  $FA$  be joined

I say that the hexagon  $ABCDEF$  is equilateral and equiangular

For since the point  $G$  is the centre of the circle  $ABCDEF$ ,

$GE$  is equal to  $GD$

Again, since the point  $D$  is the centre of the circle  $GCH$ ,

$DE$  is equal to  $DG$

But  $GE$  was proved equal to  $GD$ ,

therefore  $GE$  is also equal to  $ED$ ,

therefore the triangle  $EGD$  is equilateral,

and therefore its three angles  $EGD$ ,  $GDE$ ,  $DEG$  are equal to one another inasmuch as in isosceles triangles, the angles at the base are equal to one another

[I 5]

And the three angles of the triangle are equal to two right angles, [I 32]

Similar

two right

angles

E

so that the angles vertical to them the angles  $BGA$ ,  $AGF$ ,  $FGE$  are equal

[I 15]

Therefore the six angles  $EGD$ ,  $DGC$ ,  $CGB$ ,  $BGA$ ,  $AGF$ ,  $FGE$  are equal to one another

But equal angles stand on equal circumferences, [III 26]  
therefore the six circumferences  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EF$ ,  $FA$  are equal to one another

And equal circumferences are subtended by equal straight lines [III 29]

therefore the six straight lines are equal to one another,

therefore the hexagon  $ABCDEF$  is equilateral

I say next that it is also equiangular

For since the circumference  $FA$  is equal to the circumference  $ED$ ,

let the circumference  $ABCD$  be added to each

therefore the whole  $FABCD$  is equal to the whole  $EDCBA$ ,

and the angle  $FED$  stands on the circumference  $FABCD$ ,



and the angle  $AFE$  on the circumference  $EDCBA$ ;

radius of the circle

hexagon has

the

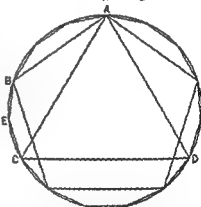
we can both inscribe a circle in a given hexagon and circumscribe one about it

### PROPOSITION 16

*In a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular*

Let  $ABCD$  be the given circle, thus it is required to inscribe in the circle  $ABCD$  a fifteen-angled figure which shall be both equilateral and equiangular

In the circle  $ABCD$  let there be inscribed a side  $AC$  of the equilateral triangle inscribed in it, and a side  $AB$  of an equilateral pentagon, therefore of the equal segments of which there are fifteen in the circle  $ABCD$ , there will be five in the circumference  $ABC$  which is one-third of the circle and there will be three in the circumference  $AB$  which is one-fifth of the circle, therefore in the remainder  $BC$  there will be two of the equal segments



Let  $BC$  be bisected at  $E$ , [III 30] therefore each of the circumferences  $BE$ ,  $EC$  is a fifteenth of the circle  $ABCD$ ,

division on the circle we draw tangents to the circle there will be circumscribed about the circle a fifteen angled figure which is equilateral and equiangular

And further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in the given fifteen angled figure and circumscribe one about it

## BOOK FIVE

### DEFINITIONS

1 A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater

2 The greater is a *multiple* of the less when it is measured by the less

3 A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind

4 Magnitudes are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another

5 Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and the second and of the third and the fourth, the former multiple is greater than the latter, or equal to it, or less than it.

than the third has to the fourth

8 A proportion in three terms is the least possible

9 When three magnitudes are proportional the first is said to have to the third the *duplicate ratio* of that which it has to the second

10 When four magnitudes are <continuously> proportional, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second, and so on continually, whatever be the proportion

11 The term *corresponding magnitudes* is used of antecedents in relation to antecedents, and of consequents in relation to consequents

12 *Alternate ratio* means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent

13 *Inverse ratio* means taking the consequent as antecedent in relation to the antecedent as consequent

14 *Composition of a ratio* means taking the antecedent together with the consequent as one in relation to the consequent by itself

15 *Separation of a ratio* means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself

16 *Conversion of a ratio* means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent

17 A ratio *ex aequali* arises when there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion as the first is to the last among the first magnitudes, so is the first to the last among the second magnitudes,

Or, in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms

18 A *perturbed proportion* arises when, there being three magnitudes and

third to the antecedent among the second magnitudes

## BOOK V PROPOSITIONS

### PROPOSITION 1

*If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one, that multiple also will all be of all*

Let any number of magnitudes whatever  $AB, CD$  be respectively equimultiples of any magnitudes  $E, F$  equal in multitude,

I say that, whatever multiple  $AB$  is of  $E$ , that multiple will  $AB, CD$  also be of  $E, F$



For, since  $AB$  is the same multiple of  $E$  that  $CD$  is of  $F$ , as many magnitudes as there are in  $AB$  equal to  $E$ , so many also are there in  $CD$  equal to  $F$

Let  $AB$  be divided into the magnitudes  $AG, GB$  equal to  $E$ ,  
and  $CD$  into  $CH, HD$  equal to  $F$ ,  
then the multitude of the magnitudes  $AG, GB$  will be equal to the multitude of the magnitudes  $CH, HD$

Now, since  $AG$  is equal to  $E$ , and  $CH$  to  $F$ ,

therefore  $AG$  is equal to  $E$ , and  $AG, CH$  to  $E, F$

For the same reason

$GB$  is equal to  $E$ , and  $GB, HD$  to  $E, F$ ,

therefore, as many magnitudes as there are in  $AB$  equal to  $E$ , so many also are there in  $AB, CD$  equal to  $E, F$ ,

therefore, whatever multiple  $AB$  is of  $E$ , that multiple will  $AB, CD$  also be of  $E, F$

Therefore etc

Q E D

### PROPOSITION 2

*If a first magnitude be the same multiple of a second that a third is of a fourth, and*

second,  $C$ , that the sum of the third and sixth,  $DH$ , is of the fourth,  $F$

For, since  $AB$  is the same multiple of  $C$  that  $DE$  is of  $F$ , therefore, as many magnitudes as there are in  $AB$  equal to  $C$ , so many also are there in  $DE$  equal to  $F$

For the same reason also, as many as there are in  $BG$  equal to  $C$ , so many are there also in  $EH$  equal to  $F$ , therefore, as many as there are in the whole  $AG$  equal to  $C$ , so many also are there in the whole  $DH$  equal to  $F$

Therefore, whatever multiple  $AG$  is of  $C$ , that multiple also is  $DH$  of  $F$   
Therefore the sum of the first and fifth,  $AG$ , is the same multiple of the second,  $C$ , that the sum of the third and sixth,  $DH$ , is of the fourth,  $F$

Therefore etc

Q E D

### PROPOSITION 3

*If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third, then also ex aequali the magnitudes taken will be equimultiples respectively, the one of the second, and the other of the fourth*

Let a first magnitude  $A$  be the same multiple of a second  $B$  that a third  $C$  is of a fourth  $D$ , and let equimultiples  $EF$ ,  $GH$  be taken of  $A$ ,  $C$ ,

I say that  $EF$  is the same multiple of  $B$  that  $GH$  is of  $D$

For, since  $EF$  is the same multiple of  $A$  that  $GH$  is of  $C$ , therefore, as many magnitudes as there are in  $EF$  equal to  $A$ , so many also are there in  $GH$  equal to  $C$

Let  $EF$  be divided into the magnitudes  $EK$ ,  $KF$  equal to  $A$ , and  $GH$  into the magnitudes  $GL$ ,  $LH$  equal to  $C$ ,

then the multitude of the magnitudes  $EK$ ,  $KF$  will be equal to the multitude of the magnitudes  $GL$ ,  $LH$

$A$  —————

$B$  ———

$E$  —————  $K$  —————  $F$

$C$  —————

$D$  ———

$G$  —————  $L$  —————  $H$

And, since  $A$  is the same multiple of  $B$  that  $C$  is of  $D$ ,

while  $EK$  is equal to  $A$ , and  $GL$  to  $C$ , therefore  $EK$  is the same multiple of  $B$  that  $GL$  is of  $D$

For the same reason

$KF$  is the same multiple of  $B$  that  $LH$  is of  $D$

Since, then, a first magnitude  $EK$  is the same multiple of a second  $B$  that a third  $GL$  is of a fourth  $D$

and a fifth  $KF$  is also the same multiple of the second  $B$  that a sixth  $LH$  is of the fourth  $D$ ,

therefore the sum of the first and fifth,  $EF$ , is also the same multiple of the second  $B$  that the sum of the third and sixth,  $GH$  is of the fourth  $D$  [V 2]

Therefore etc

Q E D

### PROPOSITION 4

*If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order*

For let a first magnitude  $A$  have to a second  $B$  the same ratio as a third  $C$  to a fourth  $D$ , and let equimultiples  $E, F$  be taken of  $A, C$ , and  $G, H$  other, chance, equimultiples of  $B, D$ ,

I say that, as  $E$  is to  $G$ , so is  $F$  to  $H$

For let equimultiples  $K, L$  be taken of  $E, F$ , and other, chance, equimultiples  $M, N$  of  $G, H$

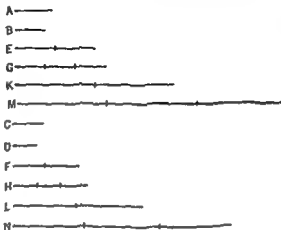
Since  $E$  is the same multiple of  $A$  that  $F$  is of  $C$ , and equimultiples  $K, L$  of  $E, F$  have been taken, therefore  $K$  is the same multiple of  $A$  that  $L$  is of  $C$

[v 3]

For the same reason

$M$  is the same multiple of  $B$  that  $N$  is of  $D$

And, since, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,



And  $K, L$  are equimultiples of  $E, F$ ,  
and  $M, N$  other, chance, equimultiples of  $G, H$ ,  
therefore, as  $E$  is to  $G$ , so is  $F$  to  $H$

[v Def 5]

[v Def 5]

Q E D

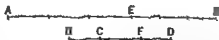
Therefore etc

### PROPOSITION 5

If a magnitude be the same multiple of a magnitude that a part subtracted is of a part subtracted the remainder will also be the same multiple of the remainder that the whole is of the whole

For let the magnitude  $AB$  be the same multiple of the magnitude  $CD$  that the part  $AE$  subtracted is of the part  $CF$  subtracted,

I say that the remainder  $EB$  is also the same multiple of the remainder  $FD$  that the whole  $AB$  is of the whole  $CD$



For whatever multiple  $AE$  is of  $CF$  let  $EB$  be made that multiple of  $CG$ . Then, since  $AE$  is the same multiple of  $CF$  that  $EB$  is of  $CG$ ,

therefore  $AE$  is the same multiple of  $CF$  that  $AB$  is of  $GF$  [v 1]

But, by the assumption  $AE$  is the same multiple of  $CF$  that  $AB$  is of  $CD$

Therefore  $AB$  is the same multiple of each of the magnitudes  $GF, CD$ ,

therefore  $GF$  is equal to  $CD$

Let  $CF$  be subtracted from each,

therefore the remainder  $GC$  is equal to the remainder  $FD$

And, since  $AE$  is the same multiple of  $CF$  that  $EB$  is of  $GC$ ,

and  $GC$  is equal to  $DF$ ,

therefore  $AE$  is the same multiple of  $CF$  that  $EB$  is of  $FD$

But, by hypothesis,

$AE$  is the same multiple of  $CF$  that  $AB$  is of  $CD$ ,

therefore  $EB$  is the same multiple of  $FD$  that  $AB$  is of  $CD$

That is, the remainder  $EB$  will be the same multiple of the remainder  $FD$  that the whole  $AB$  is of the whole  $CD$

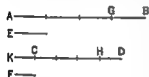
Therefore etc.

Q E D

### PROPOSITION 6

*If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders also are either equal to the same or equimultiples of them*

For let two magnitudes  $AB$ ,  $CD$  be equimultiples of two magnitudes  $E$ ,  $F$ , and let  $AG$ ,  $CH$  subtracted from them be equimultiples of the same two  $E$ ,  $F$ ,



I say that the remainders also,  $GB$ ,  $HD$ , are either equal to  $E$ ,  $F$  or equimultiples of them

For, first let  $GB$  be equal to  $E$ ,

I say that  $HD$  is also equal to  $F$

For let  $CK$  be made equal to  $F$

Since  $AG$  is the same multiple of  $E$  that  $CH$  is of  $F$ ,

while  $GB$  is equal to  $E$  and  $KC$  to  $F$ ,

therefore  $AB$  is the same multiple of  $E$  that  $KH$  is of  $F$  [v 2]

But, by hypothesis  $AB$  is the same multiple of  $E$  that  $CD$  is of  $F$ ,

therefore  $KH$  is the same multiple of  $F$  that  $CD$  is of  $F$

Since then each of the magnitudes  $KH$ ,  $CD$  is the same multiple of  $F$ ,

therefore  $KH$  is equal to  $CD$

Let  $CH$  be subtracted from each,

therefore the remainder  $KC$  is equal to the remainder  $HD$

But  $F$  is equal to  $KC$ ,

therefore  $HD$  is also equal to  $F$

Hence, if  $GB$  is equal to  $E$   $HD$  is also equal to  $F$

Similarly we can prove that, even if  $GB$  be a multiple of  $E$ ,  $HD$  is also the same multiple of  $F$

Therefore etc

Q E D

### PROPOSITION 7

*Equal magnitudes have to the same the same ratio as also has the same to equal magnitudes*

Let  $A$ ,  $B$  be equal magnitudes and  $C$  any other, chance, magnitude,

I say that each of the magnitudes  $A$ ,  $B$  has the same ratio to  $C$ , and  $C$  has the same ratio to each of the magnitudes  $A$ ,  $B$

For let equimultiples  $D$ ,  $E$  of  $A$ ,  $B$  be taken, and of  $C$  another, chance multiple  $F$

Then since  $D$  is the same multiple of  $A$  that  $E$  is of  $B$ , while  $A$  is equal to  $B$ , therefore  $D$  is equal to  $E$

But  $F$  is another, chance, magnitude

If therefore  $D$  is in excess of  $F$ ,  $E$  is also in excess of  $F$ , if equal to it, equal, and if less less

And  $D$ ,  $E$  are equimultiples of  $A$ ,  $B$ , while  $F$  is another, chance multiple of  $C$ , therefore, as  $A$  is to  $C$ , so is  $B$  to  $C$



[v Def 5]

I say next that  $C$  also has the same ratio to each of the magnitudes  $A$ ,  $B$ . For, with the same construction, we can prove similarly that  $D$  is equal to  $E$ , and  $F$  is some other magnitude

If therefore  $F$  is in excess of  $D$ , it is also in excess of  $E$ , if equal equal, and if less, less

And  $F$  is a multiple of  $C$ , while  $D$ ,  $E$  are other, chance, equimultiples of  $A$ ,  $B$ , therefore, as  $C$  is to  $A$ , so is  $C$  to  $B$  [v Def 5]

Therefore etc

**PORISM** From this it is manifest that, if any magnitudes are proportional, they will also be proportional inversely Q E D

### PROPOSITION 8

*Of unequal magnitudes, the greater has to the same a greater ratio than the less has, and the same has to the less a greater ratio than it has to the greater*

Let  $AB$ ,  $C$  be unequal magnitudes, and let  $AB$  be greater, let  $D$  be another, chance magnitude,

I say that  $AB$  has to  $D$  a greater ratio than  $C$  has to  $D$  and  $D$  has to  $C$  a greater ratio than it has to  $AB$

For since  $AB$  is greater than  $C$ , let  $BE$  be made equal to  $C$ ,

then the less of the magnitudes  $AE$ ,  $EB$ , if multiplied will sometime be greater than  $D$

[v Def 4]

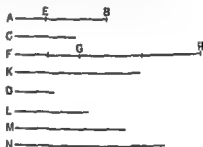
First let  $AE$  be less than  $EB$ ,

let  $AE$  be multiplied and let

$FG$  be a multiple of it which is greater than  $D$ ,

then whatever multiple  $FG$  is of  $AE$ , let  $GH$  be made the same multiple of  $EB$  and  $K$  of  $C$ ,

and let  $L$  be taken double of  $D$ ,  $M$  triple of it and successive multiples increasing by one until what is taken is a multiple of  $D$  and the first that is greater than  $K$ . Let it be taken and let it be  $N$  which is quadruple of  $D$  and the first multiple of it that is greater than  $K$



Then, since  $K$  is less than  $N$  first,

therefore  $K$  is not less than  $M$

And, since  $FG$  is the same multiple of  $AE$  that  $GH$  is of  $EB$

therefore  $FG$  is the same multiple of  $AE$  that  $FH$  is of  $AB$  [v 1]

But  $FG$  is the same multiple of  $AE$  that  $K$  is of  $C$ ,

therefore  $FH$  is the same multiple of  $AB$  that  $K$  is of  $C$ ,

therefore  $FH$ ,  $K$  are equimultiples of  $AB$ ,  $C$

Again, since  $GH$  is the same multiple of  $EB$  that  $K$  is of  $C$ ,

and  $EB$  is equal to  $C$ ,

But  $K$  is not less than  $M$ , therefore  $GH$  is equal to  $K$

And  $FG$  is greater than  $D$ , therefore neither is  $GH$  less than  $M$

But  $D, M$  together are equal to  $N$ , inasmuch as  $M$  is triple of  $D$ , and  $M, D$  together are quadruple of  $D$ , while  $N$  is also quadruple of  $D$ , whence  $M, D$  together are equal to  $N$

But  $FH$  is greater than  $M, D$ , therefore  $FH$  is in excess of  $N$ , while  $K$  is not in excess of  $N$

And  $FH, K$  are equimultiples of  $AB, C$ , while  $N$  is another, chance, multiple of  $D$ , therefore  $AB$  has to  $D$  a greater ratio than  $C$  has to  $D$  [v Def 7]

I say next, that  $D$  also has to  $C$  a greater ratio than  $D$  has to  $AB$

For, with the same construction, we can prove similarly that  $N$  is in excess of  $K$ , while  $N$  is not in excess of  $FH$

And  $N$  is a multiple of  $D$ , while  $FH, K$  are other, chance, equimultiples of  $AB, C$ , therefore  $D$  has to  $C$  a greater ratio than  $D$  has to  $AB$  [v Def 7]

Again, let  $AE$  be greater than  $EB$ . Then the less,  $EB$ , if multiplied, will sometime be greater than  $D$  [v Def 4]

Let it be multiplied, and let  $GH$  be a multiple of  $EB$  and greater than  $D$ , and, whatever multiple  $GH$  is of  $EB$ , let  $FG$  be made the same multiple of  $AE$ , and  $K$  of  $C$

Then we can prove similarly that  $FH, K$  are equimultiples of  $AB, C$ , and similarly, let  $N$  be taken a multiple of  $D$  but the first that is greater than  $FG$ , so that  $FG$  is again not less than  $M$

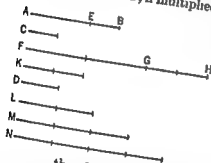
But  $GH$  is greater than  $D$ , so that  $FG$  is again not less than  $M$

Therefore the whole  $FH$  is in excess of  $N$ , inasmuch as  $FG$  also, which is greater than  $GH$ , is in excess of  $N$ , inasmuch as  $M$  is triple of  $D$ , and  $M, D$  together are quadruple of  $D$ , while  $N$  is also quadruple of  $D$ , whence  $M, D$  together are equal to  $N$

Now  $K$  is not in excess of  $N$ , inasmuch as  $M$  is triple of  $D$ , and  $M, D$  together are quadruple of  $D$ , while  $N$  is also quadruple of  $D$ , whence  $M, D$  together are equal to  $N$

And in the same manner, by following the above argument, we complete the demonstration

Therefore etc



PROPOSITION 9

Q E D

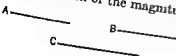
Magnitudes which have the same ratio to the same are equal to one another and magnitudes to which the same has the same ratio are equal

For let each of the magnitudes  $A, B$  have the same ratio to  $C$ ,

I say that  $A$  is equal to  $B$

For otherwise each of the magnitudes  $A, B$  would not have had the same ratio to  $C$ , but it has, therefore  $A$  is equal to  $B$

[v 8]





Again, let  $C$  have the same ratio to each of the magnitudes  $A$ ,  $B$ ,

I say that  $A$  is equal to  $B$

For otherwise,  $C$  would not have had the same ratio to each of the magnitudes  $A$ ,  $B$ , [v 5]

but it has,

therefore  $A$  is equal to  $B$ .

Therefore etc

Q E D

### PROPOSITION 10

*Of magnitudes which have a ratio to the same, that which has a greater ratio is greater, and that to which the same has a greater ratio is less*

For let  $A$  have to  $C$  a greater ratio than  $B$  has to  $C$ ,

I say that  $A$  is greater than  $B$



For, if not,  $A$  is either equal to  $B$  or less

Now  $A$  is not equal to  $B$ ,

for in that case each of the magnitudes  $A$ ,  $B$  would have had the same ratio to  $C$ , [v 7]

but they have not,

therefore  $A$  is not equal to  $B$

Nor again is  $A$  less than  $B$ ,

for in that case  $A$  would have had to  $C$  a less ratio than  $B$  has to  $C$ , [v 8]

but it has not,

therefore  $A$  is not less than  $B$

But it was proved not to be equal either,

therefore  $A$  is greater than  $B$

Again, let  $C$  have to  $B$  a greater ratio than  $C$  has to  $A$ ,

I say that  $B$  is less than  $A$

For, if not, it is either equal or greater

Now  $B$  is not equal to  $A$ ,

for in that case  $C$  would have had the same ratio to each of the magnitudes  $A$ ,  $B$ , [v 7]

but it has not,

therefore  $A$  is not equal to  $B$

Nor again is  $B$  greater than  $A$ ,

for in that case  $C$  would have had to  $B$  a less ratio than it has to  $A$ , [v 8]

but it has not

therefore  $B$  is not greater than  $A$

But it was proved that it is not equal either,

therefore  $B$  is less than  $A$

Therefore etc

Q E D

### PROPOSITION 11

*Ratios which are the same with the same ratio are also the same with one another*

For, as  $A$  is to  $B$ , so let  $C$  be to  $D$ ,

and as  $C$  is to  $D$ , so let  $E$  be to  $F$ ,

I say that, as  $A$  is to  $B$ , so is  $E$  to  $F$

For of  $A, C, E$  let equmultiples  $G, H, K$  be taken and of  $B, D, F$  other,

$A$ _____	$C$ _____	$E$ _____
$B$ _____	$D$ _____	$F$ _____
$G$ _____	$H$ _____	$K$ _____
$L$ _____	$M$ _____	$N$ _____

chance, equmultiples  $L, M, N$

Then since, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,

and of  $A, C$  equmultiples  $G, H$  have been taken,

and of  $B, D$  other, chance, equmultiples  $L, M$ ,

therefore, if  $G$  is in excess of  $L$ ,  $H$  is also in excess of  $M$ ,

if equal, equal,

and if less, less

Again, since as  $C$  is to  $D$ , so is  $E$  to  $F$ ,

and of  $C, E$  equmultiples  $H, K$  have been taken,

and of  $D, F$  other, chance, equmultiples  $M, N$ ,

therefore, if  $H$  is in excess of  $M$ ,  $K$  is also in excess of  $N$ ,

if equal, equal,

and if less, less

But we saw that, if  $H$  was in excess of  $M$ ,  $G$  was also in excess of  $L$ , if equal, equal, and if less less,

so that, in addition, if  $G$  is in excess of  $L$ ,  $K$  is also in excess of  $N$ ,

if equal equal,

and if less, less

And  $G, K$  are equmultiples of  $A, E$ ,

while  $L, N$  are other, chance, equmultiples of  $B, F$ ,

therefore, as  $A$  is to  $B$ , so is  $E$  to  $F$

Therefore etc

Q E D

# PROPOSITION 12

If any number of magnitudes be proportional as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents

Let any number of magnitudes  $A, B, C, D, E, F$  be proportional, so that, as  $A$  is to  $B$ , so is  $C$  to  $D$  and  $E$  to  $F$ ,

I say that as  $A$  is to  $B$ , so are  $A, C, E$  to  $B, D, F$

$A$ _____	$B$ _____	$C$ _____
$D$ _____	$E$ _____	$F$ _____
$G$ _____	$L$ _____	
$H$ _____	$M$ _____	
$K$ _____	$N$ _____	

For of  $A, C, E$  let equmultiples  $G, H, K$  be taken and of  $B, D, F$  other, chance, equmultiples  $L, M, N$

Then since, as  $A$  is to  $B$ , so is  $C$  to  $D$ , and  $E$  to  $F$ , and of  $A, C, E$  equmultiples  $G, H, K$  have been taken,

and if less, less,  
so that, in addition,  
if  $G$  is in excess of  $L$ , then  $G, H, K$  are in excess of  $L, M, N$ ,  
if equal, equal,  
and if less, less

Now  $G$  and  $G, H, K$  are equimultiples of  $A$  and  $A, C, E$ , since, if any number of magnitudes whatever are respectively equimultiples of any magnitudes equal in multitude, whatever multiple one of the magnitudes is of one that multiple also will all be of all [v 1]

For the same reason

$L$  and  $L, M, N$  are also equimultiples of  $B$  and  $B, D, F$ ,  
therefore, as  $A$  is to  $B$ , so are  $A, C, E$  to  $B, D, F$  [v Def 5]

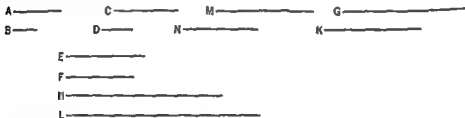
Therefore etc Q E D

### PROPOSITION 13

*If a first magnitude have to a second the same ratio as a third to a fourth and the third have to the fourth a greater ratio than a fifth has to a sixth, the first will also have to the second a greater ratio than the fifth to the sixth*

For let a first magnitude  $A$  have to a second  $B$  the same ratio as a third  $C$  has to a fourth  $D$ ,  
and let the third  $C$  have to the fourth  $D$  a greater ratio than a fifth  $E$  has to a sixth  $F$ ,

I say that the first  $A$  will also have to the second  $B$  a greater ratio than the fifth  $E$  to the sixth  $F$



For since there are some equimultiples of  $C, E$ ,  
and of  $D, F$  other, chance, equimultiples, such that the multiple of  $C$  is in excess of the multiple of  $D$ ,

while the multiple of  $E$  is not in excess of the multiple of  $F$ , [v Def 7].  
let them be taken

and let  $G, H$  be equimultiples of  $C, E$ ,

and  $K, L$  other chance equimultiples of  $D, F$ ,

so that  $G$  is in excess of  $K$ , but  $H$  is not in excess of  $L$ ,

and, whatever multiple  $G$  is of  $C$ , let  $M$  be also that multiple of  $A$ ,

and, whatever multiple  $K$  is of  $D$ , let  $N$  be also that multiple of  $B$

Now, since, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,

and of  $A, C$  equimultiples  $M, G$  have been taken

and of  $B, D$  other, chance, equimultiples  $N, K$ ,

therefore, if  $M$  is in excess of  $N$ ,  $G$  is also in excess of  $K$ ,

if equal, equal,

and if less, less

[v Def 5]

But  $G$  is in excess of  $K$ ,therefore  $M$  is also in excess of  $N$ But  $H$  is not in excess of  $L$ ,and  $M, H$  are equimultiples of  $A, E$ ,and  $N, L$  other, chance, equimultiples of  $B, F$ ,therefore  $A$  has to  $B$  a greater ratio than  $E$  has to  $F$ . [v Def 7]

Therefore etc

Q E D

## PROPOSITION 14

If a first magnitude have to a second the same ratio as a third has to a fourth, and the first be greater than the third, the second will also be greater than the fourth; if equal, equal, and if less, less

For let  $\equiv$  first magnitude  $A$  have the same ratio to a second  $B$  as a third  $C$  has to a fourth  $D$ , and let  $A$  be greater than  $C$ ,

I say that  $B$  is also greater than  $D$ 

$A$ ————— $B$ —————	$C$ ————— $D$ —————	For, since $A \equiv$ greater than $C$ , and $B$ is another, chance, magnitude, therefore $A$ has to $B$ a greater ratio than $C$ has to $B$
		[v 6]

But, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,therefore  $C$  has also to  $D$  a greater ratio than  $C$  has to  $B$  [v 13]

But that to which the same has a greater ratio is less,

[v 10]

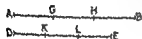
therefore  $D \equiv$  less than  $B$ ,so that  $B$  is greater than  $D$ Similarly we can prove that if  $A$  be equal to  $C$ ,  $B$  will also be equal to  $D$ ,and, if  $A$  be less than  $C$ ,  $B$  will also be less than  $D$ 

Therefore etc

Q E D

## PROPOSITION 15

Parts have the same ratio as the same multiples of them taken in corresponding order

For let  $AB$  be the same multiple of  $C$  that  $DE$  is of  $F$ ,I say that, as  $C$  is to  $F$ , so is  $AB$  to  $DE$ 

For since  $AB \equiv$  the same multiple of  $C$  that  $DE$  is of  $F$ , as many magnitudes as there are in  $AB$  equal to  $C$  so many are there also in  $DE$  equal to  $F$

th

that is the magnitudes  $DA, AL, LB$ And since  $AG, GH, HB$  are equal to one another,and  $DK, KL, LE$  are also equal to one anothertherefore, as  $AG$  is to  $DK$ , so is  $GH$  to  $KL$  and  $HB$  to  $LE$  [v 7]Therefore, as one of the antecedents  $\equiv$  to one of the consequents, so will all the antecedents be to all the consequents,

[v 12]

therefore as  $AG$  is to  $DK$ , so is  $AB$  to  $DE$

But  $AG$  is equal to  $C$  and  $DK$  to  $F$ ,  
therefore, as  $C$  is to  $F$ , so is  $AB$  to  $DE$

Therefore etc

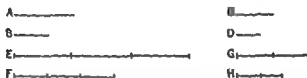
Q E D

### PROPOSITION 16

*If four magnitudes be proportional, they will also be proportional alternately*

Let  $A, B, C, D$  be four proportional magnitudes,  
so that, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,

I say that they will also be so alternately, that is, as  $A$  is to  $C$ , so is  $B$  to  $D$



For of  $A, B$  let equmultiples  $E, F$  be taken,  
and of  $C, D$  other, chance, equmultiples  $G, H$ .

Then, since  $E$  is the same multiple of  $A$  that  $F$  is of  $B$ ,  
and parts have the same ratio as the same multiples of them, [v 15]  
therefore as  $A$  is to  $B$ , so is  $E$  to  $F$

But as  $A$  is to  $B$  so is  $C$  to  $D$ ,  
therefore also, as  $C$  is to  $D$ , so is  $E$  to  $F$ . [v 11]

Again, since  $G, H$  are equmultiples of  $C, D$ ,  
therefore, as  $C$  is to  $D$ , so is  $G$  to  $H$  [v 15]

But, as  $C$  is to  $D$ , so is  $E$  to  $F$ ,  
therefore also as  $E$  is to  $F$ , so is  $G$  to  $H$  [v 11]

But, if four magnitudes be proportional and the first be greater than the third,

the second will also be greater than the fourth,  
if equal equal,  
and if less, less [v 14]

Therefore if  $E$  is in excess of  $G$ ,  $F$  is also in excess of  $H$ ,  
if equal, equal,  
and if less, less

Now  $E, F$  are equmultiples of  $A, B$ ,  
and  $G, H$  other, chance, equmultiples of  $C, D$ ;  
therefore as  $A$  is to  $C$ , so is  $B$  to  $D$  [v Def 5]

Therefore etc

Q E D

### PROPOSITION 17

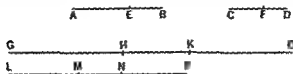
*If magnitudes be proportional componendo, they will also be proportional separando*

Let  $AB, BE, CD, DF$  be magnitudes proportional *componendo*, so that, as  $AB$  is to  $BE$ , so is  $CD$  to  $DF$ ,

I say that they will also be proportional *separando*, that is, as  $AE$  is to  $EB$ , so is  $CF$  to  $DF$

For of  $AE, EB, CF, FD$  let equmultiples  $GH, HK, LM, MN$  be taken,  
and of  $EB, FD$  other, chance, equmultiples,  $KO, NP$

Then, since  $GH$  is the same multiple of  $AE$  that  $HK$  is of  $EB$ ,  
 therefore  $GH$  is the same multiple of  $AE$  that  $GK$  is of  $AB$  [v 1]  
 But  $GH$  is the same multiple of  $AE$  that  $LM$  is of  $CF$ ,  
 therefore  $GK$  is the same multiple of  $AB$  that  $LM$  is of  $CF$



Again, since  $LM$  is the same multiple of  $CF$  that  $MN$  is of  $FD$ ,  
 therefore  $LM$  is the same multiple of  $CF$  that  $LN$  is of  $CD$  [v 1]  
 But  $LM$  was the same multiple of  $CF$  that  $GK$  is of  $AB$ ,  
 therefore  $GK$  is the same multiple of  $AB$  that  $LN$  is of  $CD$   
 Therefore  $GK, LN$  are equimultiples of  $AB, CD$   
 Again, since  $HK$  is the same multiple of  $EB$  that  $MN$  is of  $FD$ ,  
 and  $KO$  is also the same multiple of  $EB$  that  $NP$  is of  $FD$ ,  
 therefore the sum  $HO$  is also the same multiple of  $EB$  that  $MP$  is of  $FD$  [v 2]

And, since as  $AB$  is to  $BE$ , so is  $CD$  to  $DF$ ,  
 and of  $AB, CD$  equimultiples  $GK, LN$  have been taken,  
 and of  $EB, FD$  equimultiples  $HO, MP$ ,  
 therefore, if  $GK$  is in excess of  $HO$ ,  $LN$  is also in excess of  $MP$ ,  
 if equal, equal,  
 and if less, less

Let  $GH$  be in excess of  $KO$ ,  
 then, if  $HK$  be added to each,  
 $GK$  is also in excess of  $HO$   
 But we saw that if  $GK$  was in excess of  $HO$ ,  $LN$  was also in excess of  $MP$ ,  
 therefore  $LN$  is also in excess of  $MP$ ,  
 and if  $MN$  be subtracted from each,  
 $LM$  is also in excess of  $NP$ ,  
 so that if  $GH$  is in excess of  $KO$   $LM$  is also in excess of  $NP$

Similarly we can prove that  
 if  $GH$  be equal to  $KO$ ,  $LM$  will also be equal to  $NP$ ,  
 and if less, less

And  $GH, LM$  are equimultiples of  $AE, CF$   
 while  $KO, NP$  are other chance equimultiples of  $EB, FD$ ,  
 therefore, as  $AE$  is to  $EB$ , so is  $CF$  to  $FD$

Therefore etc

Q E D

# PROPOSITION 18

If magnitudes be proportional *separando* they will also be proportional *componendo*

Let  $AE, EB, CF, FD$  be magnitudes proportional *separando* so that, as  $AE$  is to  $EB$ , so is  $CF$  to  $FD$ ,

I say that they will also be proportional *componendo*, that is, as  $AB$  is to  $BE$ , so is  $CD$  to  $FD$

For, if  $CD$  be not to  $DF$  as  $AB$  to  $BE$ , then, as  $AB$  is to  $BE$ , so will  $CD$  be either to some magnitude less than  $DF$  or to a greater

First let it be in that ratio to a less magnitude  $DG$

Then, since, as  $AB$  is to  $BE$ , so is  $CD$  to  $DG$ ,

they are magnitudes proportional *compendo*, so that they will also be proportional *separando*

[v 17]

Therefore, as  $AE$  is to  $EB$ , so is  $CG$  to  $GD$

But also, by hypothesis,

as  $AE$  is to  $EB$ , so is  $CF$  to  $FD$

Therefore also, as  $CG$  is to  $GD$ , so is  $CF$  to  $FD$

[v 11]

But the first  $CG$  is greater than the third  $CF$ ,

therefore the second  $GD$  is also greater than the fourth  $FD$  [v 14]

But it is also less which is impossible

Therefore, as  $AB$  is to  $BE$ , so is not  $CD$  to a less magnitude than  $FD$

Similarly we can prove that neither is it in that ratio to a greater,

it is therefore in that ratio to  $FD$  itself

Therefore etc

Q E D

### PROPOSITION 19

If, as a whole is to a whole so is a part subtracted to a part subtracted, the remainder will also be to the remainder as whole to whole

For, as the whole  $AB$  is to the whole  $CD$ , so let the part  $AE$  subtracted be to the part  $CF$  subtracted,

I say that the remainder  $EB$  will also be to the remainder  $FD$  as the whole  $AB$  to the whole  $CD$

For since as  $AB$  is to  $CD$ , so is  $AE$  to  $CF$ ,

alternately also, as  $BA$  is to  $AE$ , so is  $DC$  to  $CF$

[v 16]

And since the magnitudes are proportional *compendo*, they will also be proportional *separando*,

[v 17]

that is, as  $BE$  is to  $EA$ , so is  $DF$  to  $CF$ ,

and, alternately,

as  $BE$  is to  $DF$  so is  $EA$  to  $FC$

[v 10]

But as  $AE$  is to  $CF$  so by hypothesis is the whole  $AB$  to the whole  $CD$

Therefore also the remainder  $EB$  will be to the remainder  $FD$  as the whole  $AB$  is to the whole  $CD$

[v 11]

Therefore etc

[PORISM] From this it is manifest that if magnitudes be proportional *compendo* they will also be proportional *convertendo* }

Q E D

### PROPOSITION 20

If there be three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio and if ex aequali the first be greater than the third the fourth will also be greater than the sixth if equal, equal and, if less, less

Let there be three magnitudes  $A B C$  and others  $D E F$  equal to them in multitude, which taken two and two are in the same ratio, so that,

as  $A$  is to  $B$  so is  $D$  to  $E$ ,

and,

as  $B$  is to  $C$ , so is  $E$  to  $F$ ,and let  $A$  be greater than  $C$  *ex aequali*,I say that  $D$  will also be greater than  $F$ , if  $A$  is equal to  $C$ , equal, and, if less, less

A —————

D —————

B ———

E ———

C —————

F —————

For, since  $A$  is greater than  $C$ ,and  $B$  is some other magnitude,

and the greater has to the same a greater ratio than the less has, [v. 8]

therefore  $A$  has to  $B$  a greater ratio than  $C$  has to  $B$ But, as  $A$  is to  $B$ , so is  $D$  to  $E$ ,and, as  $C$  is to  $B$ , inversely, so is  $F$  to  $E$ ,therefore  $D$  has also to  $E$  a greater ratio than  $F$  has to  $E$  [v. 13]

But, of magnitudes which have a ratio to the same, that which has a greater ratio is greater, [v. 10]

therefore  $D$  is greater than  $F$ Similarly we can prove that, if  $A$  be equal to  $C$ ,  $D$  will also be equal to  $F$ , and if less, less

Therefore etc

Q E D

## PROPOSITION 21

If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then, if *ex aequali* the first magnitude is greater than the third, the fourth will also be greater than the sixth, if equal, equal, and if less, less

Let there be three magnitudes  $A, B, C$ , and others  $D, E, F$  equal to them in multitude, which taken two and two are in the same ratio, and let the proportion of them be perturbed, so that,

as  $A$  is to  $B$ , so is  $E$  to  $F$ ,

and,

as  $B$  is to  $C$ , so is  $D$  to  $E$ ,and let  $A$  be greater than  $C$  *ex aequali*,I say that  $D$  will also be greater than  $F$ , if  $A$  is equal to  $C$ , equal, and if less, less

A —————

D —————

B ———

E —————

C —————

F —————

For since  $A$  is greater than  $C$ ,and  $B$  is some other magnitude,therefore  $A$  has to  $B$  a greater ratio than  $C$  has to  $B$  [v. 8]But, as  $A$  is to  $B$ , so is  $E$  to  $F$ ,and, as  $C$  is to  $B$ , inversely, so is  $F$  to  $E$ Therefore also  $E$  has to  $F$  a greater ratio than  $E$  has to  $D$  [v. 13]

But that to which the same has a greater ratio is less, [v. 10]

therefore  $F$  is less than  $D$ ,therefore  $D$  is greater than  $F$ Similarly we can prove that, if  $A$  be equal to  $C$ ,  $D$  will also be equal to  $F$ , and if less, less

Therefore etc

Q E D



## PROPOSITION 22

If there be any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, they will also be in the same ratio *ex aequali*

Let there be any number of magnitudes  $A, B, C$ , and others  $D, E, F$  equal to them in multitude, which taken two and two together are in the same ratio, so that,

as  $A$  is to  $B$ , so is  $D$  to  $E$ ,

and, as  $B$  is to  $C$ , so is  $E$  to  $F$ ,

I say that they will also be in the same ratio *ex aequali*,

<that is, as  $A$  is to  $C$ , so is  $D$  to  $F$ >

For of  $A, D$  let equimultiples  $G, H$  be taken,

and of  $B, E$  other, chance, equimultiples  $K, L$ ;

and, further, of  $C, F$  other, chance, equimultiples  $M, N$ .



Then, since, as  $A$  is to  $B$ , so is  $D$  to  $E$ ,

and of  $A, D$  equimultiples  $G, H$  have been taken,

and of  $B, E$  other, chance, equimultiples  $K, L$ ,

therefore, as  $G$  is to  $K$ , so is  $H$  to  $L$

[v 4]

For the same reason also,

as  $K$  is to  $M$ , so is  $L$  to  $N$

Since then, there are three magnitudes  $G, K, M$ , and others  $H, L, N$  equal to them in multitude, which taken two and two together are in the same ratio,

therefore, *ex aequali*, if  $G$  is in excess of  $M$ ,  $H$  is also in excess of  $N$ ,

if equal, equal, and if less, less

[v 20]

And  $G, H$  are equimultiples of  $A, D$ ,

and  $M, N$  other, chance, equimultiples of  $C, F$

Therefore as  $A$  is to  $C$ , so is  $D$  to  $F$

[v Def 5]

Therefore etc

Q E D

## PROPOSITION 23

If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio and the proportion of them be perturbed, they will also be in the same ratio *ex aequali*

Let there be three magnitudes  $A, B, C$ , and others equal to them in multitude, which taken two and two together are in the same proportion, namely  $D, E, F$ , and let the proportion of them be perturbed so that,

as  $A$  is to  $B$ , so is  $E$  to  $F$ ,

and, as  $B$  is to  $C$  so is  $D$  to  $E$ ,

I say that as  $A$  is to  $C$  so is  $D$  to  $F$

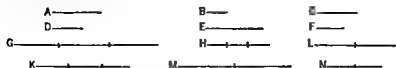
Of  $A, B, D$  let equimultiples  $G, H, K$  be taken

and of  $C, E, F$  other chance equimultiples  $L, M, N$

Then, since  $G, H$  are equimultiples of  $A, B$ ,

and parts have the same ratio as the same multiples of them, [v 15]  
therefore, as  $A$  is to  $B$ , so is  $G$  to  $H$

For the same reason also,



as  $E$  is to  $F$ , so is  $M$  to  $N$

And, as  $A$  is to  $B$ , so is  $E$  to  $F$ ,  
therefore also, as  $G$  is to  $H$ , so is  $M$  to  $N$  [v 11]

Next, since, as  $B$  is to  $C$ , so is  $D$  to  $E$ ,  
alternately, also, as  $B$  is to  $D$ , so is  $C$  to  $E$  [v 16]

And, since  $H$ ,  $K$  are equmultiples of  $B$ ,  $D$ ,  
and parts have the same ratio as their equmultiples,  
therefore, as  $B$  is to  $D$ , so is  $H$  to  $K$  [v 15]

But, as  $B$  is to  $D$ , so is  $C$  to  $E$ ,  
therefore also, as  $H$  is to  $K$ , so is  $C$  to  $E$  [v 11]

Again, since  $L$ ,  $M$  are equmultiples of  $C$ ,  $E$ ,  
therefore, as  $C$  is to  $E$ , so is  $L$  to  $M$  [v 15]

But, as  $C$  is to  $E$ , so is  $H$  to  $K$ ,  
therefore also, as  $H$  is to  $K$ , so is  $L$  to  $M$ , [v 11]

and, alternately, as  $H$  is to  $L$ , so is  $K$  to  $M$  [v 16]

But it was also proved that,  
as  $G$  is to  $H$ , so is  $M$  to  $N$

Since, then, there are three magnitudes  $G$ ,  $H$ ,  $L$ , and others equal to them in multitude  $K$ ,  $M$ ,  $N$ , which taken two and two together are in the same ratio, and the proportion of them is perturbed,

therefore, *ex aequali*, if  $G$  is in excess of  $L$ ,  $K$  is also in excess of  $N$ ,  
if equal, equal, and if less, less [v 21]

And  $G$ ,  $K$  are equmultiples of  $A$ ,  $D$ ,  
and  $L$ ,  $N$  of  $C$ ,  $F$ .

Therefore, as  $A$  is to  $C$ , so is  $D$  to  $F$

Therefore etc

Q E D

### PROPOSITION 24

If a first magnitude have to a second the same ratio as a third has to a fourth and also a fifth have to the second the same ratio as a sixth to the fourth the first and fifth added together will have to the second the same ratio as the third and sixth have to the fourth

Let a first magnitude  $AB$  have to a second  $C$  the same ratio as a third  $DE$  has to a fourth  $F$ ,

and let also a fifth  $BG$  have to the second  $C$  the same ratio as a sixth  $EH$  has to the fourth  $F$ ,

as the third and sixth  $DH$ , has to the fourth  $F$

For since, as  $BG$  is to  $C$ , so is  $EH$  to  $F$ ,

inversely, as  $C$  is to  $BG$ , so is  $F$  to  $EH$

Since, then, as  $AB$  is to  $C$ , so is  $DE$  to  $F$ ,

and as  $C$  is to  $BG$ , so is  $F$  to  $EH$ ,

therefore, *ex aequali*, as  $AB$  is to  $BG$ , so is  $DE$  to  $EH$  [v 22]

And, since the magnitudes are proportional *separando*, they will also be proportional *componendo*, [v 18]

therefore, as  $AG$  is to  $GB$ , so is  $DH$  to  $HE$

But also, as  $BG$  is to  $C$ , so is  $EH$  to  $F$ ,

therefore, *ex aequali*, as  $AG$  is to  $C$ , so is  $DH$  to  $F$  [v 22]

Therefore etc

Q E D

### PROPOSITION 25

If four magnitudes be proportional the greatest and the least are greater than the remaining two

Let the four magnitudes  $AB$ ,  $CD$ ,  $E$ ,  $F$  be proportional so that as  $AB$  is to  $CD$ , so is  $E$  to  $F$ , and let  $AB$  be the greatest of them and  $F$  the least,

I say that  $AB$ ,  $F$  are greater than  $CD$ ,  $E$

For let  $AG$  be made equal to  $E$ , and  $CH$  equal to



Since, as  $AB$  is to  $CD$ , so is  $E$  to  $F$ ,

and  $E$  is equal to  $AG$ , and  $F$  to  $CH$ ,

therefore as  $AB$  is to  $CD$ , so is  $AG$  to  $CH$

And since, as the whole  $AB$  is to the whole  $CD$ , so is the part  $AG$  subtracted to the part  $CH$  subtracted

the remainder  $GB$  will also be to the remainder  $HD$  as the whole  $AB$  is to the whole  $CD$  [v 19]

But  $AB$  is greater than  $CD$ ,

therefore  $GB$  is also greater than  $HD$

And, since  $AG$  is equal to  $E$ , and  $CH$  to  $F$ ,

therefore  $AG$ ,  $F$  are equal to  $CH$ ,  $E$

And if  $GB$ ,  $HD$  being unequal and  $GB$  greater,  $AG$ ,  $F$  be added to  $GB$  and  $CH$ ,  $E$  be added to  $HD$ ,

it follows that  $AB$ ,  $F$  are greater than  $CD$ ,  $E$

Therefore etc

Q E D

## BOOK SIX

### DEFINITIONS

- 1 *Similar rectilineal figures* are such as have their angles severally equal and the sides about the equal angles proportional
- 2 A straight line is said to have been cut in *extreme and mean ratio* when as the whole line is to the greater segment, so is the greater to the less
- 3 The *height* of any figure is the perpendicular drawn from the vertex to the base

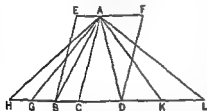
### BOOK VI PROPOSITIONS

#### PROPOSITION 1

*Triangles and parallelograms which are under the same height are to one another as their bases*

Let  $ABC$ ,  $ACD$  be triangles and  $EC$ ,  $CF$  parallelograms under the same height,

I say that, as the base  $BC$  is to the base  $CD$ , so is the triangle  $ABC$  to the triangle  $ACD$ , and the parallelogram  $EC$  to the parallelogram  $CF$



For let  $BD$  be produced in both directions to the points  $H$ ,  $L$  and let [any number of straight lines]  $BG$ ,  $GH$  be made equal to the base  $BC$ , and any number of straight lines  $DK$ ,  $KL$  equal to the base  $CD$ ,

let  $AG$ ,  $AH$ ,  $AK$ ,  $AL$  be joined  
Then, since  $CB$ ,  $BG$ ,  $GH$  are equal to one another,

the triangles  $ABC$ ,  $AGB$ ,  $AHG$  are also equal to one another [I 38]

Therefore, whatever multiple the base  $HC$  is of the base  $BC$ , that multiple also is the triangle  $AHC$  of the triangle  $ABC$

For the same reason  
whatever multiple the base  $LC$  is of the base  $CD$ , that multiple also is the triangle  $ALC$  of the triangle  $ACD$

and if the base  $HC$  is equal to the base  $CL$  the triangle  $AHC$  is also equal to the triangle  $ALC$

if the base  $HC$  is in excess of the base  $CL$  the triangle  $AHC$  is also in excess of the triangle  $ALC$

and if less less

Thus there being four magnitudes two bases  $BC$ ,  $CD$  and two triangles  $ABC$ ,  $ACD$ ,

equimultiples have been taken of the base  $BC$  and the triangle  $ABC$ , namely the base  $HC$  and the triangle  $AHC$ , and of the base  $CD$  and the triangle  $ADC$  other, chance, equimultiples, namely the base  $LC$  and the triangle  $ALC$ , and it has been proved that, if the base  $HC$  is in excess of the base  $CL$ , the triangle  $AHC$  is also in excess of the triangle  $ALC$ , if equal, equal, and, if less, less

Therefore, as the base  $BC$  is to the base  $CD$ , so  $\equiv$  the triangle  $ABC$  to the triangle  $ACD$  [v Def 5] [i 41]

Next, since the parallelogram  $EC$  is double of the triangle  $ABC$ , and the parallelogram  $FC$  is double of the triangle  $ACD$ , while parts have the same ratio as the same multiples of them, [v 15] therefore as the triangle  $ABC$  is to the triangle  $ACD$ , so is the parallelogram  $EC$  to the parallelogram  $FC$

Since, then, it was proved that, as the base  $BC$  is to  $CD$ , so is the triangle  $ABC$  to the triangle  $ACD$ , and, as the triangle  $ABC$  is to the triangle  $ACD$ , so is the parallelogram  $EC$  to the parallelogram  $FC$  therefore also, as the base  $BC$  is to the base  $CD$ , so is the parallelogram  $EC$  to the parallelogram  $FC$  [v 11]

Therefore etc

Q E D

### PROPOSITION 2

If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally, and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle

For let  $DE$  be drawn parallel to  $BC$ , one of the sides of the triangle  $ABC$  I say that as  $BD$  is to  $DA$  so is  $CE$  to  $EA$

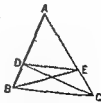
Therefore the triangle  $BDE$  is equal to the triangle  $CDE$ , for they are on the same base  $DE$  and in the same parallels  $DE$   $BC$  [i 38]

And the triangle  $ADE$  is another area But equals have the same ratio to the same, [v 7] therefore as the triangle  $BDE$  is to the triangle  $ADE$  so is the triangle  $CDE$  to the triangle  $ADE$

But, as the triangle  $BDE$  is to  $ADE$  so is  $BD$  to  $DA$ , for being under the same height the perpendicular drawn from  $E$  to  $AB$ , they are to one another as their bases [vi 1] Therefore the triangle  $CDE$  is to  $ADE$  so is  $CE$  to  $EA$

Again, let the sides  $AB$   $AC$  of the triangle  $ABC$  be cut proportionally, so that as  $BD$  is to  $DA$  so  $\equiv$   $CE$  to  $EA$  and let  $DE$  be joined [v 11]

I say that  $DE$  is parallel to  $BC$



For, with the same construction,

since, as  $BD$  is to  $DA$ , so is  $CE$  to  $EA$ ,

but, as  $BD$  is to  $DA$ , so is the triangle  $BDE$  to the triangle  $ADE$ ,

and, as  $CE$  is to  $EA$ , so is the triangle  $CDE$  to the triangle  $ADE$ , [vi 1]

therefore also,

as the triangle  $BDE$  is to the triangle  $ADE$ , so is the triangle  $CDE$  to the triangle  $ADE$  [v 11]

Therefore each of the triangles  $BDE$ ,  $CDE$  has the same ratio to  $ADE$

Therefore the triangle  $BDE$  is equal to the triangle  $CDE$ , [v 9]

and they are on the same base  $DE$

But equal triangles which are on the same base are also in the same parallels

[i 39]

Therefore  $DE$  is parallel to  $BC$

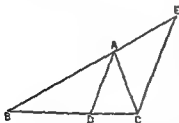
Therefore etc

Q E D

### PROPOSITION 3

*If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle, and, if the segments of the base have the same ratio as the remaining sides of the triangle, the straight line joined from the vertex to the point of section will bisect the angle of the triangle*

Let  $ABC$  be a triangle, and let the angle  $BAC$  be bisected by the straight line  $AD$ ,



I say that, as  $BD$  is to  $CD$ , so is  $BA$  to  $AC$

For let  $CE$  be drawn through  $C$  parallel to  $DA$ , and let  $BA$  be carried through and meet it at  $E$

Then, since the straight line  $AC$  falls upon the parallels  $AD$ ,  $EC$ ,  
the angle  $ACE$  is equal to the angle  $CAD$

[i 29]

But the angle  $CAD$  is by hypothesis equal to the angle  $BAD$ ,

therefore the angle  $BAD$  is also equal to the angle  $ACE$

Again, since the straight line  $BAE$  falls upon the parallels  $AD$ ,  $EC$ ,

the exterior angle  $BAD$  is equal to the interior angle  $AEC$  [i 29]

But the

[i 6]

And, since  $AD$  has been drawn parallel to  $EC$ , one of the sides of the triangle  $BCE$ ,

therefore, proportionally, as  $BD$  is to  $DC$ , so is  $BA$  to  $AE$

But  $AE$  is equal to  $AC$ ,

[vi 2]

therefore as  $BD$  is to  $DC$ , so is  $BA$  to  $AC$

Again let  $BA$  be to  $AC$  as  $BD$  to  $DC$ , and let  $AD$  be joined,

I say that the angle  $BAC$  has been bisected by the straight line  $AD$

For, with the same construction,

since, as  $BD$  is to  $DC$ , so is  $BA$  to  $AC$ ,

and also, as  $BD$  is to  $DC$  so is  $BA$  to  $AE$  for  $AD$  has been drawn parallel to

*EC*, one of the sides of the triangle *BCE*

therefore also, as *BA* is to *AC*, so is *BA* to *AE*

Therefore *AC* is equal to *AE*,

so that the angle *AEC* is also equal to the angle *ACE*

But the angle *AEC* is equal to the exterior angle *BAD*,

Therefore etc

[vi 2]

[v 11]

[v 9]

[i 5]

[i 29]

[id]

Q E D

#### PROPOSITION 4

*In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles*

Let *ABC*, *DCE* be equiangular triangles having the angle *ABC* equal to the

sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles

For let *BC* be placed in a straight line with *CE*

Then, since the angles *ABC*, *ACB* are less than two right angles,

[i 17]

and the angle *ACB* is equal to the angle *DEC*,

therefore the angles *ABC*, *DEC* are less than two right angles,

therefore *BA*, *ED*, when produced, will meet

[i Post 5]

Let them be produced and meet at *F*

Now, since the angle *DCE* is equal to the angle *ABC*,

*BF* is parallel to *CD*

[i 28]

Again, since the angle *ACB* is equal to the angle *DEC*,

*AC* is parallel to *FE*

[i 28]

Therefore *FACD* is a parallelogram,

therefore *FA* is equal to *DC*, and *AC* to *FD*

[i 34]

And since *AC* has been drawn parallel to *FE*, one side of the triangle *FBE*,

therefore, as *BA* is to *AF*, so is *BC* to *CE*

[vi 2]

But *AF* is equal to *CD*,

therefore, as *BA* is to *CD*, so is *BC* to *CE*,

and alternately, as *AB* is to *BC*, so is *DC* to *CE*

[v 16]

Again, since *CD* is parallel to *BF*,

therefore, as *BC* is to *CE*, so is *FD* to *DE*

[vi 2]

But *FD* is equal to *AC*,

therefore as *BC* is to *CE*, so is *AC* to *DE*,

and alternately, as *BC* is to *CA*, so is *CE* to *ED*

[v 16]

Since, then, it was proved that,

as *AB* is to *BC*, so is *DC* to *CE*,

as *BC* is to *CA*, so is *CE* to *ED*,

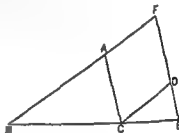
and,

therefore, *ex aequali*, as *BA* is to *AC*, so is *CD* to *DE*

[v 22]

Therefore etc

Q E D



## PROPOSITION 5

If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend

Let  $ABC$ ,  $DEF$  be two triangles having their sides proportional, so that,  
 as  $AB$  is to  $BC$ , so is  $DE$  to  $EF$ ,  
 as  $BC$  is to  $CA$ , so is  $EF$  to  $FD$ ,  
 and further, as  $BA$  is to  $AC$ , so is  $ED$  to  $DF$ ,

the angle  $ACB$ ,

[I 23]

the angle  $ACB$  is equal to the angle  $DFE$  at  $C$

[I 32]



as  $ABC$ ,  
 $GEF$  the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles, [I 4]  
 therefore, as  $AB$  is to  $BC$ , so is  $GE$  to  $EF$

But, as  $AB$  is to  $BC$ , so by hypothesis  $DE$  to  $EF$ ,

therefore, as  $DE$  is to  $EF$ , so is  $GE$  to  $EF$  [V 11]

Therefore each of the straight lines  $DE$ ,  $GE$  has the same ratio to  $EF$ ,

therefore  $DE$  is equal to  $GE$  [V 9]

For the same reason

$DF$  is also equal to  $GF$

Since then  $DE$  is equal to  $EG$ ,

and  $EF$  is common,

the two sides  $DE$ ,  $EF$  are equal to the two sides  $GE$ ,  $EF$ ,

and the base  $DF$  is equal to the base  $FG$ ,

therefore the angle  $DEF$  is equal to the angle  $GEF$ , [I 8]

and the triangle  $DEF$  is equal to the triangle  $GEF$ ,

and the remaining angles are equal to the remaining angles, namely those which the equal sides subtend [I 4]

Therefore the angle  $DFE$  is also equal to the angle  $GFE$ ,

and the angle  $EDF$  to the angle  $EGF$

And, since the angle  $FED$  is equal to the angle  $GEF$ ,

while the angle  $GEF$  is equal to the angle  $ABC$ ,

therefore the angle  $ABC$  is also equal to the angle  $DEF$

For the same reason

the angle  $ACB$  is also equal to the angle  $DFE$ ,

and further, the angle at  $A$  to the angle at  $D$ ,

therefore the triangle  $ABC$  is equiangular with the triangle  $DEF$

Therefore etc

Q E D



## PROPOSITION 6

If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend

Let  $ABC$ ,  $DEF$  be two triangles having one angle  $BAC$  equal to one angle  $EDF$  and the sides about the equal angles proportional so that,  
as  $BA$  is to  $AC$ , so is  $ED$  to  $DF$ ,

I say that the triangle  $ABC$  is equiangular with the triangle  $DEF$ , and will have the angle  $ABC$  equal to the angle  $DEF$ , and the angle  $ACB$  to the angle  $DFE$

Therefore the triangle  $ABC$  is equiangular with the triangle  $DEF$ , and will have the angle  $ABC$  equal to the angle  $DEF$ , and the angle  $ACB$  to the angle  $DFE$  [I 23]  
at  $G$  [I 32]  
[VI 4]

Therefore the triangle  $ABC$  is equiangular with the triangle  $DEF$ , and will have the angle  $ABC$  equal to the angle  $DEF$ , and the angle  $ACB$  to the angle  $DFE$  [I 23]  
Therefore, proportionally [VI 4]  
But, by hypothesis, as  $L$   
therefore also, as  $ED$  is to  $DF$ , so is  $GD$  to  $DF$ . [V 11]

Therefore  $ED$  is equal to  $GD$ , [V 9]  
and  $DF$  is common,  
therefore the two sides  $ED$ ,  $DF$  are equal to the two sides  $GD$ ,  $DF$ , and the angle  $EDF$  is equal to the angle  $GDF$ ,



therefore the base  $EF$  is equal to the base  $GF$ ,  
[I 4]  
les, namely those [I 4]

therefore the angle  $ACB$  is also equal to the angle  $DFE$   
And by hypothesis, the angle  $BAC$  is also equal to the angle  $EDF$ ,  
therefore the remaining angle at  $B$  is also equal to the remaining angle at  $E$ , [I 32]

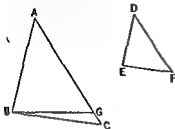
therefore the triangle  $ABC$  is equiangular with the triangle  $DEF$   
Therefore etc [Q E D]

## PROPOSITION 7

If two triangles have one angle equal to one angle, the sides about other angles proportional and the remaining angles either both less or both not less than a right angle, the triangles will be equiangular and will have those angles equal, the sides about which are proportional

Let  $ABC$ ,  $DEF$  be two triangles having one angle  $BAC$  equal to one angle  $EDF$  and the sides about other angles proportional, so that,  
remaining angles at  $C$ ,  $F$  less than a right angle,

I say that the triangle  $ABC$  is equiangular with the triangle  $DEF$ , the angle  $ABC$  will be equal to the angle  $DEF$ , and the remaining angle namely the angle at  $C$ , equal to the remaining angle the angle at  $F$



For if the angle  $ABC$  is unequal to the angle  $DEF$ , one of them is greater

Let the angle  $ABC$  be greater, and on the straight line  $AB$  and at the point  $B$  on it let the angle  $ABG$  be constructed equal to the angle  $DEF$  [I 23]

Then since the angle  $A$  is equal to  $D$  and the angle  $ABG$  to the angle  $DEF$ ,

therefore the angle  $ACB$  is equal to the angle  $DFE$

But as  $DE$  is to  $EF$  so by hypothesis is  $AB$  to  $BC$  therefore  $AB$  has the same ratio to each of the straight lines  $BC$ ,  $BG$ , [V 11]  
therefore  $BC$  is equal to  $BG$  [V 9]

so that the angle at  $C$  is also equal to the angle  $BGC$  [I 5]

But by hypothesis the angle at  $C$  is less than a right angle therefore the angle  $BGC$  is also less than a right angle, so that the angle  $AGB$  adjacent to it is greater than a right angle [I 13]

therefore it is equal to it

angle,

I say again that in this case too the triangle  $ABC$  is equiangular with the triangle  $DEF$

For with the same construction, we can prove similarly that

$BC$  is equal to  $BG$ ,

so that the angle at  $C$  is also equal to the angle  $BGC$  [I 5]

But the angle at  $C$  is not less than a right angle,

therefore neither is the angle  $BGC$  less than a right angle

Thus in the triangle  $BGC$  the two angles are not less than two right angles which is impossible [I 17]

Therefore once more the angle  $ABC$  is not unequal to the angle  $DEF$ , therefore it is equal to it

[I 32]

Therefore etc

Q E D

## PROPOSITION 8

If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another

Let  $ABC$  be a right-angled triangle having the angle  $BAC$  right, and let  $AD$  be drawn from  $A$  perpendicular to  $BC$ ,

I say that each of the triangles  $ABD$ ,  $ADC$  is similar to the whole  $ABC$  and, further, they are similar to one another

For, since the angle  $BAC$  is equal to the angle  $ADB$ , for each is right, and the angle at  $B$  is common to the two triangles  $ABC$  and  $ABD$ ,

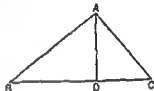
therefore the remaining angle  $ACB$  is equal to the remaining angle  $BAD$ , [I 32]

therefore the triangle  $ABC$  is equiangular with the triangle  $ABD$

Therefore, as  $BC$  which subtends the right angle in the triangle  $ABC$  is to  $BA$  which subtends the right angle in the triangle  $ABD$ , so is

$AB$  itself which subtends the angle at  $C$  in the triangle  $ABC$  to  $BD$  which subtends the equal angle  $BAD$  in the triangle  $ABD$ , and so also is  $AC$  to  $AD$  [VI 4]

and



Therefore the triangle  $ABC$  is similar to the triangle  $ABD$  [VI Def 1]

Similarly we can prove that

the triangle  $ABC$  is also similar to the triangle  $ADC$ ,

therefore each of the triangles  $ABD$ ,  $ADC$  is similar to the whole  $ABC$

I say next that the triangles  $ABD$ ,  $ADC$  are also similar to one another

For, since the right angle  $BDA$  is equal to the right angle  $ADC$ ,

and moreover the angle  $BAD$  was also proved equal to the angle at  $C$ ,

therefore the remaining angle at  $B$  is also equal to the remaining angle  $DAC$ , [I 32]

therefore the triangle  $ABD$  is similar to the triangle  $ADC$

TI

DA

so is

subt

also

Therefore etc

Porism From this it is clear that, if in a right-angled triangle a perpendicular be drawn from the right angle to the base the straight line so drawn is a mean proportional between the segments of the base Q E D

## PROPOSITION 9

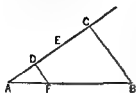
From a given straight line to cut off a prescribed part

Let  $AB$  be the given straight line,

thus it is required to cut off from  $AB$  a prescribed part

Let the third part be that prescribed

Let a straight line  $AC$  be drawn through from  $A$  containing with  $AB$  any angle,



let a point  $D$  be taken at random on  $AC$ , and let  $DE$ ,  $EC$  be made equal to  $AD$  [I 3]

Let  $BC$  be joined, and through  $D$  let  $DF$  be drawn parallel to it [I 31]

Then, since  $FD$  has been drawn parallel to  $BC$ , one of the sides of the triangle  $ABC$ , therefore, proportionally, as  $CD$  is to  $DA$ , so is  $BF$  to  $FA$  [VI 2]

But  $CD$  is double of  $DA$ ,

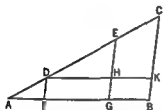
therefore  $BF$  is also double of  $FA$ ,

therefore  $BA$  is triple of  $AF$

Therefore from the given straight line  $AB$  the prescribed third part  $AF$  has been cut off Q E F

### PROPOSITION 10

To . . . . .



tain any angle,

let  $CB$  be joined, and through  $D$ ,  $E$  let  $DF$ ,  $EG$  be drawn parallel to  $BC$ , and through  $D$  let  $DHK$  be drawn parallel to  $AB$  [I 31]

Therefore each of the figures  $FH$ ,  $HB$  is a parallelogram, therefore  $DH$  is equal to  $FG$  and  $HK$  to  $GB$  [I 34]

Now, since the straight line  $HE$  has been drawn parallel to  $KC$ , one of the sides of the triangle  $DKC$ ,

therefore, proportionally, as  $CE$  is to  $ED$ , so is  $KH$  to  $HD$  [VI 2]

But  $KH$  is equal to  $BG$ , and  $HD$  to  $GF$ ,

therefore, as  $CE$  is to  $ED$ , so is  $BG$  to  $GF$

Again, since  $FD$  has been drawn parallel to  $GE$ , one of the sides of the triangle  $AGE$ ,

therefore, proportionally, as  $ED$  is to  $DA$ , so is  $GF$  to  $FA$  [VI 2]

But it was also proved that,

as  $CE$  is to  $ED$ , so is  $BG$  to  $GF$ ,

therefore, as  $CE$  is to  $ED$ , so is  $BG$  to  $GF$ ,

and, as  $ED$  is to  $DA$ , so is  $GF$  to  $FA$

Therefore the given uncut straight line  $AB$  has been cut similarly to the given cut straight line  $AC$  Q E F

### PROPOSITION 11

To two given straight lines to find a third proportional

Let  $BA$ ,  $AC$  be the two given straight lines, and let them be placed so as to contain any angle,

thus it is required to find a third proportional to  $BA$ ,  $AC$

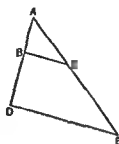
For let them be produced to the points  $D, E$ , and let  $BD$  be made equal to  $AC$ ,  
 let  $BC$  be joined, and through  $D$  let  $DE$  be drawn parallel to it [I 31]

Since then,  $BC$  has been drawn parallel to  $DE$ , one of the sides of the triangle  $ADE$ , proportionally, as  $AB$  is to  $BD$ , so is  $AC$  to  $CE$  [VI 2]

But  $BD$  is equal to  $AC$ ,

therefore, as  $AB$  is to  $AC$ , so is  $AC$  to  $CE$

Therefore to two given straight lines  $AB, AC$  a third proportional to them,  $CE$ , has been found Q E F

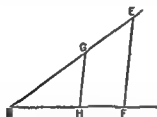


### PROPOSITION 12

To three given straight lines to find a fourth proportional

Let  $A, B, C$  be the three given straight lines,

thus it is required to find a fourth proportional to  $A, B, C$ .



Let two straight lines  $DE, DF$  be set out containing any angle  $EDF$ , let  $DG$  be made equal to  $A$ ,  $GE$  equal to  $B$ , and further  $DH$  equal to  $C$ ,

let  $GH$  be joined, and let  $EF$  be drawn through  $E$  parallel to it [I 31]

Since, then,  $GH$  has been drawn parallel to  $EF$ , one of the sides of the triangle  $DEF$ ,

therefore as  $DG$  is to  $GE$ , so is  $DH$  to  $HF$  [VI 2]

But  $DG$  is equal to  $A$ ,  $GE$  to  $B$ , and  $DH$  to  $C$ ,

therefore as  $A$  is to  $B$ , so is  $C$  to  $HF$

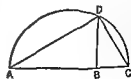
Therefore to the three given straight lines  $A, B, C$  a fourth proportional  $HF$  has been found Q E F

### PROPOSITION 13

To two given straight lines to find a mean proportional

Let  $AB, BC$  be the two given straight lines, thus it is required to find a mean proportional to  $AB, BC$

Let them be placed in a straight line, and let the semicircle  $ADC$  be described on  $AC$ , let  $BD$  be drawn from the point  $B$  at right angles to  $AC$

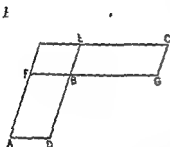


therefore  $DB$  is a mean proportional between the segments of the base,  $AB$ ,  $BC$  [VI 8, For]

Therefore to the two given straight lines  $AB$ ,  $BC$  a mean proportional  $DB$  has been found Q E F

## PROPOSITION 14

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional and conversely



therefore  $FB$ ,  $BG$  are also in a straight line [I 14]

I say that, in  $AB$ ,  $BC$ , the sides about the equal angles are reciprocally proportional, that is to say, that, as  $DB$  is to  $BE$ , so is  $GB$  to  $BF$

For let the parallelogram  $FE$  be completed

Since, then, the parallelogram  $AB$  is equal to the parallelogram  $BC$ ,

and  $FE$  is another area,

therefore, as  $AB$  is to  $FE$ , so is  $BC$  to  $FE$  [v 7]

But, as  $AB$  is to  $FE$ , so is  $DB$  to  $BE$ , [vi 1]

and, as  $BC$  is to  $FE$ , so is  $GB$  to  $BF$  [id]

therefore also, as  $DB$  is to  $BE$ , so is  $GB$  to  $BF$  [v 11]

Therefore in the parallelograms  $AB$ ,  $BC$  the sides about the equal angles are reciprocally proportional

Next, let  $GB$  be to  $BF$  as  $DB$  to  $BE$ ,

I say that the parallelogram  $AB$  is equal to the parallelogram  $BC$

For since, as  $DB$  is to  $BE$ , so is  $GB$  to  $BF$ ,

while, as  $DB$  is to  $BE$ , so is the parallelogram  $AB$  to the parallelogram  $FE$ , [vi 1]

and, as  $GB$  is to  $BF$ , so is the parallelogram  $BC$  to the parallelogram  $FE$ , [vi 1]

therefore also, as  $AB$  is to  $FE$ , so is  $BC$  to  $FE$ , [v 11]

therefore the parallelogram  $AB$  is equal to the parallelogram  $BC$  [v 9]

Therefore etc Q E D

## PROPOSITION 15

In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional and conversely

Let  $ABC$ ,  $ADE$  be equal triangles having one angle, namely the angle  $BAC$

I say that in the triangles the sides about the equal angles are reciprocally proportional

For let the

Let  $BD$  be joined

Since, then, the triangle  $ABC$  is equal to the triangle  $ADE$ , and  $BAD$  is another area,

therefore, as the triangle  $CAB$  is to the triangle  $BAD$ , so is the triangle  $EAD$  to the triangle  $BAD$  [v 7]

But, as  $CAB$  is to  $BAD$ , so is  $CA$  to  $AD$ , [vi 1]

and, as  $EAD$  is to  $BAD$ , so is  $EA$  to  $AB$  [id]

Therefore also, as  $CA$  is to  $AD$ , so is  $EA$  to  $AB$  [v 11]

Therefore in the triangles  $ABC$ ,  $ADE$  the sides about the equal angles are reciprocally proportional

Next, let the sides of the triangles  $ABC$ ,  $ADE$  be reciprocally proportional, that is to say, let  $EA$  be to  $AB$  as  $CA$  to  $AD$ ,

I say that the triangle  $ABC$  is equal to the triangle  $ADE$

For, if  $BD$  be again joined,

$CA$  is to  $AD$  as  $EA$  to  $AB$

...

...

to the triangle  $BAD$

[v 11]

Therefore each of the triangles  $ABC$ ,  $EAD$  has the same ratio to  $BAD$

Therefore the triangle  $ABC$  is equal to the triangle  $EAD$  [v 9]

Therefore etc

Q E D

### PROPOSITION 16

If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means, and, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportional

Let the four straight lines  $AB$ ,  $CD$ ,  $E$ ,  $F$  be proportional, so that, as  $AB$  is to  $CD$ , so is  $E$  to  $F$ ,

I say that the rectangle contained by  $AB$ ,  $F$  is equal to the rectangle contained by  $CD$ ,  $E$



Let  $AG$ ,  $CH$  be drawn from the points  $A$ ,  $C$  at right angles to the straight lines  $AB$ ,  $CD$ , and let  $AG$  be made equal to  $F$ , and  $CH$  equal to  $E$

Let the parallelograms  $BG$ ,  $DH$  be completed

Then since, as  $AB$  is to  $CD$ , so is  $E$  to  $F$ ,

while  $E$  is equal to  $CH$ , and  $F$  to  $AG$ ,

therefore, as  $AB$  is to  $CD$ , so is  $CH$  to  $AG$

Therefore in the parallelograms  $BG$ ,  $DH$  the sides about the equal angles are reciprocally proportional

But those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal, [vi 14]

parallelogram  $DH$

equal to  $CH$ ,

therefore the rectangle contained by  $AB, F$  is equal to the rectangle contained by  $CD, E$

Next, let the rectangle contained by  $AB, F$  be equal to the rectangle contained by  $CD, E$ ,

I say that the four straight lines will be proportional, so that, as  $AB$  is to  $CD$ , so is  $E$  to  $F$

For, with the same construction,

the rectangle  $CD, E$ ,

is equal to  $F$ ,

is equal to  $E$ ,

And they are equiangular

But in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional [VI 14]

Therefore, as  $AB$  is to  $CD$ , so is  $CH$  to  $AG$

But  $CH$  is equal to  $E$ , and  $AG$  to  $F$ ,

therefore, as  $AB$  is to  $CD$ , so is  $E$  to  $F$

Therefore etc

Q E D

### PROPOSITION 17

If three straight lines be proportional, the rectangle contained by the extremes is

equal to the square on the mean

I say that the rectangle contained by  $A, C$  is equal to the square on  $B$

$A$  —————

Let  $D$  be made equal to  $B$

$B$  —————

$D$  —————

Then, since, as  $A$  is to  $B$ , so is  $B$  to  $C$ ,

$C$  —————

and  $B$  is equal to  $D$ ,

therefore, as  $A$  is to  $B$ , so is  $D$  to  $C$

But, if four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means [VI 16]

Therefore the rectangle  $A, C$  is equal to the rectangle  $B, D$

to  $B$ , so is  $B$  to  $C$

For, with the same construction,

since the rectangle  $A, C$  is equal to the square on  $B$ ,

while the square on  $B$  is the rectangle  $B, D$ , for  $B$  is equal to  $D$ ,



But  $B$  is equal to  $D$ ,

therefore, as  $A$  is to  $B$ , so is  $B$  to  $C$ .

Therefore etc

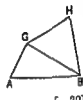
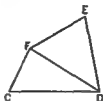
Q E D

### PROPOSITION 18

*On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure*

Let  $DF$  be joined, and on the straight line  $AB$ , and at the points  $A, B$  on it, let the angle  $GAB$  be constructed equal to the angle at  $C$ , and the angle  $ABG$  equal to the angle  $CDF$  [I 23]

Therefore the remaining angle  $CFD$



$BC$

angle  $FDE$

[I 23]

Therefore the remaining angle at  $E$  is equal to the remaining angle at  $H$ ;

[I 32]

therefore the triangle  $FDE$  is equiangular with the triangle  $GBH$ , therefore, proportionally, as  $FD$  is to  $GB$ , so is  $FE$  to  $GH$ , and  $ED$  to  $HB$  [VI 4]

But it was also proved that as  $FD$  is to  $GB$ , so is  $FC$  to  $GA$ , and  $CD$  to  $AB$ , therefore also, as  $FC$  is to  $AG$ , so is  $CD$  to  $AB$ , and  $FE$  to  $GH$ , and further  $ED$  to  $HB$

And, since the angle  $CFD$  is equal to the angle  $AGB$ ,  
and the angle  $DFE$  to the angle  $BGH$ ,

therefore the whole angle  $CPE$  is equal to the whole angle  $AGH$

For the same reason

[VI Def 1]

Therefore on the given straight line  $AB$  the rectilineal figure  $AH$  has been described similar and similarly situated to the given rectilineal figure  $CE$

Q E F

### PROPOSITION 19

*Similar triangles are to one another in the duplicate ratio of the corresponding sides*

Let  $ABC, DEF$  be similar triangles having the angle at  $B$  equal to the angle

at  $E$  and such that, as  $AB$  is to  $BC$ , so is  $DE$  to  $EF$ , so that  $BC$  corresponds to  $EF$ , [v Def 11]

I say that the triangle  $ABC$  has to the triangle  $DEF$  a ratio duplicate of that which  $BC$  has to  $EF$



For let a third proportional  $BG$  be taken to  $BC$ ,  $EF$ , so that as  $BC$  is to  $EF$ , so is  $EF$  to  $BG$ , [vi 11]

and let  $AG$  be joined

Since then, as  $AB$  is to  $BC$ , so is  $DE$  to  $EF$ ,

therefore, alternately, as  $AB$  is to  $DE$ , so is  $BC$  to  $EF$  [v 16]

But as  $BC$  is to  $EF$ , so is  $EF$  to  $BG$ ,

therefore also, as  $AB$  is to  $DE$ , so is  $EF$  to  $BG$  [v 11]

Therefore in the triangles  $ABG$ ,  $DEF$  the sides about the equal angles are reciprocally proportional

But those triangles which have one angle equal to one angle and in which the sides about the equal angles are reciprocally proportional are equal, [vi 15]

therefore the triangle  $ABG$  is equal to the triangle  $DEF$

Now since as  $BC$  is to  $EF$ , so is  $EF$  to  $BG$ ,

and if three straight lines be proportional, the first has to the third a ratio duplicate of that which it has to the second, [v Def 9]

therefore  $BC$  has to  $BG$  a ratio duplicate of that which  $CB$  has to  $EF$

But as  $CB$  is to  $BG$  so is the triangle  $ABC$  to the triangle  $ABG$ , [vi 1]

therefore the triangle  $ABC$  also has to the triangle  $ABG$  a ratio duplicate of that which  $BC$  has to  $EF$

But as  $BC$  has to  $EF$ , so is the triangle  $ABC$  to the triangle  $DEF$ ,

so the triangle  $ABC$  has to the triangle  $DEF$  a ratio duplicate of

Therefore etc

**PORISM** From this it is manifest that if three straight lines be proportional, then as the first is to the third so is the figure described on the first to that which is similar and similarly described on the second Q E D

### PROPOSITION 20

Similar polygons are divided into similar triangles and into triangles equal in

Let  $BE$ ,  $EC$ ,  $GL$ ,  $LH$  be joined

No-

$\triangle GHKL$ ,

[vi Def 1]

Since then  $ABE$   $FGL$  are two triangles having one angle equal to one angle

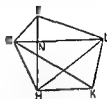
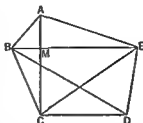
and the sides about the equal angles proportional,

therefore the triangle  $ABE$  is equiangular with the triangle  $FGL$ , [vi 6]

so that it is also similar; [vi 4 and Def 1]

therefore the angle  $ABE$  is equal to the angle  $FGL$

But the whole angle  $ABC$  is also equal to the whole angle  $FGH$  because of the similarity of the polygons, therefore the remaining angle  $EBC$  is equal to the angle  $LGH$



And, since, because of the similarity of the triangles  $ABE$ ,  $FGL$ ,

as  $EB$  is to  $BA$ , so is  $LG$  to  $GF$ ,

and moreover also, because of the similarity of the polygons,

as  $AB$  is to  $BC$ , so is  $FG$  to  $GH$ ,

therefore, *ex aequali*, as  $EB$  is to  $BC$ , so is  $LG$  to  $GH$ , [v 22]

that is, the sides about the equal angles  $EBC$ ,  $LGH$  are proportional,

therefore the triangle  $EBC$  is equiangular with the triangle  $LGH$ , [vi 6]

so that the triangle  $EBC$  is also similar to the triangle  $LGH$  [vi 4 and Def 1]

For the same reason

the triangle  $ECD$  is also similar to the triangle  $LHK$

Therefore the similar polygons  $ABCDE$ ,  $FGHKL$  have been divided into similar triangles, and into triangles equal in multitude

I say that they are also in the same ratio as the wholes, that is, in such manner that the triangles are proportional, and  $ABE$ ,  $EBC$ ,  $ECD$  are antecedents, while  $FGL$ ,  $LGH$ ,  $LHK$  are their consequents, and that the polygon  $ABCDE$  has to the polygon  $FGHAL$  a ratio duplicate of that which the corresponding side has to the corresponding side, that is  $AB$  to  $FG$

For let  $AC$ ,  $FH$  be joined

Then since, because of the similarity of the polygons

the angle  $ABC$  is equal to the angle  $FGH$ ,

and as  $AB$  is to  $BC$ , so is  $FG$  to  $GH$ ,

therefore, *ex aequali*, as  $AB$  is to  $BC$ , so is  $FG$  to  $GH$ , [vi 6]

And, since the angle  $BAM$  is equal to the angle  $GFN$ ,

and the angle  $ABM$  is also equal to the angle  $GFN$ ,

therefore the remaining angle  $AMB$  is also equal to the remaining angle  $FNG$ , [i 32]

therefore the triangle  $ABM$  is equiangular with the triangle  $GFN$

Similarly we can prove that

the triangle  $MBC$  is similar to the triangle  $GNH$

Therefore, the triangle  $ABM$  is similar to the triangle  $GNH$

and the triangle  $MBC$  is similar to the triangle  $GNH$

so that in addition, *ex aequali*

as  $AM$  is to  $MC$ , so is  $FN$  to  $NH$

But, as  $AM$  is to  $MC$ , so is the triangle  $ABM$  to  $MBC$ , and  $AME$  to  $EMC$ , for they are to one another as their bases [vi 1]

Therefore also, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [v 12]

therefore, as the triangle  $AMB$  is to  $BMC$ , so is  $ABE$  to  $CBE$

But, as  $AMB$  is to  $BMC$ , so is  $AM$  to  $MC$ ,

therefore also, as  $AM$  is to  $MC$ , so is the triangle  $ABE$  to the triangle  $EBC$ .

For the same reason also,

as  $FN$  is to  $NH$ , so is the triangle  $FGL$  to the triangle  $GLH$

And, as  $AM$  is to  $MC$ , so is  $FN$  to  $NH$ ,

therefore also, as the triangle  $ABE$  is to the triangle  $BEC$ , so is the triangle  $FGL$  to the triangle  $GLH$ ,

and, alternately, as the triangle  $ABE$  is to the triangle  $FGL$ , so is the triangle  $BEC$  to the triangle  $GLH$

therefore also as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [v 12]

Therefore the polygon  $ABCDE$  also has to the polygon  $FGHAL$  a ratio duplicate of that which the corresponding side  $AB$  has to the corresponding side  $FG$

Therefore etc

PROPOSITION Similarly also it can be proved in the case of quadrilaterals that they are in the duplicate ratio of the corresponding sides And it was also proved in the case of triangles, therefore also, generally, similar rectilinear figures are to one another in the duplicate ratio of the corresponding sides

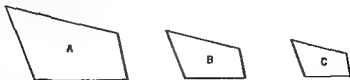
Q E D

### PROPOSITION 21

Figures which are similar to the same rectilinear figure are also similar to one another

For let each of the rectilinear figures  $A$ ,  $B$  be similar to  $C$ , I say that  $A$  is also similar to  $B$

For, since  $A$  is similar to  $C$ ,



it is equiangular with it and has the sides about the equal angles proportional [vi Def 1]

Again, since  $B$  is similar to  $C$ ,

it is equiangular with it and has the sides about the equal angles proportional  
 Therefore each of the figures  $A$ ,  $B$  is equiangular with  $C$  and with  $C$  has the  
 sides about the equal angles proportional,  
 therefore  $A$  is similar to  $B$  Q E D

## PROPOSITION 22

Let the four straight lines  $AB$ ,  $CD$ ,  $EF$ ,  $GH$  be proportional,

so that, as  $AB$  is to  $CD$ , so is  $EF$  to  $GH$ ,  
 and let there be described on  $AB$ ,  $CD$  the similar and similarly situated rectilineal figures  $KAB$ ,  $LCD$ ,  
 and on  $EF$ ,  $GH$  the similar and similarly situated rectilineal figures  $MF$ ,  $NH$ ,

I say that, as  $KAB$  is to  $LCD$ , so is  $MF$  to  $NH$

For let there be taken a third proportional  $O$  to  $AB$ ,  $CD$ , and a third proportional  $P$  to  $EF$ ,  $GH$  [vi 11]

Then since, as  $AB$  is to  $CD$ , so is  $EF$  to  $GH$ ,

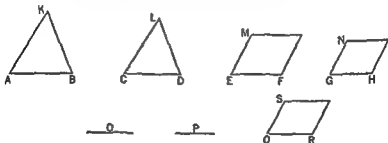
and, as  $CD$  is to  $O$ , so is  $GH$  to  $P$ ,

therefore, *ex aequali*, as  $AB$  is to  $O$ , so is  $EF$  to  $P$  [v 22]

But, as  $AB$  is to  $O$ , so is  $KAB$  to  $LCD$ , [vi 19, Por]

and as  $EF$  is to  $P$ , so is  $MF$  to  $NH$ ,

therefore also, as  $KAB$  is to  $LCD$ , so is  $MF$  to  $NH$  [v 11]



Next let  $MF$  be to  $NH$  as  $KAB$  is to  $LCD$ ,

I say also that, as  $AB$  is to  $CD$ , so is  $EF$  to  $GH$

For if  $EF$  is not to  $GH$  as  $AB$  to  $CD$ ,

let  $EF$  be to  $QR$  as  $AB$  to  $CD$ , [vi 12]

and on  $QR$  let the rectilineal figure  $SR$  be described similar and similarly situated to either of the two  $MF$ ,  $NH$  [vi 18]

Since then as  $AB$  is to  $CD$ , so is  $EF$  to  $QR$ ,

and there have been described on  $AB$ ,  $CD$  the similar and similarly situated figures  $KAB$ ,  $LCD$ ,

and on  $EF$ ,  $QR$  the similar and similarly situated figures  $MF$ ,  $SR$ ,

therefore, as  $KAB$  is to  $LCD$  so is  $MF$  to  $SR$

But also, by hypothesis,

as  $KAB$  is to  $LCD$ , so is  $MF$  to  $NH$ ,

therefore also as  $MF$  is to  $SR$ , so is  $MF$  to  $NH$  [v 11]

Therefore  $MF$  has the same ratio to each of the figures  $NH$ ,  $SR$ ,  
therefore  $NH$  is equal to  $SR$

[v 9]

But it is also similar and similarly situated to it,  
therefore  $GH$  is equal to  $QR$

And, since, as  $AB$  is to  $CD$ , so is  $EF$  to  $QR$ ,  
while  $QR$  is equal to  $GH$ ,  
therefore, as  $AB$  is to  $CD$ , so is  $EF$  to  $GH$ .

Therefore etc

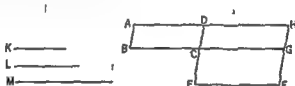
Q E D

## PROPOSITION 23

*Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides*

Let  $AC$ ,  $CF$  be equiangular parallelograms having the angle  $BCD$  equal to the angle  $ECG$ ,

I say that the parallelogram  $AC$  has to the parallelogram  $CF$  the ratio compounded of the ratios of the sides



For let them be placed so that  $BC$  is in a straight line with  $CG$ ,  
therefore  $DC$  is also in a straight line with  $CE$

Let the parallelogram  $DG$  be completed,

let a straight line  $K$  be set out, and let it be contrived that,

as  $BC$  is to  $CG$ , so is  $K$  to  $L$ ,

and as  $DC$  is to  $CE$ , so is  $L$  to  $M$  [vi 12]

Then the ratios of  $K$  to  $L$  and of  $L$  to  $M$  are the same as the ratios of the sides, namely of  $BC$  to  $CG$  and of  $DC$  to  $CE$

But the ratio of  $K$  to  $M$  is compounded of the ratio of  $K$  to  $L$  and of that of  $L$  to  $M$ ,

so that  $K$  has also to  $M$  the ratio compounded of the ratios of the sides

Now since, as  $BC$  is to  $CG$ , so is the parallelogram  $AC$  to the parallelogram  $CH$ , [vi 1]

while, as  $BC$  is to  $CG$ , so is  $K$  to  $L$ ,

therefore also, as  $K$  is to  $L$ , so is  $AC$  to  $CH$  [v 11]

Again, since, as  $DC$  is to  $CE$ , so is the parallelogram  $CH$  to  $CF$  [vi 1]

while, as  $DC$  is to  $CE$ , so is  $L$  to  $M$ ,

therefore also, as  $L$  is to  $M$ , so is the parallelogram  $CH$  to the parallelogram  $CF$  [v 11]

Since, then, it was proved that, as  $K$  is to  $L$ , so is the parallelogram  $AC$  to the parallelogram  $CH$ ,

and as  $L$  is to  $M$ , so is the parallelogram  $CH$  to the parallelogram  $CF$ ,

therefore, *ex aequali* as  $K$  is to  $M$ , so is  $AC$  to the parallelogram  $CF$

But  $K$  has to  $M$  the ratio compounded of the ratios of the sides,  
therefore  $AC$  also has to  $CF$  the ratio compounded of the ratios of the sides

Therefore etc

Q E D

## PROPOSITION 24

*In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another*

Let  $ABCD$  be a parallelogram, and  $AC$  its diameter, and let  $EG$ ,  $HK$  be parallelograms about  $AC$ ,

I say that each of the parallelograms  $EG$   $HK$  is similar both to the whole  $ABCD$  and to the other

For, since  $EF$  has been drawn parallel to  $BC$ , one of the sides of the triangle  $ABC$ ,

proportionally, as  $BE$  is to  $EA$ , so is  $CF$  to  $FA$

[vi 2]

Again since  $FG$  has been drawn parallel to  $CD$ , one of the sides of the triangle  $ACD$ ,

proportionally, as  $CF$  is to  $FA$ , so is  $DG$  to  $GA$

[vi 2]

But it was proved that

as  $CF$  is to  $FA$ , so also is  $BE$  to  $EA$ ,

therefore also as  $BE$  is to  $EA$  so is  $DG$  to  $GA$ ,

and therefore *componendo*

as  $BA$  is to  $AE$ , so is  $DA$  to  $AG$ ,

[v 18]

and alternately,

as  $BA$  is to  $AD$ , so is  $EA$  to  $AG$

[v 16]

Therefore in the parallelograms  $ABCD$ ,  $EG$ , the sides about the common angle  $BAD$  are proportional

And since  $GF$  is parallel to  $DC$ ,

the angle  $AFG$  is equal to the angle  $DCA$ ,

and the angle  $DAC$  is common to the two triangles  $ADC$ ,  $AGF$ ,

therefore the triangle  $ADC$  is equiangular with the triangle  $AGF$

For the same reason

as  $AD$  is to  $DC$ , so is  $AG$  to  $GF$ ,

as  $DC$  is to  $CA$ , so is  $GF$  to  $FA$ ,

as  $AC$  is to  $CB$ , so is  $AF$  to  $FE$

and further as  $CB$  is to  $BA$ , so is  $FE$  to  $EA$

And since it was proved that

as  $DC$  is to  $CA$  so is  $GF$  to  $FA$ ,

and, as  $AC$  is to  $CB$ , so is  $AF$  to  $FE$ ,

therefore, *ex aequali* as  $DC$  is to  $CB$  so is  $GF$  to  $FE$

[v 22]

Therefore in the parallelograms  $ABCD$   $EG$  the sides about the equal angles are proportional,

therefore the parallelogram  $ABCD$  is similar to the parallelogram  $EG$

[vi Def 1]

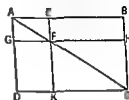
For the same reason

the parallelogram  $ABCD$  is also similar to the parallelogram  $HK$ ,

therefore each of the parallelograms  $EG$   $HK$  is similar to  $ABCD$

But figures similar to the same rectilineal figure are also similar to one another,

[vi 21]

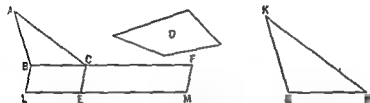


therefore the parallelogram  $EG$  is also similar to the parallelogram  $HK$   
Therefore etc Q E D

## PROPOSITION 25

*To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure*

Let  $ABC$  be the given rectilineal figure to which the figure to be constructed must be similar, and  $D$  that to which it must be equal,  
thus it is required to construct one and the same figure similar to  $ABC$  and equal to  $D$



Let there be applied to  $BC$  the parallelogram  $BE$  equal to the triangle  $ABC$   
[I 44] and to  $CE$  the parallelogram  $CM$  equal to  $D$  in the angle  $FCE$  which is  
[I 45]

1th  $EM$

VI 13] and on  $GH$  let  
[VI 18]

and, if three straight lines be proportional, as the first is to the third, so is the figure on the first to the similar and similarly situated figure described on the second,  
[VI 19, Por.]

therefore, as  $BC$  is to  $CF$ , so is the triangle  $ABC$  to the triangle  $KGH$

But, as  $BC$  is to  $CF$ , so also is the parallelogram  $BE$  to the parallelogram  $EF$   
[VI 1]

Therefore also as the triangle  $ABC$  is to the triangle  $KGH$ , so is the parallelogram  $BE$  to the parallelogram  $EF$ ,

therefore, alternately, as the triangle  $ABC$  is to the parallelogram  $BE$ , so is the triangle  $KGH$  to the parallelogram  $EF$   
[V 16]

But the triangle  $ABC$  is equal to the parallelogram  $BE$ ,

therefore the triangle  $KGH$  is also equal to the parallelogram  $EF$

But the parallelogram  $EF$  is equal to  $D$ ,

therefore  $KGH$  is also equal to  $D$

And  $KGH$  is also similar to  $ABC$

Therefore one and the same figure  $KGH$  has been constructed similar to the given rectilineal figure  $ABC$  and equal to the other given figure  $D$  Q E D

## PROPOSITION 26

*If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, it is about the same diameter with the whole*

For from the parallelogram  $ABCD$  let there be taken away the parallelo-



gram  $AF$  similar and similarly situated to  $ABCD$ , and having the angle  $DAB$  common with it,

I say that  $ABCD$  is about the same diameter with  $AF$

For suppose it is not, but, if possible, let  $AHC$  be the diameter < of  $ABCD$  >, let  $GF$  be produced and carried through to  $H$ , and let  $HK$  be drawn through  $H$  parallel to either of the straight lines  $AD$ ,  $BC$

[I 31]

Since, then,  $ABCD$  is about the same diameter with  $KG$ , therefore, as  $DA$  is to  $AB$ , so is  $GA$  to  $AK$

[VI 24]

But also, because of the similarity of  $ABCD$ ,  $EG$ ,

as  $DA$  is to  $AB$ , so is  $GA$  to  $AE$ ,

therefore also, as  $GA$  is to  $AK$ , so is  $GA$  to  $AE$

[V 11]

Therefore  $GA$  has the same ratio to each of the straight lines  $AK$ ,  $AE$

Therefore  $AE$  is equal to  $AK$  [V 9] the less to the greater which is impossible

Therefore  $ABCD$  cannot but be about the same diameter with  $AF$ , therefore the parallelogram  $ABCD$  is about the same diameter with the parallelogram  $AF$

Therefore etc

Q E D

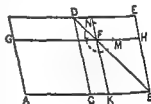
### PROPOSITION 27

*Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect*

Let  $AB$  be a straight line and let it be bisected at  $C$ , let there be applied to the straight line  $AB$  the parallelogram  $AD$  deficient by the parallelogrammic figure  $DB$  described on the half of  $AB$ , that is,  $CB$ ,

I say that of the parallelograms applied to  $AB$  and deficient by parallelogrammic figures similar and similarly situated to  $DB$ ,  $AD$  is greatest

For let there be applied to the straight line  $AB$  the parallelogram  $AF$  deficient by the parallelogrammic figure  $FB$  similar and similarly situated to  $DB$ ,



I say that  $AD$  is greater than  $AF$

For, since the parallelogram  $DB$  is similar to the parallelogram  $FB$ , they are about the same diameter

[VI 26]

Let their diameter  $DB$  be drawn and let the figure be described

Then, since  $CF$  is equal to  $FE$ ,

[I 43]

and  $FB$  is common

therefore the whole  $CH$  is equal to the whole  $KE$

But  $CH$  is equal to  $CG$ , since  $AC$  is also equal to  $CB$

[I 36]

Therefore  $GC$  is also equal to  $EK$

Let  $CF$  be added to each

therefore the whole  $AF$  is equal to the gnomon  $LMN$ ,

so that the parallelogram  $DB$ , that is,  $AD$ , is greater than the parallelogram  $AF$ .

Therefore etc

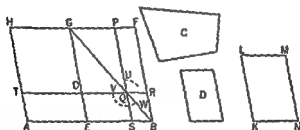
Q E D

### PROPOSITION 28

To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect

thus it is required to apply to the given straight line  $AB$  a parallelogram equal to the given rectilinear figure  $C$  and deficient by a parallelogrammic figure which is similar to  $D$ .

Let  $AB$  be bisected at the point  $E$ , and on  $EB$  let  $EBFG$  be described similar and similarly situated to  $D$ . [VI 18]



But, if not, let  $HE$  be greater than  $C$

Now  $HE$  is equal to  $GB$ .

therefore  $GB$  is also greater than  $C$

Let  $KLMN$  be constructed at once equal to the excess by which  $GB$  is greater than  $C$  and similar and similarly situated to  $D$  [VI 25]

But  $D$  is similar to  $GB$ .

therefore  $KM$  is also similar to  $GB$

Let, then,  $KL$  correspond to  $GE$ , and  $LM$  to  $GF$

Now, since  $GB$  is equal to  $C$ ,  $KM$ ,

therefore  $GB$  is greater than  $KM$ .

therefore also  $GE$  is greater than  $KL$ , and  $GF$  than  $LM$

Let  $GO$  be made equal to  $KL$ , and  $GP$  equal to  $LM$ , and let the parallelogram  $OGPQ$  be completed.

therefore it is equal and similar to  $KM$

Therefore  $GQ$  is also similar to  $GB$ .

[٧١ 21]

therefore  $GQ$  is about the same diameter with  $GB$

[VI 26]

Let  $GQB$  be their diameter, and let the figure be described

Then, since  $BG$  is equal to  $C$ ,  $KM$ ,

and in them  $GQ$  is equal to  $KM$ ,

therefore the remainder, the gnomon  $UWV$ , is equal to the remainder  $C$

And, since  $PR$  is equal to  $OS$ ,

let  $QB$  be added to each,

therefore the whole  $PB$  is equal to the whole  $OB$

But  $OB$  is equal to  $TE$ , since the side  $AE$  is also equal to the side  $EB$ , [I 36]

therefore  $TE$  is also equal to  $PB$

Let  $OS$  be added to each,

therefore the whole  $TS$  is equal to the whole, the gnomon  $VWU$

But the gnomon  $VWU$  was proved equal to  $C$ ,

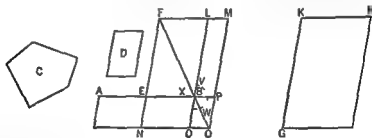
therefore  $TS$  is also equal to  $C$

allelo-  
allelo-

### PROPOSITION 29

To a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one

thus it is required to apply to the straight line  $AB$  a parallelogram equal to the rectilineal figure  $C$  and exceeding by a parallelogrammic figure similar to  $D$



Let  $AB$  be bisected at  $E$ ,  
let there be described on  $EB$  the parallelogram  $BF$  similar and similarly situated to  $D$ ,

and let  $GH$  be constructed at once equal to the sum of  $BF$ ,  $C$  and similar and similarly situated to  $D$  [VI 25]

Let  $AH$  correspond to  $FL$  and  $AG$  to  $FE$

Now, since  $GH$  is greater than  $FB$ ,

therefore  $AH$  is also greater than  $FL$ , and  $KG$  than  $FE$

Let  $FL$ ,  $FE$  be produced

let  $FLM$  be equal to  $KH$  and  $FEN$  to  $KG$ ,

and let  $MN$  be completed,

therefore  $MN$  is both equal and similar to  $GH$

But  $GH$  is similar to  $EL$ ,

therefore  $MN$  is also similar to  $EL$ , [VI 21]

therefore  $EL$  is about the same diameter with  $MN$  [VI 26]

Let their diameter  $FO$  be drawn and let the figure be described

Since  $GH$  is equal to  $EL$ ,  $C$ ,

while  $GH$  is equal to  $MN$ ,

therefore  $MN$  is also equal to  $EL$ ,  $C$

Let  $EL$  be subtracted from each,

therefore the remainder, the gnomon  $\Lambda H V$ , is equal to  $C$

Now, since  $AE$  is equal to  $EB$ ,

$AN$  is also equal to  $NB$  [I 36], that is to  $LP$  [I 43]

Let  $EO$  be added to each,

therefore the whole  $AO$  is equal to the gnomon  $VHX$

But the gnomon  $VHX$  is equal to  $C$ ,

therefore  $AO$  is also equal to  $C$

Q E F

### PROPOSITION 30

To cut a given finite straight line in extreme and mean ratio

Let  $AB$  be the given finite straight line,

thus it is required to cut  $AB$  in extreme and mean ratio



On  $AB$  let the square  $BC$  be described, and let there be applied to  $AC$  the parallelogram  $CD$  equal to  $BC$  and exceeding by the figure  $AD$  similar to  $BC$  [VI 29]

Now  $BC$  is a square,

therefore  $AD$  is also a square

And since  $BC$  is equal to  $CD$ ,

let  $CE$  be subtracted from each,

therefore the remainder  $BF$  is equal to the remainder  $AD$

But it is also equiangular with it,

therefore in  $BF$ ,  $AD$  the sides about the equal angles are reciprocally proportional, [VI 14]

therefore, as  $FE$  is to  $ED$ , so is  $AE$  to  $EB$

But  $FE$  is equal to  $AB$ , and  $ED$  to  $AE$

Therefore, as  $BA$  is to  $AE$ , so is  $AE$  to  $EB$

And  $AB$  is greater than  $AE$

therefore  $AE$  is also greater than  $EB$

Therefore the straight line  $AB$  has been cut in extreme and mean ratio at  $E$ , and the greater segment of it is  $AE$  Q E F

### PROPOSITION 31

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle

Let  $ABC$  be a right angled triangle having the angle  $BAC$  right,

I say that the figure on  $BC$  is equal to the similar and similarly described figures on  $BA$ ,  $AC$

Let  $AD$  be drawn perpendicular

Then since, in the right-angled triangle  $ABC$ ,  $AD$  has been drawn from the right angle at  $A$  perpendicular to the base  $BC$ , the triangles  $ABD$ ,  $ADC$  adjoining the perpendicular are similar both to the whole  $ABC$  and to one another [vi 8]

And, since  $ABC$  is similar to  $ABD$ , therefore, as  $CB$  is to  $BA$ , so is  $AB$  to  $BD$  [vi Def 1]

And, since three straight lines are proportional, as the first is to the third, so is the figure on the first to the similar and similarly described figure on the second [vi 19, Por]

Therefore, as  $CB$  is to  $BD$ , so is the figure on  $CB$  to the similar and similarly described figure on  $BA$

For the same reason also, as  $BC$  is to  $CD$ , so is the figure on  $BC$  to that on  $CA$ ; so that, in addition, as  $BC$  is to  $BD$ ,  $DC$ , so is the figure on  $BC$  to the similar and similarly described figures on  $BA$ ,  $AC$

But  $BC$  is equal to  $BD$ ,  $DC$ , therefore the figure on  $BC$  is also equal to the similar and similarly described figures on  $BA$ ,  $AC$

Therefore etc

Q E D

### PROPOSITION 32

If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangles will be in a straight line

Let  $ABC$ ,  $DCE$  be two triangles having the two sides  $BA$ ,  $AC$  proportional to the two sides  $DC$ ,  $DE$ , so that, as  $AB$  is to  $AC$ , so is  $DC$  to  $DE$ , and  $AB$  parallel to  $DC$ , and  $AC$  to  $DE$ ,

I say that  $BC$  is in a straight line with  $CE$

For, since  $AB$  is parallel to  $DC$ , and the straight line  $AC$  has fallen upon them the alternate angles  $BAC$ ,  $ACD$  are equal to one another [i 29]

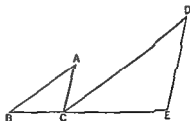
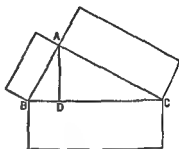
For the same reason the angle  $CDE$  is also equal to the angle  $ACD$ , so that the angle  $BAC$  is equal to the angle  $CDE$

And, since  $ABC$ ,  $DCE$  are two triangles having one angle, the angle at  $A$ , equal to one angle the angle at  $D$ ,

and the sides about the equal angles proportional,

so that, as  $BA$  is to  $AC$ , so is  $CD$  to  $DE$ ,

therefore the triangle  $ABC$  is equiangular with the triangle  $DCE$ , [vi 6]



therefore the angle  $ABC$  is equal to the angle  $DCE$

But the angle  $ACD$  was also proved equal to the angle  $BAC$ ,

therefore the whole angle  $ACE$  is equal to the two angles  $ABC, BAC$

Let the angle  $ACB$  be added to each,

therefore the angles  $ACE, ACB$  are equal to the angles  $BAC, ACB, CBA$

But the angles  $BAC, ABC, ACB$  are equal to two right angles, [1 32]

therefore the angles  $ACE, ACB$  are also equal to two right angles

Therefore with a

lines  $B$

equal t

therefore  $BC$  is in a straight line with  $CE$

[1 14]

Therefore etc

Q E D

### PROPOSITION 33

In equal circles angles have the same ratio as the circumferences on which they stand, whether they be angles at the centres or at the circumferences,

Let  $ABC, L$  be angles at their centres

I say that, as the circumference  $BC$  is to the circumference  $EF$ , so is the angle  $BGC$  to the angle  $EHF$ , and the angle  $BAC$  to the angle  $EDF$



For let any number of consecutive circumferences  $CK, KL$  be made equal to the circumference  $BC$ , and any number of consecutive circumferences  $FM, MN$  equal to the circumference  $EF$ , and let  $GK, GL, HM, HN$  be joined

Then since the circumferences  $BC, CK, KL$  are equal to one another

the angles  $BGC, CG$

therefore whatever multiple

is the angle  $BGL$  of the angle

For the

is of  $EF$  that multiple also is the

It then the

$BGL$

if the

is also

and if less less

There being then four magnitudes two circumferences  $BC, EF$ , and two angles  $BGC, FHF$ ,

there have been taken of the circumference  $BC$  and the angle  $BGC$  equimultiples namely the circumference  $BL$  and the angle  $BGL$

and of the circumference  $EF$  and the angle  $EHF$  equimultiples, namely the circumference  $EN$  and the angle  $EHN$

And it has been proved that,

if the circumference  $BL$  is in excess of the circumference  $EN$ ,

the angle  $BGL$  is also in excess of the angle  $EHN$ ,  
 if equal, equal,  
 and if less, less

Therefore, as the circumference  $BC$  is to  $EF$ , so is the angle  $BGC$  to the angle  $EHF$  [v Def 5]

But, as the angle  $BGC$  is to the angle  $EHF$ , so is the angle  $BAC$  to the angle  $EDF$ , for they are doubles respectively

Therefore also, as the circumference  $BC$  is to the circumference  $EF$ , so is the angle  $BGC$  to the angle  $EHF$ , and the angle  $BAC$  to the angle  $EDF$

Therefore etc

Q E D

## BOOK SEVEN

### DEFINITIONS

- 1 An *unit* is that by virtue of which each of the things that exist is called one
- 2 A *number* is a multitude composed of units
- 3 A number is a *part* of a number, the less of the greater, when it measures the greater,
  - 4 but parts when it does not measure it
- 5 The greater number is a *multiple* of the less when it is measured by the less
- 6 An *even number* is that which is divisible into two equal parts
- 7 An *odd number* is that which is not divisible into two equal parts or that which differs by an unit from an even number
- 8 An *even times even number* is that which is measured by an even number according to an even number
- 9 An *even-times odd number* is that which is measured by an even number according to an odd number
- 10 An *odd times odd number* is that which is measured by an odd number according to an odd number
- 11 A *prime number* is that which is measured by an unit alone
- 12 Numbers *prime to one another* are those which are measured by an unit alone as a common measure
- 13 A *composite number* is that which is measured by some number
- 14 Numbers *composite to one another* are those which are measured by some number as a common measure
- 15 A number is said to *multiply* a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced
- 16 And, when two numbers having multiplied one another make some number the number so produced is called *plane*, and its *sides* are the numbers which have multiplied one another
- 17 And, when three numbers having multiplied one another make some number, the number so produced is *solid*, and its *sides* are the numbers which have multiplied one another
- 18 A *square number* is equal multiplied by equal, or a number which is contained by two equal numbers
- 19 And a *cube* is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers
- 20 Numbers are *proportional* when the first is the same multiple, or the same part, or the same parts of the second that the third is of the fourth



21 *Similar plane and solid numbers* are those which have their sides proportional

22 A *perfect number* is that which is equal to its own parts

## BOOK VII PROPOSITIONS

### PROPOSITION 1

*Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left the original numbers will be prime to one another*

For, the less of two unequal numbers  $AB$ ,  $CD$  being continually subtracted from the greater let the number which is left never measure the one before it until an unit is left,

I say that  $AB$ ,  $CD$  are prime to one another, that is, that an unit alone measures  $AB$ ,  $CD$

For, if  $AB$ ,  $CD$  are not prime to one another, some number will measure them

Let a number measure them, and let it be  $E$ , let  $CD$ , measuring  $BF$ , leave  $FA$  less than itself

let  $AF$ , measuring  $DG$ , leave  $GC$  less than itself,

and let  $GC$ , measuring  $FH$ , leave an unit  $HA$

Since then  $E$  measures  $CD$ , and  $CD$  measures  $BF$ ,

therefore  $E$  also measures  $BF$

But it also measures the whole  $BA$ ,

therefore it will also measure the remainder  $AF$

But  $AF$  measures  $DG$ ,

therefore  $E$  also measures  $DG$

But it also measures the whole  $DC$

therefore it will also measure the remainder  $CG$

But  $CG$  measures  $FH$ ,

therefore  $E$  also measures  $FH$

But it also measures the whole  $FA$ ,

therefore it will also measure the remainder the unit  $HA$ , though it is a number which is impossible

Therefore no number will measure the numbers  $AB$ ,  $CD$ , therefore  $AB$ ,  $CD$  are prime to one another

[VII Def 12]  
Q E D



### PROPOSITION 2

*Given two numbers not prime to one another to find their greatest common measure*

Let  $AB$ ,  $CD$  be the two given numbers not prime to one another

Thus it is required to find the greatest common measure of  $AB$ ,  $CD$

If now  $CD$  measures  $AB$ —and it also measures itself— $CD$  is a common measure of  $CD$ ,  $AB$

And it is manifest that it is also the greatest, for no greater number than  $CD$  will measure  $CD$

But, if  $CD$  does not measure  $AB$  then the less of the numbers  $AB$ ,  $CD$  being continually subtracted from the greater, some number will be left which will measure the one before it

For an unit will not be left, otherwise  $AB$ ,  $CD$  will be prime to one another [VII 1], which is contrary to the hypothesis

Therefore some number will be left which will measure the one before it

Now let  $CD$ , measuring  $BE$ , leave  $EA$  less than itself,

let  $EA$ , measuring  $DF$ , leave  $FC$  less than itself,

and let  $CF$  measure  $AE$

Since then,  $CF$  measures  $AE$ , and  $AE$  measures  $DF$ ,

therefore  $CF$  will also measure  $DF$

But it also measures itself,

therefore it will also measure the whole  $CD$

But  $CD$  measures  $BE$ ,

therefore  $CF$  also measures  $BE$

But it also measures  $EA$ ,

therefore it will also measure the whole  $BA$ .

But it also measures  $CD$ ,

therefore  $CF$  measures  $AB$ ,  $CD$

Therefore  $CF$  is a common measure of  $AB$ ,  $CD$

I say next that it is also the greatest

For if  $CF$  is not the greatest common measure of  $AB$ ,  $CD$ , some number which is greater than  $CF$  will measure the numbers  $AB$ ,  $CD$

Let such a number measure them and let it be  $G$

Now, since  $G$  measures  $CD$ , while  $CD$  measures  $BE$ ,  $G$  also measures  $BE$

But it also measures the whole  $BA$ ;

therefore it will also measure the remainder  $AE$

But  $AE$  measures  $DF$ ,

therefore  $G$  will also measure  $DF$

But it also measures the whole  $DC$ ,

therefore it will also measure the remainder  $CF$ , that is, the greater will measure the less which is impossible

Therefore no number which is greater than  $CF$  will measure the numbers  $AB$ ,  $CD$ ,

therefore  $CF$  is the greatest common measure of  $AB$ ,  $CD$

**PORISM** From this it is manifest that, if a number measure two numbers it will also measure their greatest common measure Q E D

### PROPOSITION 3

Given three numbers not prime to one another, to find their greatest common measure

Let  $A$ ,  $B$ ,  $C$  be the three given numbers not prime to one another,

thus it is required to find the greatest common measure of  $A$ ,  $B$ ,  $C$

For let the greatest common measure  $D$ , of the two numbers  $A$ ,  $B$  be taken, [VII 2]

then  $D$  either measures or does not measure,  $C$

First let it measure it

But it measures  $A$ ,  $B$  also,

therefore  $D$  measures  $A$ ,  $B$ ,  $C$ ,

therefore  $D$  is a common measure of  $A$ ,  $B$ ,  $C$

I say that it is also the greatest

For, if  $D$  is not the greatest common measure of  $A$ ,  $B$ ,  $C$ , some number which

21 *Similar plane and solid numbers* are those which have their sides proportional

22 A *perfect number* is that which is equal to its own parts

## BOOK VII PROPOSITIONS

### PROPOSITION 1

*Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another*

For, the less of two unequal numbers  $AB$ ,  $CD$  being continually subtracted

that is, that an unit alone

measures  $AB$ ,  $CD$

For, if  $AB$ ,  $CD$  are not prime to one another, some number will measure them

Let a number measure them, and let it be  $E$ , let  $CD$ , measuring  $BF$ , leave  $FA$  less than itself,

let  $AF$ , measuring  $DG$ , leave  $GC$  less than itself,

and let  $GC$ , measuring  $FH$ , leave an unit  $HA$ .

Since, then,  $E$  measures  $CD$ , and  $CD$  measures  $BF$ ,

therefore  $E$  also measures  $BF$

But it also measures the whole  $BA$ ,

therefore it will also measure the remainder  $AF$ .

But  $AF$  measures  $DG$ ,

therefore  $E$  also measures  $DG$

But it also measures the whole  $DC$ ,

therefore it will also measure the remainder  $CG$

But  $CG$  measures  $FH$ ,

therefore  $E$  also measures  $FH$

But it also measures the whole  $FA$ ,

therefore it will also measure the remainder, the unit  $HA$ , though it is a number which is impossible

Therefore no number will measure the numbers  $AB$ ,  $CD$ , therefore  $AB$ ,  $CD$  are prime to one another

[VII Def 12]  
Q E D

### PROPOSITION 2

*Given two numbers not prime to one another to find their greatest common measure*

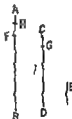
Let  $AB$ ,  $CD$  be the two given numbers not prime to one another

Thus it is required to find the greatest common measure of  $AB$ ,  $CD$

If now  $CD$  measures  $AB$ —and it also measures itself— $CD$  is a common measure of  $CD$ ,  $AB$

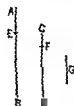
And it is manifest that it is also the greatest, for no greater number than  $CD$  will measure  $CD$

But, if  $CD$  does not measure  $AB$ , then the less of the numbers  $AB$ ,  $CD$  being continually subtracted from the greater, some number will be left which will measure the one before it



For an unit will not be left, otherwise  $AB, CD$  will be prime to one another [VII 1], which is contrary to the hypothesis

Therefore some number will be left, which will measure the  
Now : . . .



Since then,  $CF$  measures  $AE$ , and  $AE$  measures  $DF$ ,  
therefore  $CF$  will also measure  $DF$

But it also measures itself,  
therefore it will also measure the whole  $CD$

But  $CD$  measures  $BE$ ,  
therefore  $CF$  also measures  $BE$

But it also measures  $EA$ ;  
therefore it will also measure the whole  $BA$ .

But it also measures  $CD$ ,  
therefore  $CF$  measures  $AB, CD$

Therefore  $CF$  is a common measure of  $AB, CD$

'D, some number  
'

Let such a number measure them, and let it be  $G$

Now, since  $G$  measures  $CD$ , while  $CD$  measures  $BE$ ,  $G$  also measures  $BE$

But it also measures the whole  $BA$ ;  
therefore it will also measure the remainder  $AE$

But  $AE$  measures  $DF$ ,  
therefore  $G$  will also measure  $DF$

But it also measures the whole  $DC$ ,  
therefore it will also measure the remainder  $CF$ , that is, the greater will measure the less which is impossible

Therefore no number which is greater than  $CF$  will measure the numbers  $AB, CD$ ,

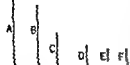
therefore  $CF$  is the greatest common measure of  $AB, CD$

PORISM From this it is manifest that, if a number measure two numbers it will also measure their greatest common measure Q.E.D.

### PROPOSITION 3

Given three numbers not prime to one another, to find their greatest common measure

Let  $A, B, C$  be the three given numbers not prime to one another,  
thus it is required to find the greatest common measure of  $A, B, C$



For let the greatest common measure  $D$  of the two numbers  $A, B$  be taken, [VII 2]

then  $D$  either measures, or does not measure,  $C$

First, let it measure it

But it measures  $A, B$  also,  
therefore  $D$  measures  $A, B, C$ ,  
therefore  $D$  is a common measure of  $A, B, C$

I say that it is also the greatest

For, if  $D$  is not the greatest common measure of  $A, B, C$ , some number which

is greater than  $D$  will measure the numbers  $A, B, C$

Let such a number measure them, and let it be  $E$

Since then  $E$  measures  $A, B, C$ ,

it will also measure  $A, B$ ,

therefore it will also measure the greatest common measure of  $A, B$

[VII 2, Por]

But the greatest common measure of  $A, B$  is  $D$ ,

therefore  $E$  measures  $D$ , the greater the less which is impossible

Therefore no number which is greater than  $D$  will measure the numbers  $A, B, C$ ,

therefore  $D$  is the greatest common measure of  $A, B, C$

Next, let  $D$  not measure  $C$ ,

I say first that  $C, D$  are not prime to one another

For, since  $A, B, C$  are not prime to one another, some number will measure them

Now that which measures  $A, B, C$  will also measure  $A, B$ , and will measure  $D$ , the greatest common measure of  $A, B$

[VII 2, Por]

But it measures  $C$  also,

therefore some number will measure the numbers  $D, C$ ,

therefore  $D, C$  are not prime to one another

Let then their greatest common measure  $E$  be taken

[VII 2]

Then, since  $E$  measures  $D$ ,

and  $D$  measures  $A, B$ ,

therefore  $E$  also measures  $A, B$

But it measures  $C$  also,

therefore  $E$  measures  $A, B, C$ ,

therefore  $E$  is a common measure of  $A, B, C$ .

I say next that it is also the greatest

For, if  $E$  is not the greatest common measure of  $A, B, C$ , some number which is greater than  $E$  will measure the numbers  $A, B, C$

Let such a number measure them and let it be  $F$

Now, since  $F$  measures  $A, B, C$ ,

it also measures  $A, B$ ,

therefore it will also measure the greatest common measure of  $A, B$

[VII 2, Por]

But the greatest common measure of  $A, B$  is  $D$ ,

therefore  $F$  measures  $D$

And it measures  $C$  also,

therefore  $F$  measures  $D, C$ ,

therefore it will also measure the greatest common measure of  $D, C$

[VII 2, Por]

But the greatest common measure of  $D, C$  is  $E$ ,

therefore  $F$  measures  $E$ , the greater the less which is impossible

Therefore no number which is greater than  $E$  will measure the numbers  $A, B, C$ ,

therefore  $E$  is the greatest common measure of  $A, B, C$

Q E D

## PROPOSITION 4

*Any number is either a part or parts of any number, the less of the greater*

Let  $A, BC$  be two numbers, and let  $BC$  be the less,

I say that  $BC$  is either a part, or parts of  $A$

For  $A, BC$  are either prime to one another or not

First, let  $A, BC$  be prime to one another

Then if  $BC$  be divided into the units in it, each unit of those in  $BC$  will be some part of  $A$ , so that  $BC \equiv$  parts of  $A$

Next let  $A, BC$  not be prime to one another, then  $BC$  either measures or does not measure,  $A$

If now  $BC$  measures  $A$ ,  $BC$  is a part of  $A$

But if not, let the greatest common measure  $D$  of  $A, BC$  be taken, [VII 2] and let  $BC$  be divided into the numbers equal to  $D$ , namely  $BL, EF, FC$

Now, since  $D$  measures  $A$ ,  $D$  is a part of  $A$

But  $D$  is equal to each of the numbers  $BE, EF, FC$ ,

therefore each of the numbers  $BE, EF, FC$  is also a part of  $A$ , so that  $BC \equiv$  parts of  $A$

Therefore etc

Q E D

## PROPOSITION 5

*If a number be a part of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the one*

For let the number  $A$  be a part of  $BC$ ,

and another,  $D$ , the same part of another  $EF$  that  $A$  is of  $BC$ ,

I say that the sum of  $A, D$  is also the same part of the sum of  $BC, EF$  that  $A$  is of  $BC$

For since, whatever part  $A$  is of  $BC$ ,  $D$  is also the same part of  $EF$ ,

therefore, as many numbers as there are in  $BC$  equal to  $A$ , so many numbers are there also in  $EF$  equal to  $D$

Let  $BC$  be divided into the numbers equal to  $A$ , namely  $BG, GC$ ,

and  $EF$  into the numbers equal to  $D$ , namely  $EH, HF$ ,

then the multitude of  $BG, GC$  will be equal to the multitude of  $EH, HF$

And, since  $BG \equiv$  equal to  $A$ , and  $EH$  to  $D$ ,

therefore  $BG, EH$  are also equal to  $A, D$

For the same reason

$GC, HF$  are also equal to  $A, D$

Therefore, as many numbers as there are in  $HC$  equal to  $A$ , so many are there also in  $BC, EF$  equal to  $A, D$

Therefore, whatever multiple  $BC$  is of  $A$ , the same multiple also is the sum of  $BC, EF$  of the sum of  $A, D$

Therefore, whatever part  $A$  is to  $BC$ , the same part also is the sum of  $A, D$  of the sum of  $BC, EF$

Q E D

## PROPOSITION 6

*If a number be parts of a number, and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one*

For let the number  $AB$  be parts of the number  $C$ , and another,  $DE$ , the same parts of another,  $F$ , that  $AB \equiv C$ ,

I say that the sum of  $AB$ ,  $DE$  is also the same parts of the sum of  $C$ ,  $F$  that  $AB$  is of  $C$ .

For since, whatever parts  $AB$  is of  $C$ ,  $DE$  is also the same parts of  $F$ ,

therefore, as many parts of  $C$  as there are in  $AB$ , so many parts of  $F$  are there also in  $DE$

Let  $AB$  be divided into the parts of  $C$ , namely  $AG$ ,  $GB$ , and  $DE$  into the parts of  $F$ , namely  $DH$ ,  $HE$ ;

thus the multitude of  $AG$ ,  $GB$  will be equal to the multitude of  $DH$ ,  $HE$

And since, whatever part  $AG$  is of  $C$ , the same part is  $DH$  of  $F$  also, therefore, whatever part  $AG$  is of  $C$ , the same part also is the sum of  $AG$ ,  $DH$  of the sum of  $C$ ,  $F$  [VII 5]

For the same reason, whatever part  $GB$  is of  $C$ , the same part also is the sum of  $GB$ ,  $HE$  of the sum of  $C$ ,  $F$

Therefore, whatever parts  $AB$  is of  $C$ , the same parts also is the sum of  $AB$ ,  $DE$  of the sum of  $C$ ,  $F$  Q E D

### PROPOSITION 7

If a number be that part of a number, which a number subtracted is of a number subtracted the remainder will also be the same part of the remainder that the whole is of the whole

For let the number  $AB$  be that part of the number  $CD$  which  $AE$  subtracted is of  $CF$  subtracted,

I say that the remainder  $EB$  is also the same part of the remainder  $FD$  that the whole  $AB$  is of the whole  $CD$



For whatever part  $AE$  is of  $CF$ , the same part also let  $EB$  be of  $CG$ .  
Now since, whatever part  $AE$  is of  $CF$ , the same part also is  $EB$  of  $CG$ , therefore whatever part  $AE$  is of  $CF$ , the same part also is  $AB$  of  $GF$  [VII 5]

But whatever part  $AE$  is of  $CF$ , the same part also, by hypothesis is  $AB$  of  $CD$ ,

therefore, whatever part  $AB$  is of  $GF$ , the same part is it of  $CD$  also, therefore  $GF$  is equal to  $CD$

Let  $CF$  be subtracted from each,

therefore the remainder  $GC$  is equal to the remainder  $FD$

Now since whatever part  $AE$  is of  $CF$ , the same part also is  $EB$  of  $GC$ , while  $GC$  is equal to  $FD$ ,

therefore, whatever part  $AE$  is of  $CF$  the same part also is  $EB$  of  $FD$

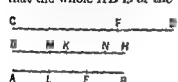
But whatever part  $AE$  is of  $CF$  the same part also is  $AB$  of  $CD$ , therefore also the remainder  $EB$  is the same part of the remainder  $FD$  that the whole  $AB$  is of the whole  $CD$  Q E D

## PROPOSITION 8

If a number be the same parts of a number that a number subtracted is of a number subtracted, the remainder will also be the same parts of the remainder that the whole is of the whole

For let the number  $AB$  be the same parts of the number  $CD$  that  $AE$  subtracted is of  $CF$  subtracted,

I say that the remainder  $EB$  is also the same parts of the remainder  $FD$  that the whole  $AB$  is of the whole  $CD$



For let  $GH$  be made equal to  $AB$

Therefore, whatever parts  $GH$  is of  $CD$ , the same parts also is  $AE$  of  $CF$

Let  $GH$  be divided into the parts of  $CD$ , namely  $GK$ ,  $KH$ , and  $AE$  into the parts of  $CF$ , namely  $AL$ ,  $LE$ ,

thus the multitude of  $GK$ ,  $KH$  will be equal to the multitude of  $AL$ ,  $LE$

Now since, whatever part  $GK$  is of  $CD$ , the same part also is  $AL$  of  $CF$ , while  $CD$  is greater than  $CF$ ,

therefore  $GK$  is also greater than  $AL$

Let  $GM$  be made equal to  $AL$

Therefore, whatever part  $GK$  is of  $CD$ , the same part also is  $GM$  of  $CF$ , therefore also the remainder  $MK$  is the same part of the remainder  $FD$  that the whole  $GK$  is of the whole  $CD$  [VII 7]

Again, since, whatever part  $KH$  is of  $CD$ , the same part also is  $EL$  of  $CF$ , while  $CD$  is greater than  $CF$ ,

therefore  $HK$  is also greater than  $EL$

Let  $KN$  be made equal to  $EL$

Therefore, whatever part  $KH$  is of  $CD$ , the same part also is  $KN$  of  $CF$ , therefore also the remainder  $NH$  is the same part of the remainder  $FD$  that the whole  $KN$  is of the whole  $CD$  [VII 7]

But the remainder  $MK$  was also proved to be the same part of the remainder  $FD$  that the whole  $GK$  is of the whole  $CD$ , therefore also the sum of  $MK$ ,  $NH$  is the same parts of  $DF$  that the whole  $HG$  is of the whole  $CD$

But the sum of  $MK$ ,  $NH$  is equal to  $EB$ ,

and  $HG$  is equal to  $BA$ ,

therefore the remainder  $EB$  is the same parts of the remainder  $FD$  that the whole  $AB$  is of the whole  $CD$  Q E D

## PROPOSITION 9

If a number be a part of a number and another be the same part of another, alternately also, whatever part or parts the first is of the third, the same part, or the same parts will the second also be of the fourth



For let the number  $A$  be a part of the number  $BC$ , and another,  $D$ , the same part of another,  $EF$ , that  $A$  is of  $BC$ ,

I say that alternately also, whatever part or parts  $A$  is of  $D$ , the same part or parts is  $BC$  of  $EF$  also

For since, whatever part  $A$  is of  $BC$ , the same part also is  $D$  of  $EF$ ,



therefore as many numbers as there are in  $BC$  equal to  $A$ , so many also are there in  $EF$  equal to  $D$

Let  $BC$  be divided into the numbers equal to  $A$ , namely  $BG, GC$ ,  
and  $EF$  into those equal to  $D$ , namely  $EH, HF$ ,

thus the multitude of  $BG, GC$  will be equal to the multitude of  $EH, HF$

Now, since the numbers  $BG, GC$  are equal to one another, and the numbers  $EH, HF$  are also equal to one another,

while the multitude of  $BG, GC$  is equal to the multitude of  $EH, HF$ ,<sup>†</sup>  
therefore whatever part or parts  $BG$  is of  $EH$ , the same part or the same parts is  $GC$  of  $HF$  also,

so that, in addition whatever part or parts  $BG$  is of  $EH$ , the same part also or the same parts, is the sum  $BC$  of the sum  $EF$  [VII 5, 6]

But  $BG$  is equal to  $A$ , and  $EH$  to  $D$ ,  
therefore, whatever part or parts  $A$  is of  $D$ , the same part or the same parts is  $BC$  of  $EF$  also Q E D

### PROPOSITION 10

*If a number be parts of a number, and another be the same parts of another, alternately also, whatever parts or part the first is of the third, the same parts or the same part will the second also be of the fourth*

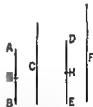
For let the number  $AB$  be parts of the number  $C$ , and another,  $DE$ , the same parts of another,  $F$ ,

I say that, alternately also, whatever parts or part  $AB$  is of  $DE$ , the same parts or the same part is  $C$  of  $F$  also

For since, whatever parts  $AB$  is of  $C$ , the same parts also is  $DE$  of  $F$ ,

therefore, as many parts of  $C$  as there are in  $AB$ , so many parts also of  $F$  are there in  $DE$

Let  $AB$  be divided into the parts of  $C$ , namely  $AG, GB$ ,  
and  $DE$  into the parts of  $F$ , namely  $DH, HE$ ,



thus the multitude of  $AG, GB$  will be equal to the multitude of  $DH, HE$

Now since, whatever part  $AG$  is of  $C$ , the same part also is  $DH$  of  $F$ ,  
alternately also, whatever part or parts  $AG$  is of  $DH$ ,  
the same part or the same parts is  $C$  of  $F$  also [VII 9]

For the same reason also,  
whatever part or parts  $GB$  is of  $HE$ , the same part or the same parts is  $C$  of  $F$  also,

so that, in addition whatever parts or part  $AB$  is of  $DE$ , the same parts also or the same part, is  $C$  of  $F$  [VII 5, 6]  
Q E D

### PROPOSITION 11

*If, as whole is to whole, so is a number subtracted to a number subtracted, the remainder will also be to the remainder as whole to whole*

As the whole  $AB$  is to the whole  $CD$ , so let  $AE$  subtracted be to  $CF$  subtracted,

I say that the remainder  $EB$  is also to the remainder  $FD$  as the whole  $AB$  to the whole  $CD$

Since, as  $AB$  is to  $CD$ , so is  $AE$  to  $CF$ ,

$\begin{array}{l} A \\ E \\ \text{---} \\ D \end{array} \quad \begin{array}{l} C \\ F \\ \text{---} \\ D \end{array}$ 
 whatever part or parts  $AB$  is of  $CD$ , the same part or the same parts is  $AE$  of  $CF$  also, [VII Def 20]  
 Therefore also the remainder  $EB$  is the same part or parts of  $FD$  that  $AB$  is of  $CD$  [VII 7, 8]  
 Therefore, as  $EB$  is to  $FD$ , so is  $AB$  to  $CD$  [VII Def 20]  
 Q E D

## PROPOSITION 12

If there be as many numbers as we please in proportion, then, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents

Let  $A, B, C, D$  be as many numbers as we please in proportion, so that,  
 as  $A$  is to  $B$ , so is  $C$  to  $D$ ,

I say that, as  $A$  is to  $B$ , so are  $A, C$  to  $B, D$

For since, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,

$\begin{array}{l} A \\ B \\ \text{---} \\ D \end{array} \quad \begin{array}{l} C \\ D \\ \text{---} \\ D \end{array}$ 
 whatever part or parts  $A$  is of  $B$ , the same part or parts is  $C$  of  $D$  also [VII Def 20]

Therefore also the sum of  $A, C$  is the same part or the same parts of the sum of  $B, D$  that  $A$  is of  $B$  [VII 5, 6]

Therefore, as  $A$  is to  $B$ , so are  $A, C$  to  $B, D$  [VII Def 20]

Q E D

## PROPOSITION 13

If four numbers be proportional, they will also be proportional alternately

Let the four numbers  $A, B, C, D$  be proportional, so that,

as  $A$  is to  $B$ , so is  $C$  to  $D$ ,

I say that they will also be proportional alternately, so that,

as  $A$  is to  $C$ , so will  $B$  be to  $D$

$\begin{array}{l} A \\ B \\ \text{---} \\ C \end{array} \quad \begin{array}{l} C \\ D \\ \text{---} \\ D \end{array}$ 
 For since, as  $A$  is to  $B$  so is  $C$  to  $D$ ,  
 therefore, whatever part or parts  $A$  is of  $B$ , the same part or the same parts is  $C$  of  $D$  also [VII Def 20]

Therefore alternately, whatever part or parts  $A$  is of  $C$ , the same part or the same parts is  $B$  of  $D$  also [VII 10]

Therefore, as  $A$  is to  $C$ , so is  $B$  to  $D$  [VII Def 20]

Q E D

## PROPOSITION 14

If there be as many numbers as we please, and others equal to them in multitude, which taken two and two are in the same ratio, they will also be in the same ratio *ex aequali*

Let there be as many numbers as we please  $A, B, C$ , and others equal to them in multitude  $D, E, F$ , which taken two and two are in the same ratio, so that,

$\begin{array}{c} \text{---} \\ A \\ \text{---} \\ B \\ \text{---} \\ C \end{array}$

$\begin{array}{c} \text{---} \\ D \\ \text{---} \\ E \\ \text{---} \\ F \end{array}$

as  $A$  is to  $B$ , so is  $D$  to  $E$ ,  
 and as  $B$  is to  $C$ , so is  $E$  to  $F$ ,

I say that, *ex aequali*

as  $A$  is to  $C$  so also is  $D$  to  $F$

For, since as  $A$  is to  $B$ , so is  $D$  to  $E$

therefore alternately,

as  $A$  is to  $D$ , so is  $B$  to  $E$

Again since as  $B$  is to  $C$ , so is  $E$  to  $F$ ,

[VII 13]

therefore, alternately,

as  $B$  is to  $E$ , so is  $C$  to  $F$ .

[VII 13]

But, as  $B$  is to  $E$ , so is  $A$  to  $D$ ,

therefore also, as  $A$  is to  $D$ , so is  $C$  to  $F$ .

Therefore, alternately,

as  $A$  is to  $C$ , so is  $D$  to  $F$

[id]

### PROPOSITION 15

*If an unit measure any number, and another number measure any other number the same number of times, alternately also, the unit will measure the third number the same number of times that the second measures the fourth*

For let the unit  $A$  measure any number  $BC$ ,  
and let another number  $D$  measure any  
other number  $EF$  the same number of  
times,



I say that, alternately also, the unit  $A$   
measures the number  $D$  the same number  
of times that  $BC$  measures  $EF$

For, since the unit  $A$  measures the number  $BC$  the same number of times  
that  $D$  measures  $EF$ ,  
therefore, as many units as there are in  $BC$ , so many numbers equal to  $D$  are  
there in  $EF$  also

Let  $BC$  be divided into the units in it,  $BG, GH, HC$ ,

and  $EF$  into the numbers  $EK, KL, LF$  equal to  $D$

Thus the multitude of  $BG, GH, HC$  will be equal to the multitude of  $EK, KL, LF$

And, since the units  $BG, GH, HC$  are equal to one another,

and the numbers  $EK, KL, LF$  are also equal to one another,

while the multitude of the units  $BG, GH, HC$  is equal to the multitude of the  
numbers  $EK, KL, LF$ ,

therefore, as the unit  $BG$  is to the number  $EK$ , so will the unit  $GH$  be to the  
number  $KL$ , and the unit  $HC$  to the number  $LF$

Therefore also as one of the antecedents is to one of the consequents so will  
all the antecedents be to all the consequents,

[VII 12]

therefore, as the unit  $BG$  is to the number  $EK$ , so is  $BC$  to  $EF$

But the unit  $BG$  is equal to the unit  $A$ ,

and the number  $EK$  to the number  $D$

Therefore, as the unit  $A$  is to the number  $D$ , so is  $BC$  to  $EF$

Therefore the unit  $A$  measures the number  $D$  the same number of times that  
 $BC$  measures  $EF$

Q E D

### PROPOSITION 16

*If two numbers by multiplying one another make certain numbers, the numbers so produced will be equal to one another*

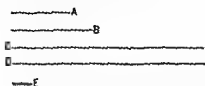
Let  $A, B$  be two numbers, and let  $A$  by multiplying  $B$  make  $C$ , and  $B$  by  
multiplying  $A$  make  $D$ ,

I say that  $C$  is equal to  $D$

For, since  $A$  by multiplying  $B$  has made  $C$ ,

therefore  $B$  measures  $C$  according to the units in  $A$

But the unit  $E$  also measures the number  $A$  according to the units in it,  
therefore the unit  $E$  measures  $A$  the same number of times that  $B$  measures  $C$



Therefore, alternately, the unit  $E$  measures the number  $B$  the same number of times that  $A$  measures  $C$  [VII 15]

Again, since  $B$  by multiplying  $A$  has made  $D$ ,  
therefore  $A$  measures  $D$  according to the units in  $B$

But the unit  $E$  also measures  $B$  according to the units in it,  
therefore the unit  $E$  measures the number  $B$  the same number of times that  $A$  measures  $D$

But the unit  $E$  measured the number  $B$  the same number of times that  $A$  measures  $C$ ,

therefore  $A$  measures each of the numbers  $C$ ,  $D$  the same number of times

Therefore  $C$  is equal to  $D$

Q E D

### PROPOSITION 17

*If a number by multiplying two numbers make certain numbers, the numbers so produced will have the same ratio as the numbers multiplied*

For let the number  $A$  be multiplying the two numbers  $B$ ,  $C$  make  $D$ ,  $E$ ,

I say that, as  $B$  is to  $C$ , so is  $D$  to  $E$

For, since  $A$  by multiplying  $B$  has made  $D$ ,

therefore  $B$  measures  $D$  according to the units in  $A$



But the unit  $F$  also measures the number  $A$  according to the units in it,  
therefore the unit  $F$  measures the number  $A$  the same number of times that  $B$  measures  $D$

Therefore, as the unit  $F$  is to the number  $A$ , so is  $B$  to  $D$  [VII Def 20]

For the same reason,

as the unit  $F$  is to the number  $A$  so also is  $C$  to  $E$ ,

therefore also, as  $B$  is to  $D$ , so is  $C$  to  $E$

Therefore alternately, as  $B$  is to  $C$ , so is  $D$  to  $E$

[VII 19]

Q E D

### PROPOSITION 18

*If two numbers by multiplying any number make certain numbers, the numbers so produced will have the same ratio as the multipliers*

For let two numbers  $A$ ,  $B$  by multiplying any number  $C$  make  $D$ ,  $E$ ,

I say that, as  $A$  is to  $B$ , so is  $D$  to  $E$

For, since  $A$  by multiplying  $C$  has made  $D$ ,

therefore also  $C$  by multiplying  $A$  has made  $D$

[VII 16]

For the same reason also

$C$  by multiplying  $B$  has made  $E$

Therefore the number  $C$  by multiplying the two numbers  $A$ ,  $B$  has made  $D$ ,  $E$

Therefore, as  $A$  is to  $B$ , so is  $D$  to  $E$

[VII 17]

Q E D

## PROPOSITION 19

*If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional*

Let  $A, B, C, D$  be four numbers in proportion, so that,

as  $A$  is to  $B$ , so is  $C$  to  $D$ ;

and let  $A$  by multiplying  $D$  make  $E$ , and let  $B$  by multiplying  $C$  make  $F$ ,

I say that  $E$  is equal to  $F$

For let  $A$  by multiplying  $C$  make  $G$

Since, then,  $A$  by multiplying  $C$  has made  $G$ , and by multiplying  $D$  has made  $E$ ,

the number  $A$  by multiplying the two numbers  $C, D$  has made  $G, E$

Therefore, as  $C$  is to  $D$ , so is  $G$  to  $E$  [VII 17]

But, as  $C$  is to  $D$ , so is  $A$  to  $B$ ,

therefore also, as  $A$  is to  $B$ , so is  $G$  to  $E$

Again, since  $A$  by multiplying  $C$  has made  $G$ ,

but, further,  $B$  has also by multiplying  $C$  made  $F$ ,

the two numbers  $A, B$  by multiplying a certain number  $C$  have made  $G, F$

Therefore, as  $A$  is to  $B$ , so is  $G$  to  $F$  [VII 18]

But further, as  $A$  is to  $B$ , so is  $G$  to  $E$  also,

therefore also, as  $G$  is to  $E$ , so is  $G$  to  $F$

Therefore  $G$  has to each of the numbers  $E, F$  the same ratio;

therefore  $E$  is equal to  $F$

[cf V 9]

Again, let  $E$  be equal to  $F$ ,

I say that, as  $A$  is to  $B$ , so is  $C$  to  $D$

For, with the same construction,

since  $E$  is equal to  $F$ ,

therefore, as  $G$  is to  $E$ , so is  $G$  to  $F$ .

[cf V 7]

But, as  $G$  is to  $E$ , so is  $C$  to  $D$ ,

[VII 17]

and, as  $G$  is to  $F$ , so is  $A$  to  $B$

[VII 18]

Therefore also, as  $A$  is to  $B$ , so is  $C$  to  $D$

Q E D

## PROPOSITION 20

*The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less*

For let  $CD, EF$  be the least numbers of those which have the same ratio with  $A, B$ ,

I say that  $CD$  measures  $A$  the same number of times that  $EF$  measures  $B$

Now  $CD$  is not parts of  $A$

For, if possible, let it be so,

therefore  $EF$  is also the same parts of  $B$  that  $CD$  is of  $A$

[VII 13 and Def 20]

Therefore, as many parts of  $A$  as there are in  $CD$ , so many parts of  $B$  are there also in  $EF$

Let  $CD$  be divided into the parts of  $A$ , namely  $CG, GD$ , and  $EF$  into the parts of  $B$ , namely  $EH, HF$ ,  
 thus the multitude of  $CG, GD$  will be equal to the multitude of  $EH, HF$

Now, since the numbers  $CG, GD$  are equal to one another,  
 and the numbers  $EH, HF$  are also equal to one another,  
 while the multitude of  $CG, GD$  is equal to the multitude of  
 $EH, HF$ ,

therefore, as  $CG$  is to  $EH$ , so is  $GD$  to  $HF$   
 Therefore also, as one of the antecedents is to one of the  
 consequents, so will all the antecedents be to all the conse-  
 quents [VII 12]

Therefore, as  $CG$  is to  $EH$ , so is  $CD$  to  $EF$   
 Therefore  $CG, EH$  are in the same ratio with  $CD, EF$ , being less than they  
 which is impossible, for by hypothesis  $CD, EF$  are the least numbers of those  
 which have the same ratio with them

Therefore  $CD$  is not parts of  $A$ ;  
 therefore it is a part of it [VII 4]  
 And  $EF$  is the same part of  $B$  that  $CD$  is of  $A$ , [VII 13 and Def 20]  
 therefore  $CD$  measures  $A$  the same number of times that  $EF$  measures  $B$   
 Q E D

## PROPOSITION 21

Numbers prime to one another are the least of those which have the same ratio with them

Let  $A, B$  be numbers prime to one another,  
 I say that  $A, B$  are the least of those which have  
 the same ratio with them  
 For, if not, there will be some numbers less than  
 $A, B$  which are in the same ratio with  $A, B$   
 Let them be  $C, D$

Since then the least numbers of those which have the same ratio measure  
 those which have the same ratio the same number of times, the greater the  
 greater and the less the less, that is, the antecedent the antecedent and the  
 consequent the consequent, [VII 20]

therefore  $C$  measures  $A$  the same number of times that  $D$  measures  $B$   
 Now, as many times as  $C$  measures  $A$ , so many units let there be in  $E$

[VII 16]

For the same reason  
 $E$  also measures  $B$  according to the units in  $D$  [VII 16]

Therefore  $E$  measures  $A, B$  which are prime to one another which is im-  
 possible [VII Def 12]

Therefore there will be no numbers less than  $A, B$  which are in the same  
 ratio with  $A, B$

Therefore  $A, B$  are the least of those which have the same ratio with them  
 Q E D

## PROPOSITION 22

*The least numbers of those which have the same ratio with them are prime to one another*

Let  $A, B$  be the least numbers of those which have the same ratio with them,

I say that  $A, B$  are prime to one another

For if they are not prime to one another, some number will measure them

Let some number measure them, and let it be  $C$

And as many times as  $C$  measures  $A$ , so many units let there be in  $D$ ,

and, as many times as  $C$  measures  $B$  so many units let there be in  $E$

Since  $C$  measures  $A$  according to the units in  $D$ ,

therefore  $C$  by multiplying  $D$  has made  $A$  [VII Def 15]

For the same reason also

$C$  by multiplying  $E$  has made  $B$

Thus the number  $C$  by multiplying the two numbers  $D, E$  has made  $A, B$ ,

therefore as  $D$  is to  $E$ , so is  $A$  to  $B$ ; [VII 17]

therefore  $D, E$  are in the same ratio with  $A, B$ , being less than they which is impossible

Therefore no number will measure the numbers  $A, B$

Therefore  $A, B$  are prime to one another

Q E D

## PROPOSITION 23

*If two numbers be prime to one another, the number which measures the one of them will be prime to the remaining number*

Let  $A, B$  be two numbers prime to one another, and let any number  $C$  measure  $A$ ,

I say that  $C, B$  are also prime to one another

For, if  $C, B$  are not prime to one another

some number will measure  $C, B$

Let a number measure them, and let it be  $D$

Since  $D$  measures  $C$  and  $C$  measures  $A$ , therefore  $D$  also measures  $A$

But it also measures  $B$ ,

therefore  $D$  measures  $A, B$  which are prime to one another which is impossible [VII Def 12]

Therefore no number will measure the numbers  $C, B$

Therefore  $C, B$  are prime to one another

Q E D

## PROPOSITION 24

*If two numbers be prime to any number their product also will be prime to the same*

For let the two numbers  $A, B$  be prime to any number  $C$ , and let  $A$  by multiplying  $B$  make  $D$ ,

I say that  $C, D$  are prime to one another

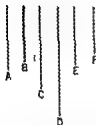
For, if  $C, D$  are not prime to one another some number will measure  $C, D$

Let a number measure them and let it be  $E$

Now, since  $C, A$  are prime to one another,

and a certain number  $E$  measures  $C$ ,  
therefore  $A, E$  are prime to one another [vii 23]

As many times then, as  $E$  measures  $D$ , so many units let there be in  $F$ ,



therefore  $F$  also measures  
 $D$  according to the units in  $E$  [vii 16]

Therefore  $E$  by multiplying  $F$  has made  $D$  [vii Def 15]

But further,  $A$  by multiplying  $B$  has also made  $D$ ,  
therefore the product of  $E, F$  is equal to the product of  
 $A, B$

But if the product of the extremes be equal to that of  
the means the four numbers are proportional, [vii 19]  
therefore as  $E$  is to  $A$ , so is  $B$  to  $F$

But  $A, E$  are prime to one another

numbers which are prime to one another are also the least of those which have  
the same ratio [vii 21]

and the least numbers of those which have the same ratio with them measure  
those which have the same ratio the same number of times the greater the  
greater and the less the less that is, the antecedent the antecedent and the  
consequent the consequent, [vii 20]

therefore  $E$  measures  $B$

But it also measures  $C$ ,

therefore  $E$  measures  $B, C$  which are prime to one another which is impossible  
[vii Def 12]

Therefore no number will measure the numbers  $C, D$

Therefore  $C, D$  are prime to one another Q E D

#### PROPOSITION 25

If two numbers be prime to one another, the product of one of them into itself will  
be prime to the remaining one

Let  $A, B$  be two numbers prime to one another

and let  $A$  by multiplying itself make  $C$ ,



I say that  $B, C$  are prime to one another

For let  $D$  be made equal to  $A$

Since  $A, B$  are prime to one another and  $A$  is equal to  $D$ ,  
therefore  $D, B$  are also prime to one another

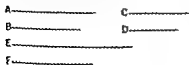
Therefore each of the two numbers  $D, A$  is prime to  $B$ ,  
therefore the product of  $D, A$  will also be prime to  $B$  [vii 24]

But the number which is the product of  $D, A$  is  $C$

Therefore  $C, B$  are prime to one another Q E D

#### PROPOSITION 26

If two numbers be prime to two numbers both to each their products also will be  
prime to one another



For let the two numbers  $A, B$  be prime  
to the two numbers  $C, D$ , both to each,  
and let  $A$  by multiplying  $B$  make  $E$  and let  
 $C$  by multiplying  $D$  make  $F$ ,

I say that  $E, F$  are prime to one another



For, since each of the numbers  $A, B$  is prime to  $C$ ,  
 therefore the product of  $A, B$  will also be prime to  $C$  [VII 24]  
 But the product of  $A, B$  is  $E$ ,  
 therefore  $E, C$  are prime to one another  
 For the same reason  
 $E, D$  are also prime to one another  
 Therefore each of the numbers  $C, D$  is prime to  $E$   
 Therefore the product of  $C, D$  will also be prime to  $E$  [VII 24]  
 But the product of  $C, D$  is  $F$   
 Therefore  $E, F$  are prime to one another Q E D

## PROPOSITION 27

*If two numbers be prime to one another, and each by multiplying itself make a certain number, the products will be prime to one another, and, if the original numbers by multiplying the products make certain numbers, the latter will also be prime to one another [and this is always the case with the extremes]*

ing  $L$  make  $F$ ,

I say that both  $C, E$  and  $D, F$  are prime to one another  
 For, since  $A, B$  are prime to one another, and  $A$  by multiplying itself has made  $C$ ,  
 therefore  $C, B$  are prime to one another [VII 25]  
 Since, then,  $C, B$  are prime to one another,  
 and  $B$  by multiplying itself has made  $E$ ,  
 therefore  $C, E$  are prime to one another [id]  
 Again, since  $A, B$  are prime to one another,  
 and  $B$  by multiplying itself has made  $E$ ,  
 therefore  $A, E$  are prime to one another [id]  
 Since then the two numbers  $A, C$  are prime to the two numbers  $B, E$ , both to each,  
 therefore also the product of  $A, C$  is prime to the product of  $B, E$  [VII 20]  
 And the product of  $A, C$  is  $D$ , and the product of  $B, E$  is  $F$   
 Therefore  $D, F$  are prime to one another Q E D

## PROPOSITION 28

*If two numbers be prime to one another, the sum will also be prime to each of them, and if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another*

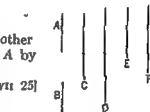
For let two numbers  $AB, BC$  prime to one another be added,  
 I say that the sum  $AC$  is also prime to each of the numbers  $AB, BC$

For, if  $CA, AB$  are not prime to one another,  
 some number will measure  $CA, AB$

Let a number measure them, and let it be  $D$

Since then  $D$  measures  $CA, AB$ ,

therefore it will also measure the remainder  $BC$ .



But it also measures  $BA$ ,  
 therefore  $D$  measures  $AB, BC$  which are prime to one another. which is impossible [VII Def 12]

Therefore no number will measure the numbers  $CA, AB$ , therefore  $CA, AB$  are prime to one another

For the same reason

$AC, CB$  are also prime to one another

Therefore  $CA$  is prime to each of the numbers  $AB, BC$

Again, let  $CA, AB$  be prime to one another,

I say that  $AB, BC$  are also prime to one another

For, if  $AB, BC$  are not prime to one another,

some number will measure  $AB, BC$

Let a number measure them, and let it be  $D$

Now since  $D$  measures each of the numbers  $AB, BC$ , it will also measure the whole  $CA$

But it also measures  $AB$ ,

therefore  $D$  measures  $CA, AB$  which are prime to one another

which is impossible

[VII Def 12]

Therefore no number will measure the numbers  $AB, BC$

Therefore  $AB, BC$  are prime to one another

Q E D

### PROPOSITION 29

Any prime number is prime to any number which it does not measure

Let  $A$  be a prime number, and let it not measure  $B$ ,

I say that  $B, A$  are prime to one another

For, if  $B, A$  are not prime to one another,

some number will measure them

Let  $C$  measure them

Since  $C$  measures  $B$ ,

and  $A$  does not measure  $B$

therefore  $C$  is not the same with  $A$

Now, since  $C$  measures  $B, A$ ,

therefore it also measures  $A$  which is prime, though it is not the same with it which is impossible

Therefore no number will measure  $B, A$

Therefore  $A, B$  are prime to one another

Q E D

### PROPOSITION 30

If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers

For let the two numbers  $A, B$  by multiplying one another make  $C$ , and let any prime number  $D$  measure  $C$ ,

I say that  $D$  measures one of the numbers  $A, B$

For let it not measure  $A$

Now  $D$  is prime,

therefore  $A, D$  are prime to one another [VII 29]

And, as many times as  $D$  measures  $C$ , so many units let there be in  $E$

Since then  $D$  measures  $C$  according to the units in  $E$

therefore  $D$  by multiplying  $E$  has made  $C$  [VII Def 15]

Further,  $A$  by multiplying  $B$  has also made  $C$ ,

therefore the product of  $D$ ,  $E$  is equal to the product of  $A$ ,  $B$

Therefore, as  $D$  is to  $A$ , so is  $B$  to  $E$  [VII 19]

But  $D$ ,  $A$  are prime to one another,

primes are also least, [VII 21]

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore  $D$  measures  $B$

Similarly we can also show that, if  $D$  does not measure  $B$ , it will measure  $A$

Therefore  $D$  measures one of the numbers  $A$ ,  $B$  Q E D

### PROPOSITION 31

*Any composite number is measured by some prime number*

Let  $A$  be a composite number,

I say that  $A$  is measured by some prime number

For, since  $A$  is composite,

some number will measure it

Let a number measure it, and let it be  $B$

Now, if  $B$  is prime, what was enjoined will have been done

$A$  \_\_\_\_\_

$B$  \_\_\_\_\_

$C$  \_\_\_\_\_

But if it is composite, some number will measure it

Let a number measure it, and let it be  $C$

Then since  $C$  measures  $B$ ,

and  $B$  measures  $A$

therefore  $C$  also measures  $A$

And if  $C$  is prime, what was enjoined will have been done

For, if it is not found, an infinite series of numbers will measure the number  $A$ , each of which is less than the other

which is impossible in numbers

Therefore some prime number will be found which will measure the one before it which will also measure  $A$

Therefore any composite number is measured by some prime number

Q E D

### PROPOSITION 32

*Any number either is prime or is measured by some prime number*

Let  $A$  be a number,

I say that  $A$  either is prime or is measured by some prime number

If now  $A$  is prime that which was enjoined will have been done

But if it is composite some prime number will measure it [VII 31]

Therefore any number either is prime or is measured by some prime number

Q E D

## PROPOSITION 33

Given as many numbers as we please, to find the least of those which have the same ratio with them

Let  $A, B, C$  be the given numbers, as many as we please,

thus it is required to find the least of those which have the same ratio with  $A, B, C$

$A, B, C$  are either prime to one another or not

Now, if  $A, B, C$  are prime to one another, they are the least of those which have the same ratio with them [VII 21]

But, if not, let  $D$  the greatest common measure of  $A, B, C$  be taken, [VII 3]

and, as many times as  $D$  measures the numbers  $A, B, C$  respectively, so many units let there be in the numbers  $E, F, G$  respectively

Therefore the numbers  $E, F, G$  measure the numbers  $A, B, C$  respectively according to the units in  $D$  [VII 16]

Therefore  $E, F, G$  measure  $A, B, C$  the same number of times,

therefore  $E, F, G$  are in the same ratio with  $A, B, C$  [VII Def 20]

I say next that they are the least that are in that ratio

For, if  $E, F, G$  are not the least of those which have the same ratio with  $A, B, C$ , there will be numbers less than  $E, F, G$  which are in the same ratio with  $A, B, C$

Let them be  $H, K, L$ , therefore  $H$  measures  $A$  the same number of times that the numbers  $K, L$  measure the numbers  $B, C$  respectively

Now, as many times as  $H$  measures  $A$ , so many units let there be in  $M$ , therefore the numbers  $K, L$  also measure the numbers  $B, C$  respectively according to the units in  $M$

And, since  $H$  measures  $A$  according to the units in  $M$ ,

therefore  $M$  also measures  $A$  according to the units in  $H$  [VII 16]

For the same reason

$M$  also measures the numbers  $B, C$  according to the units in the numbers  $K, L$  respectively,

Therefore  $M$  measures  $A, B, C$

Now, since  $H$  measures  $A$  according to the units in  $M$ ,

therefore  $H$  by multiplying  $M$  has made  $A$  [VII Def 15]

For the same reason also

$E$  by multiplying  $D$  has made  $A$

Therefore the product of  $E, D$  is equal to the product of  $H, M$

Therefore, as  $E$  is to  $H$ , so is  $M$  to  $D$  [VII 19]

But  $E$  is greater than  $H$ ,

therefore  $M$  is also greater than  $D$

And it measures  $A, B, C$

which is impossible, for by hypothesis  $D$  is the greatest common measure of  $A, B, C$

Therefore there cannot be any numbers less than  $E, F, G$  which are in the same ratio with  $A, B, C$

Therefore  $E, F, G$  are the least of those which have the same ratio with  $A, B, C$  Q E D

### PROPOSITION 34

*Given two numbers, to find the least number which they measure*

Let  $A, B$  be the two given numbers,

thus it is required to find the least number which they measure

Now  $A, B$  are either prime to one another or not

First, let  $A, B$  be prime to one another, and let  $A$

by multiplying  $B$  make  $C$ ,

therefore also  $B$  by multiplying  $A$  has made  $C$

Therefore  $A, B$  measure  $C$

I say next that it is also the least number they measure

For, if not,  $A, B$  will measure some number which is less than  $C$

Let them measure  $D$

Then, as many times as  $A$  measures  $D$ , so many units let there be in  $E$ , and as many times as  $B$  measures  $D$ , so many units let there be in  $F$ ;

therefore  $A$  by multiplying  $E$  has made  $D$ ,

and  $B$  by multiplying  $F$  has made  $D$ , [VII Def 15]

therefore the product of  $A, E$  is equal to the product of  $B, F$

Therefore, as  $A$  is to  $B$ , so is  $F$  to  $E$  [VII 19]

But  $A, B$  are prime,

primes are also least, [VII 21]

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, [VII 20]

therefore  $B$  measures  $E$ , as consequent consequent

And since  $A$  by multiplying  $B, E$  has made  $C, D$ ,

therefore, as  $B$  is to  $E$ , so is  $C$  to  $D$  [VII 17]

But  $B$  measures  $E$ ,

therefore  $C$  also measures  $D$ , the greater the less which is impossible

Therefore  $A, B$  do not measure any number less than  $C$ ,

therefore  $C$  is the least that is measured by  $A, B$

Next, let  $A, B$  not be prime to one another,

and let  $F, E$ , the least numbers of those which have the same ratio with  $A, B$ , be taken, [VII 33]

therefore the product of  $A, E$  is equal to the product of  $B, F$  [VII 19]

And let  $A$  by multiplying  $E$  make  $C$ ,

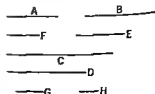
therefore also  $B$  by multiplying  $F$  has made  $C$ ,

therefore  $A, B$  measure  $C$

I say next that it is also the least number that they measure

For, if not  $A, B$  will measure some number which is less than  $C$

Let them measure  $D$



And, as many times as  $A$  measures  $D$ , so many units let there be in  $G$ ,  
and, as many times as  $B$  measures  $D$ , so many units let there be in  $H$ .

Therefore  $A$  by multiplying  $G$  has made  $D$ ,

and  $B$  by multiplying  $H$  has made  $D$

Therefore the product of  $A$ ,  $G$  is equal to the product of  $B$ ,  $H$ ;

therefore, as  $A$  is to  $B$ , so is  $H$  to  $G$

[VII 19]

But, as  $A$  is to  $B$ , so is  $F$  to  $E$

Therefore also, as  $F$  is to  $E$ , so is  $H$  to  $G$ .

But  $F$ ,  $E$  are least,

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less,

[VII. 20]

therefore  $E$  measures  $G$

And, since  $A$  by multiplying  $E$ ,  $G$  has made  $C$ ,  $D$ ,

therefore, as  $E$  is to  $G$ , so is  $C$  to  $D$

[VII 17]

But  $E$  measures  $G$ ,

therefore  $C$  also measures  $D$ , the greater the less

which is impossible

Therefore  $A$ ,  $B$  will not measure any number which is less than  $C$

Therefore  $C$  is the least that is measured by  $A$ ,  $B$

Q E D

### PROPOSITION 35

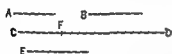
If two numbers measure any number, the least number measured by them will also measure the same

For let the two numbers  $A$ ,  $B$  measure any number  $CD$ ,

and let  $E$  be the least that they measure,

I say that  $E$  also measures  $CD$

For, if  $E$  does not measure  $CD$ , let  $E$ , measuring  $DF$ , leave  $CF$  less than itself



Now, since  $A$ ,  $B$  measure  $E$ ,

and  $E$  measures  $DF$ ,

therefore  $A$ ,  $B$  will also measure  $DF$ .

But they also measure the whole  $CD$ ,

therefore they will also measure the remainder  $CF$  which is less than  $E$ :

which is impossible

Therefore  $E$  cannot fail to measure  $CD$ ,

therefore it measures it

Q E D

### PROPOSITION 36

Given three numbers, to find the least number which they measure

Let  $A$ ,  $B$ ,  $C$  be the three given numbers,

thus it is required to find the least number which they measure

A —————

B —————

C —————

D —————

E —————

Let  $D$ , the least number measured by the two numbers  $A$ ,  $B$ , be taken

[VII 34]

Then  $C$  either measures, or does not measure,  $D$

First, let it measure it

But  $A$ ,  $B$  also measure  $D$ ,

therefore  $A$ ,  $B$ ,  $C$  measure  $D$

I say next that it is also the least that they measure

For, if not,  $A, B, C$  will measure some number which is less than  $D$

Let them measure  $E$

Since  $A, B, C$  measure  $E$ ,

therefore also  $A, B$  measure  $E$

Therefore the least number measured by  $A, B$  will also measure  $E$  [VII 35]

But  $D$  is the least number measured by  $A, B$ ,

therefore  $D$  will measure  $E$ , the greater the less

which is impossible

Therefore  $A, B, C$  will not measure any number which is less than  $D$ ,

therefore  $D$  is the least that  $A, B, C$  measure

Again, let  $C$  not measure  $D$ ,

and let  $E$ , the least number measured by  $C, D$ , be taken

[VII 34]

Since  $A, B$  measure  $D$ ,

and  $D$  measures  $E$ ,

therefore also  $A, B$  measure  $E$

But  $C$  also measures  $E$ ,

therefore also  $A, B, C$  measure  $E$

I say next that it is also the least that they measure

For, if not,  $A, B, C$  will measure some number which is less than  $E$

Let them measure  $F$

Since  $A, B, C$  measure  $F$ ,

therefore also  $A, B$  measure  $F$ ,

therefore the least number measured by  $A, B$  will also measure  $F$  [VII 35]

But  $D$  is the least number measured by  $A, B$ ,

therefore  $D$  measures  $F$

But  $C$  also measures  $F$ ,

therefore  $D, C$  measure  $F$ ,

so that the least number measured by  $D, C$  will also measure  $F$

But  $E$  is the least number measured by  $C, D$ ,

therefore  $E$  measures  $F$ , the greater the less

which is impossible

Therefore  $A, B, C$  will not measure any number which is less than  $E$

Therefore  $E$  is the least that is measured by  $A, B, C$

Q E D

### PROPOSITION 37

If a number be measured by any number, the number which is measured will have a part called by the same name as the measuring number

For let the number  $A$  be measured by any number  $B$ ,

I say that  $A$  has a part called by the same name as  $B$

For as many times as  $B$  measures  $A$  so many units let there be in  $C$

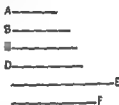
Since  $B$  measures  $A$  according to the units in  $C$ ,

and the unit  $D$  also measures the number  $C$  according to the units in it

therefore the unit  $D$  measures the number  $C$  the same number of times as  $B$  measures  $A$

Therefore alternately the unit  $D$  measures the number  $B$  the same number of times as  $C$  measures  $A$ ,

[VII 15]



therefore, whatever part the unit  $D$  is of the number  $B$ , the same part in  $C$  of  $A$  also

But the unit  $D$  is a part of the number  $B$  called by the same name as it,  
therefore  $C$  is also a part of  $A$  called by the same name as  $B$ ,  
so that  $A$  has a part  $C$  which is called by the same name as  $B$  Q E D

## PROPOSITION 38

*If a number have any part whatever, it will be measured by a number called by the same name as the part*

For let the number  $A$  have any part whatever,  $B$ ,  
and let  $C$  be a number called by the same name as the part  $B$ ,  
A \_\_\_\_\_ I say that  $C$  measures  $A$   
\_\_\_\_\_ B For, since  $B$  is a part of  $A$  called by the same name  
\_\_\_\_\_ C as  $C$ ,  
\_\_\_\_\_ D and the unit  $D$  is also a part of  $C$  called by the same  
name as it,  
therefore, whatever part the unit  $D$  is of the number  $C$ ,  
the same part is  $B$  of  $A$  also,

therefore the unit  $D$  measures the number  $C$  the same number of times that  $B$  measures  $A$

Therefore, alternately, the unit  $D$  measures the number  $B$  the same number of times that  $C$  measures  $A$  [VII 15]

Therefore  $C$  measures  $A$  Q E D

## PROPOSITION 39

*To find the number which is the least that will have given parts*

Let  $A, B, C$  be the given parts,  
thus it is required to find the number which is the least that will have the parts  $A, B, C$

Let  $D, E, F$  be numbers called by the same name as the parts  $A, B, C$ ,  
and let  $G$ , the least number measured by  $D, E, F$ , be taken [VII 36]  
Therefore  $G$  has parts called by the same name as  $D, E, F$  [VII 37]  
But  $A, B, C$  are parts called by the same name as  $D, E, F$ ,  
therefore  $G$  has the parts  $A, B, C$

I say next that it is also the least number that has  
For if not, there will be some number less than  $G$  which will have the parts  $A, B, C$

Let it be  $H$   
Since  $H$  has the parts  $A, B, C$ ,  
therefore  $H$  will be measured by numbers called by the same name as the parts  $A, B, C$  [VII 38]

But  $D, E, F$  are numbers called by the same name as the parts  $A, B, C$ ,  
therefore  $H$  is measured by  $D, E, F$

And it is less than  $G$  which is impossible  
Therefore there will be no number less than  $G$  that will have the parts  $A, B, C$  Q E D



# BOOK EIGHT

## PROPOSITION 1

If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them

Let there be as many numbers as we please,  $A, B, C, D$ , in continued proportion, and let the extremes of them  $A, D$  be prime to one another,

I say that  $A, B, C, D$  are the least of those which have the same ratio with them

For, if not, let  $E, F, G, H$  be less than  $A, B, C, D$ , and in the same ratio with them

Now, since  $A, B, C, D$  are in the same ratio with  $E, F, G, H$ , and the multitude of the numbers  $A, B, C, D$  is equal to the multitude of the numbers  $E, F, G, H$ ,

therefore *ex aequali*,  
as  $A$  is to  $D$ , so is  $E$  to  $H$

[VII 14]

But  $A, D$  are prime,

primes are also least

[VII 21]

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent

[VII 20]

Therefore  $A$  measures  $E$ , the greater the less

which is impossible

Therefore  $E, F, G, H$  which are less than  $A, B, C, D$  are not in the same ratio with them

Therefore  $A, B, C, D$  are the least of those which have the same ratio with them

Q. E. D.

## PROPOSITION 2

To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio

I let the ratio of  $A$  to  $B$  be the given ratio in least numbers, thus it is required to find numbers in continued proportion, as many as may be prescribed and the least that are in the ratio of  $A$  to  $B$

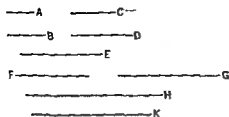
Let four be prescribed,

let  $A$  ~ let it make  $D$ ,

$G, H$

and let  $B$  by multiplying  $E$  make  $K$

Now, since  $A$  by multiplying itself has made  $C$ ,



and by multiplying  $B$  has made  $D$ ,  
therefore, as  $A$  is to  $B$ , so is  $C$  to  $D$   
[VII 17]

Again, since  $A$  by multiplying  $B$   
has made  $D$ ,  
and  $B$  by multiplying itself has made  
 $E$ ,  
therefore the numbers  $A, B$  by mul-  
tiplying  $B$  have made the numbers  
 $D, E$  respectively

Therefore, as  $A$  is to  $B$ , so is  $D$  to  $E$ , [VII 18]

But, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,

therefore also, as  $C$  is to  $D$ , so is  $D$  to  $E$

And since  $A$  by multiplying  $C, D$  has made  $F, G$ ,

therefore, as  $C$  is to  $D$ , so is  $F$  to  $G$  [VII 17]

But, as  $C$  is to  $D$ , so was  $A$  to  $B$ ,

therefore also, as  $A$  is to  $B$ , so is  $F$  to  $G$

Again since  $A$  by multiplying  $D, E$  has made  $G, H$ ,

therefore, as  $D$  is to  $E$ , so is  $G$  to  $H$ . [VII 17]

But, as  $D$  is to  $E$ , so is  $A$  to  $B$

Therefore also, as  $A$  is to  $B$ , so is  $G$  to  $H$

And, since  $A, B$  by multiplying  $E$  have made  $H, K$ ,

therefore, as  $A$  is to  $B$ , so is  $H$  to  $K$  [VII 18]

But as  $A$  is to  $B$ , so is  $F$  to  $G$ , and  $G$  to  $H$

Therefore also, as  $F$  is to  $G$ , so is  $G$  to  $H$ , and  $H$  to  $K$ ,

therefore  $C, D, E$ , and  $F, G, H, K$  are proportional in the ratio of  $A$  to  $B$

I say next that they are the least numbers that are so

For, since  $A, B$  are the least of those which have the same ratio with them,  
and the least of those which have the same ratio are prime to one another,  
[VII 22]

therefore  $A, B$  are prime to one another

And the numbers  $A, B$  by multiplying themselves respectively have made  
the numbers  $C, E$ , and by multiplying the numbers  $C, E$  respectively have  
made the numbers  $F, K$ ,

therefore  $C, E$  and  $F, K$  are prime to one another respectively [VII 27]

But if there be as many numbers as we please in continued proportion, and  
the extremes of them be prime to one another, they are the least of those which  
have the same ratio with them [VIII 1]

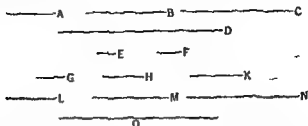
Therefore  $C, D, E$  and  $F, G, H, K$  are the least of those which have the same  
ratio with  $A, B$  Q E D

PROPOSITION 3  
From this it is manifest that, if three numbers in continued propor-  
tion be the least of those which have the same ratio with them the extremes of  
them are squares, and, if four numbers, cubes

### PROPOSITION 3

If as many numbers as we please in continued proportion be the least of those which  
have the same ratio with them, the extremes of them are prime to one another.

Let as many numbers as we please,  $A, B, C, D$ , in continued proportion be the least of those which have the same ratio with them;



I say that the extremes of them  $A, D$  are prime to one another

For let two numbers  $E, F$ , the least that are in the ratio of  $A, B, C, D$ , be taken, [VII 33]

then three others  $G, H, K$  with the same property,

and others more by one continually, [VII 2]

until the multitude taken becomes equal to the multitude of the numbers  $A, B, C, D$

Let them be taken and let them be  $L, M, N, O$

Now, since  $E, F$  are the least of those which have the same ratio with them they are prime to one another [VII 22]

And since the numbers  $E, F$  by multiplying themselves respectively have made the numbers  $G, K$ , and by multiplying the numbers  $G, K$  respectively have made the numbers  $L, O$ , [VIII 2 Por]

therefore both  $G, K$  and  $L, O$  are prime to one another [VII 27]

And since  $A, B, C, D$  are the least of those which have the same ratio with them,

while  $L, M, N, O$  are the least that are in the same ratio with  $A, B, C, D$ , and the multitude of the numbers  $A, B, C, D$  is equal to the multitude of the numbers  $L, M, N, O$

therefore the numbers  $A, B, C, D$  are equal to the numbers  $L, M, N, O$  respectively,

therefore  $A$  is equal to  $L$ , and  $D$  to  $O$

And  $L, O$  are prime to one another

Therefore  $A, D$  are also prime to one another

Q E D

#### PROPOSITION 4

Given as many ratios as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios

Let the given ratios in least numbers be that of  $A$  to  $B$ , that of  $C$  to  $D$ , and that of  $E$  to  $F$ ,

thus it is required to find numbers in continued proportion which are the least that are in the ratio of  $A$  to  $B$ , in the ratio of  $C$  to  $D$ , and in the ratio of  $E$  to  $F$

Let  $G$  the least number be taken [VII 34]

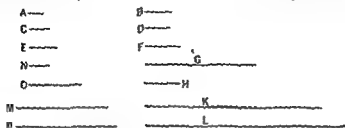
times also let  $A$  measure  $H$ ,

mes also let  $D$  measure  $K$

Now  $E$  either measures or does not measure  $A$

First, let it measure it

And, as many times as  $E$  measures  $K$ , so many times let  $F$  measure  $L$  also  
 Now, since  $A$  measures  $H$  the same number of times that  $B$  measures  $G$ ,  
 therefore, as  $A$  is to  $B$ , so is  $H$  to  $G$  [VII Def 20, VII 13]



For the same reason also

as  $C$  is to  $D$ , so is  $G$  to  $K$ ,

and further, as  $E$  is to  $F$ , so is  $K$  to  $L$ ,

therefore  $H, G, K, L$  are continuously proportional in the ratio of  $A$  to  $B$ , in the ratio of  $C$  to  $D$ , and in the ratio of  $E$  to  $F$

I say next that they are also the least that have this property

For, if  $H, G, K, L$  are not the least numbers continuously proportional in the ratios of  $A$  to  $B$ , of  $C$  to  $D$ , and of  $E$  to  $F$ , let them be  $N, O, M, P$

Then since, as  $A$  is to  $B$ , so is  $N$  to  $O$ ,

while  $A, B$  are least

and the least numbers measure those which have the same ratio the same number of times the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent,

therefore  $B$  measures  $O$

[VII 20]

For the same reason

$C$  also measures  $O$ ,

therefore  $B, C$  measure  $O$ ,

therefore the least number measured by  $B, C$  will also measure  $O$  [VII 35]

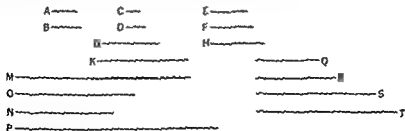
But  $G$  is the least number measured by  $B, C$ ,

therefore  $G$  measures  $O$ , the greater the less

which is impossible

Therefore there will be no numbers less than  $H, G, K, L$  which are continuously in the ratio of  $A$  to  $B$ , of  $C$  to  $D$ , and of  $E$  to  $F$

Next, let  $E$  not measure  $K$



Let  $M$ , the least number measured by  $E, K$ , be taken

And as many times as  $K$  measures  $M$ , so many times let  $H, G$  measure  $N, O$  respectively,

and, as many times as  $E$  measures  $M$ , so many times let  $F$  measure  $P$  also  
Since  $H$  measures  $N$  the same number of times that  $G$  measures  $O$ ,  
therefore, as  $H$  is to  $G$ , so is  $N$  to  $O$  [vii 13 and Def 20]  
But, as  $H$  is to  $G$ , so is  $A$  to  $B$ ,  
therefore also, as  $A$  is to  $B$ , so is  $N$  to  $O$   
For the same reason also,  
as  $C$  is to  $D$ , so is  $O$  to  $M$   
Again since  $E$  measures  $M$  the same number of times that  $F$  measures  $P$ ,  
therefore as  $E$  is to  $F$ , so is  $M$  to  $P$ , [vii 13 and Def 20]  
therefore  $N, O, M, P$  are continuously proportional in the ratios of  $A$  to  $B$ , of  
 $C$  to  $D$ , and of  $E$  to  $F$   
I say next that they are also the least that are in the ratios  $A, B, C, D, E, F$   
For, if not there will be some numbers less than  $N, O, M, P$  continuously  
proportional in the ratios  $A, B, C, D, E, F$   
Let them be  $Q, R, S, T$   
Now since, as  $Q$  is to  $R$ , so is  $A$  to  $B$   
while  $A, B$  are least,  
and the least numbers measure those which have the same ratio with them the  
same number of times, the antecedent the antecedent and the consequent the  
consequent, [vii 20]

therefore  $B$  measures  $R$   
For the same reason  $C$  also measures  $R$ ,  
therefore  $B, C$  measure  $R$   
Therefore the least number measured by  $B, C$  will also measure  $R$  [vii 35]  
But  $G$  is the least number measured by  $B, C$ ,  
therefore  $G$  measures  $R$   
And as  $G$  is to  $R$  so is  $K$  to  $S$  [vii 13]  
therefore  $K$  also measures  $S$   
But  $E$  also measures  $S$ ,  
therefore  $E, K$  measure  $S$   
Therefore the least number measured by  $E, K$  will also measure  $S$  [vii 35]  
But  $M$  is the least number measured by  $E, K$ ,  
therefore  $M$  measures  $S$ , the greater the less  
which is impossible  
Therefore there will not be any numbers less than  $N, O, M, P$  continuously  
proportional in the ratios of  $A$  to  $B$ , of  $C$  to  $D$ , and of  $E$  to  $F$ ,  
therefore  $N, O, M, P$  are the least numbers continuously proportional in the  
ratios  $A, B, C, D, E, F$  Q E D

PROPOSITION 5

Plane numbers have to one another the ratio compounded of the ratios of their sides  
Let  $A, B$  be plane numbers, and let the numbers  $C, D$  be the sides of  $A$ , and  
 $E, F$  of  $B$ ,  
I say that  $A$  has to  $B$  the ratio compounded of the ratios of the sides  
For, the ratios being given which  $C$  has to  $E$  and  $D$  to  $F$ , let the least num-  
bers  $G, H, K$  that are continuously in the ratios  $C, E, D, F$  be taken so that  
as  $C$  is to  $E$ , so is  $G$  to  $H$ ,  
and, as  $D$  is to  $F$  so is  $H$  to  $K$  [viii 4]  
And let  $D$  by multiplying  $E$  make  $L$

Now, since  $D$  by multiplying  $C$  has made  $A$ , and by multiplying  $E$  has made  $L$ ,

therefore, as  $C$  is to  $E$ , so is  $A$  to  $L$  [VII 17]

But, as  $C$  is to  $E$ , so is  $G$  to  $H$ ,

therefore also, as  $G$  is to  $H$ , so is  $A$  to  $L$

—————  $A$

———  $B$

———  $C$

———  $E$

—————  $D$

———  $F$

—————  $G$

—————  $H$

———  $K$

—————  $L$

Again, since  $E$  by multiplying  $D$  has made  $L$ , and further by multiplying  $F$  has made  $B$ ,

therefore, as  $D$  is to  $F$ , so is  $L$  to  $B$  [VII 17]

But, as  $D$  is to  $F$ , so is  $H$  to  $K$ ,

therefore also, as  $H$  is to  $K$ , so is  $L$  to  $B$

But it was also proved that,

as  $G$  is to  $H$ , so is  $A$  to  $L$ ,

therefore, *ex aequali*,

as  $G$  is to  $K$ , so is  $A$  to  $B$  [VII 14]

But  $G$  has to  $K$  the ratio compounded of the ratios of the sides, therefore  $A$  also has to  $B$  the ratio compounded of the ratios of the sides

Q E D

### PROPOSITION 6

If there be as many numbers as we please in continued proportion, and the first do not measure the second, neither will any other measure any other

Let there be as many numbers as we please,  $A, B, C, D, E$ , in continued proportion and let  $A$  not measure  $B$ ,

I say that neither will any other measure any other

—————  $A$

—————  $B$

—————  $C$

—————  $D$

—————  $E$

———  $F$

———  $G$

———  $H$

Now it is manifest that  $A, B, C, D, E$  do not measure one another in order, for  $A$  does not even measure  $B$

I say, then, that neither will any other measure any other

For, if possible, let  $A$  measure  $C$

And, however many  $A, B, C$  are let as many numbers  $F, G, H$ , the least of those which have

the same ratio with  $A, B, C$  be taken

[VII 33]

Now, since  $F, G, H$  are in the same ratio with  $A, B, C$ , and the multitude of the numbers  $A, B, C$  is equal to the multitude of the numbers  $F, G, H$ ,

therefore, *ex aequali*, as  $A$  is to  $C$ , so is  $F$  to  $H$  [VII 14]

And since, as  $A$  is to  $B$ , so is  $F$  to  $G$ ,

while  $A$  does not measure  $B$ ,

therefore neither does  $F$  measure  $G$ , [VII Def 20]

therefore  $F$  is not an unit, for the unit measures any number

Now  $F, H$  are prime to one another [VIII 3]

And, as  $F$  is to  $H$  so is  $A$  to  $C$ ,

therefore neither does  $A$  measure  $C$

Similarly we can prove that neither will any other measure any other

Q E D

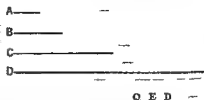
## PROPOSITION 7

*If there be as many numbers as we please in continued proportion, and the first measure the last, it will measure the second also*

Let there be as many numbers as we please,  $A, B, C, D$ , in continued proportion, and let  $A$  measure  $D$ ,

I say that  $A$  also measures  $B$

For, if  $A$  does not measure  $B$ , neither will any other of the numbers measure any other [VII 6]



But  $A$  measures  $D$

Therefore  $A$  also measures  $B$

## PROPOSITION 8

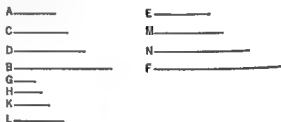
*If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers*

Let the numbers  $C, D$  fall between the two numbers  $A, B$  in continued proportion with them, and let  $E$  be made in the same ratio to  $F$  as  $A$  is to  $B$ ,

I say that, as many numbers as have fallen between  $A, B$  in continued proportion, so many will also fall between  $E, F$  in continued proportion

For, as many as  $A, B, C, D$  are in multitude, let so many numbers  $G, H, K, L$ , the least of those which have the same ratio with  $A, C, D, B$ , be taken, [VII 33]

therefore the extremes of them  $G, L$  are prime to one another [VIII 3]



Now, since  $A, C, D, B$  are in the same ratio with  $G, H, K, L$ , and the multitude of the numbers  $A, C, D, B$  is equal to the multitude of the numbers  $G, H, K, L$

therefore, *ex aequali*, as  $A$  is to  $B$ , so is  $G$  to  $L$  [VII 14]

But, as  $A$  is to  $B$ , so is  $E$  to  $F$ ,

therefore also, as  $G$  is to  $L$ , so is  $E$  to  $F$

But  $G, L$  are prime,

primes are also least, [VII 21]

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent [VII 20]

Therefore  $G$  measures  $E$  the same number of times as  $L$  measures  $F$

Next, as many times as  $G$  measures  $E$ , so many times let  $H, K$  also measure  $M, N$  respectively,

therefore  $G, H, K, L$  measure  $E, M, N, F$  the same number of times

Therefore  $G, H, K, L$  are in the same ratio with  $E, M, N, F$  [VII Def 20]

But  $G, H, K, L$  are in the same ratio with  $A, C, D, B$ ,

therefore  $A, C, D, B$  are also in the same ratio with  $E, M, N, F$

But  $A, C, D, B$  are in continued proportion,  
therefore  $E, M, N, F$  are also in continued proportion

Therefore, as many numbers as have fallen between  $A, B$  in continued proportion with them, so many numbers have also fallen between  $E, F$  in continued proportion Q E D

## PROPOSITION 9

*If two numbers be prime to one another, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many will also fall between each of them and an unit in continued proportion*

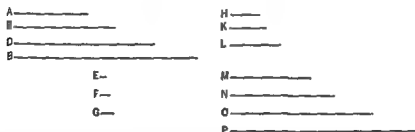
Let  $A, B$  be two numbers prime to one another, and let  $C, D$  fall between them in continued proportion,

and let the unit  $E$  be set out,

I say that, as many numbers as fall between  $A, B$  in continued proportion so many will also fall between either of the numbers  $A, B$  and the unit in continued proportion

For let two numbers  $F, G$ , the least that are in the ratio of  $A, C, D, B$ , be taken

three numbers  $H, K, L$  with the same property,  
and others more by one continually, until their multitude is equal to the multitude of  $A, C, D, B$  [VIII 2]



Let them be taken and let them be  $M, N, O, P$

It is now manifest that  $F$  by multiplying itself has made  $H$  and by multiplying  $H$  has made  $M$ , while  $G$  by multiplying itself has made  $L$  and by multiplying  $L$  has made  $P$  [VIII 2, For]

And, since  $M, N, O, P$  are the least of those which have the same ratio with  $F, G$

and  $A, C, D, B$  are also the least of those which have the same ratio with  $F, G$ , [VIII 1]

while the multitude of the numbers  $M, N, O, P$  is equal to the multitude of the numbers  $A, C, D, B$

therefore  $M, N, O, P$  are equal to  $A, C, D, B$  respectively,

therefore  $M$  is equal to  $A$  and  $P$  to  $B$

Now, since  $F$  by multiplying itself has made  $H$ ,

therefore  $F$  measures  $H$  according to the units in  $F$

But the unit  $E$  also measures  $F$  according to the units in it,  
therefore the unit  $E$  measures the number  $F$  the same number of times as  $F$  measures  $H$



Therefore, as the unit  $E$  is to the number  $F$ , so is  $F$  to  $H$  [VII Def 20]

Again, since  $F$  by multiplying  $H$  has made  $M$ ,

therefore  $H$  measures  $M$  according to the units in  $F$

But the unit  $E$  also measures the number  $F$  according to the units in it, therefore the unit  $E$  measures the number  $F$  the same number of times as  $H$  measures  $M$

Therefore, as the unit  $E$  is to the number  $F$ , so is  $H$  to  $M$

But it was also proved that, as the unit  $E$  is to the number  $F$ , so is  $F$  to  $H$ , therefore also, as the unit  $E$  is to the number  $F$ , so is  $F$  to  $H$ , and  $H$  to  $M$

But  $M$  is equal to  $A$ ,

therefore, as the unit  $E$  is to the number  $F$ , so is  $F$  to  $H$  and  $H$  to  $A$

For the same reason also,

as the unit  $E$  is to the number  $G$ , so is  $G$  to  $L$  and  $L$  to  $B$

Therefore as many numbers as have fallen between  $A$ ,  $B$  in continued proportion, so many numbers also have fallen between each of the numbers  $A$ ,  $B$  and the unit  $E$  in continued proportion Q E D

### PROPOSITION 10

*If numbers fall between each of two numbers and an unit in continued proportion however many numbers fall between each of them and an unit in continued proportion so many also will fall between the numbers themselves in continued proportion*

For let the numbers  $D$ ,  $E$  and  $F$ ,  $G$  respectively fall between the two numbers  $A$ ,  $B$  and the unit  $C$  in continued proportion,

I say that as many numbers as have fallen between each of the numbers  $A$ ,  $B$  and the unit  $C$  in continued proportion so many numbers will also fall between  $A$ ,  $B$  in continued proportion

For let  $D$  by multiplying  $F$  make  $H$ , and let the numbers  $D$ ,  $F$  by multiplying  $H$  make  $K$ ,  $L$  respectively

Now since, as the unit  $C$  is to the number  $D$  so is  $D$  to  $E$ ,

therefore the unit  $C$  measures the number  $D$  the same number of times as  $D$  measures  $E$

[VII Def 20]

But the unit  $C$  measures the number  $D$  according to the units in  $D$ ,

therefore the number  $D$  also measures  $E$  according to the units in  $D$ , therefore  $D$  by multiplying itself has made  $E$

Again since as  $C$  is to the number  $D$  so is  $E$  to  $A$ , therefore the unit  $C$  measures the number  $D$  the same number of times as  $E$  measures  $A$

But the unit  $C$  measures the number  $D$  according to the units in  $D$ , therefore  $E$  also measures  $A$  according to the units in  $D$ , therefore  $D$  by multiplying  $E$  has made  $A$

For the same reason also

$F$  by multiplying itself has made  $G$  and by multiplying  $G$  has made  $B$

C—

D—

E—

F—

G—

A—

H—

H—

K—

L—

And since  $D$  by multiplying itself has made  $E$  and by multiplying  $F$  has made  $H$ ,

therefore, as  $D$  is to  $F$ , so is  $E$  to  $H$  [VII 17]

For the same reason also,

as  $D$  is to  $F$ , so is  $H$  to  $G$  [VII 18]

Therefore also, as  $E$  is to  $H$ , so is  $H$  to  $G$

Again, since  $D$  by multiplying the numbers  $E, H$  has made  $A, K$  respectively,

therefore, as  $E$  is to  $H$ , so is  $A$  to  $K$  [VII 17]

But, as  $E$  is to  $H$ , so is  $D$  to  $F$ ,

therefore also, as  $D$  is to  $F$ , so is  $A$  to  $K$

Again, since the numbers  $D, F$  by multiplying  $H$  have made  $K, L$  respectively,

therefore, as  $D$  is to  $F$ , so is  $K$  to  $L$  [VII 18]

But, as  $D$  is to  $F$ , so is  $A$  to  $K$ ,

therefore also, as  $A$  is to  $K$ , so is  $K$  to  $L$

Further, since  $F$  by multiplying the numbers  $H, G$  has made  $L, B$  respectively,

therefore, as  $H$  is to  $G$ , so is  $L$  to  $B$ , [VII 17]

But, as  $H$  is to  $G$ , so is  $D$  to  $F$ ,

therefore also, as  $D$  is to  $F$ , so is  $L$  to  $B$

But it was also proved that,

as  $D$  is to  $F$ , so is  $A$  to  $K$  and  $K$  to  $L$ ,

therefore also, as  $A$  is to  $K$ , so is  $K$  to  $L$  and  $L$  to  $B$

Therefore  $A, K, L, B$  are in continued proportion

Therefore, as many numbers as fall between each of the numbers  $A, B$  and the unit  $C$  in continued proportion, so many also will fall between  $A, B$  in continued proportion

Q E D

### PROPOSITION 11

*Between two square numbers there is one mean proportional number, and the square has to the square the ratio duplicate of that which the side has to the side*

Let  $A, B$  be square numbers

and let  $C$  be the side of  $A$ , and  $D$  of  $B$ ,

I say that between  $A, B$  there is one mean proportional number, and  $A$  has to  $B$  the ratio duplicate of that which  $C$  has to  $D$

$A$  \_\_\_\_\_

$B$  \_\_\_\_\_

$C$  \_\_\_\_\_  $D$  \_\_\_\_\_

$E$  \_\_\_\_\_

For let  $C$  by multiplying  $D$  make  $E$

Now, since  $A$  is a square and  $C$  is its side,

therefore  $C$  by multiplying itself has made  $A$

For the same reason also

$D$  by multiplying itself has made  $B$

Since, then,  $C$  by multiplying the numbers  $C, D$  has made  $A, E$  respectively,

therefore, as  $C$  is to  $D$ , so is  $A$  to  $E$  [VII 17]

For the same reason also,

as  $C$  is to  $D$ , so is  $E$  to  $B$

[VII 18]

Therefore also, as  $A$  is to  $E$ , so is  $E$  to  $B$

Therefore between  $A, B$  there is one mean proportional number

I say next that  $A$  also has to  $B$  the ratio duplicate of that which  $C$  has to  $D$

For, since  $A, E, B$  are three numbers in proportion,

therefore  $A$  has to  $B$  the ratio duplicate of that which  $A$  has to  $E$  [v Def 9]

But, as  $A$  is to  $E$ , so is  $C$  to  $D$

Therefore  $A$  has to  $B$  the ratio duplicate of that which the side  $C$  has to  $D$   
Q E D

### PROPOSITION 12

*Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side*

Let  $A, B$  be cube numbers,

and let  $C$  be the side of  $A$ , and  $D$  of  $B$ ,

I say that between  $A, B$  there are two mean proportional numbers, and  $A$  has to  $B$  the ratio triplicate of that which  $C$  has to  $D$

For let  $C$  by multiplying itself make  $E$ , and by multiplying  $D$  let it make  $F$ ,

let  $D$  by multiplying itself make  $G$ ,

and let the numbers  $C, D$  by multiplying  $F$  make  $H, K$  respectively



Now, since  $A$  is a cube, and  $C$  its side,

and  $C$  by multiplying itself has made  $E$ ,

therefore  $C$  by multiplying itself has made  $E$  and by multiplying  $E$  has made  $A$

For the same reason also

$D$  by multiplying itself has made  $G$  and by multiplying  $G$  has made  $B$

And, since  $C$  by multiplying the numbers  $C, D$  has made  $E, F$  respectively,

therefore, as  $C$  is to  $D$ , so is  $E$  to  $F$ . [vii 17]

For the same reason also,

as  $C$  is to  $D$ , so is  $F$  to  $G$

[vii 18]

Again since  $C$  by multiplying the numbers  $E, F$  has made  $A, H$  respectively,

therefore as  $E$  is to  $F$ , so is  $A$  to  $H$

[vii 17]

But, as  $E$  is to  $F$ , so is  $C$  to  $D$

Therefore also, as  $C$  is to  $D$ , so is  $A$  to  $H$

Again, since the numbers  $C, D$  by multiplying  $F$  have made  $H, K$  respectively,

therefore, as  $C$  is to  $D$ , so is  $H$  to  $K$

[vii 18]

Again since  $D$  by multiplying each of the numbers  $F, G$  has made  $K, B$  respectively,

therefore, as  $F$  is to  $G$ , so is  $K$  to  $B$

[vii 17]

But, as  $F$  is to  $G$ , so is  $C$  to  $D$ ,

therefore also as  $C$  is to  $D$ , so is  $A$  to  $H$ ,  $H$  to  $K$ , and  $K$  to  $B$

Therefore  $H, K$  are two mean proportionals between  $A, B$

I say next that  $A$  also has to  $B$  the ratio triplicate of that which  $C$  has to  $D$

For, since  $A, H, K, B$  are four numbers in proportion

therefore  $A$  has to  $B$  the ratio triplicate of that which  $A$  has to  $H$  [v Def 10]

But, as  $A$  is to  $H$ , so is  $C$  to  $D$ ,

therefore  $A$  also has to  $B$  the ratio triplicate of that which  $C$  has to  $D$   
Q E D

## PROPOSITION 13

If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional, and, if the original numbers by multiplying the products make certain numbers, the latter will also be proportional.

Let there be as many numbers as we please,  $A, B, C$ , in continued proportion, so that, as  $A$  is to  $B$ , so is  $B$  to  $C$ , let  $A, B, C$  by multiplying themselves make  $D, E, F$ , and by multiplying  $D, E, F$  let them make  $G, H, K$ ,

I say that  $D, E, F$  and  $G, H, K$  are in continued proportion

A _____	G _____
B _____	H _____
C _____	K _____
D _____	
E _____	M _____
F _____	N _____
L _____	P _____
O _____	Q _____

For let  $A$  by multiplying  $B$  make  $L$ ,

and let the numbers  $A, B$  by multiplying  $L$  make  $M, N$  respectively

And again let  $B$  by multiplying  $C$  make  $O$ ,

and let the numbers  $B, C$  by multiplying  $O$  make  $P, Q$  respectively.

Then, in manner similar to the foregoing, we can prove that

$D, L, E$  and  $G, M, N, H$  are continuously proportional in the ratio of  $A$  to  $B$ , and further  $E, O, F$  and  $H, P, Q, K$  are continuously proportional in the ratio of  $B$  to  $C$

Now, as  $A$  is to  $B$ , so is  $B$  to  $C$ ,

therefore  $D, L, E$  are also in the same ratio with  $E, O, F$ ,

and further  $G, M, N, H$  in the same ratio with  $H, P, Q, K$

And the multitude of  $D, L, E$  is equal to the multitude of  $E, O, F$  and that of  $G, M, N, H$  to that of  $H, P, Q, K$ ,

therefore, *ex aequali*,

as  $D$  is to  $E$ , so is  $E$  to  $F$ ,

and,

as  $G$  is to  $H$ , so is  $H$  to  $K$ .

[VII 14]

Q E D

## PROPOSITION 14

If a square measure a square, the side will also measure the side, and, if the side measure the side, the square will also measure the square

Let  $A, B$  be square numbers, let  $C, D$  be their sides and let  $A$  measure  $B$ ,

I say that  $C$  also measures  $D$

A \_\_\_\_\_

B \_\_\_\_\_

C \_\_\_\_\_ D \_\_\_\_\_

E \_\_\_\_\_

For let  $C$  by multiplying  $D$  make  $E$ ,

therefore  $A, E, B$  are continuously proportional in the ratio of  $C$  to  $D$  [VIII 11]

And since  $A, E, B$  are continuously proportional, and  $A$  measures  $B$ ,

therefore  $A$  also measures  $E$

Q E D

therefore  $A$  has to  $B$  the ratio duplicate of that which  $A$  has to  $E$  [v Def 9]

But, as  $A \equiv$  to  $E$ , so  $\equiv C$  to  $D$

Therefore  $A$  has to  $B$  the ratio duplicate of that which the side  $C$  has to  $D$   
Q E D

### PROPOSITION 12

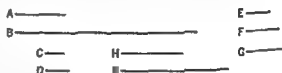
*Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side*

Let  $A, \equiv$  be cube numbers,

and let  $C$  be the side of  $A$ , and  $D$  of  $B$ ;

I say that between  $A, B$  there are two mean proportional numbers, and  $A$  has to  $B$  the ratio triplicate of that which  $C$  has to  $D$

For let  $C$  by multiplying itself make  $E$ , and by multiplying  $D$  let it make  $F$ , let  $D$  by multiplying itself make  $G$ , and let the numbers  $C, D$  by multiplying  $F$  make  $H, K$  respectively



Now, since  $A \equiv$  a cube, and  $C$  its side,

and  $C$  by multiplying itself has made  $E$ ,

therefore  $C$  by multiplying itself has made  $E$  and by multiplying  $E$  has made  $A$

For the same reason also

$D$  by multiplying itself has made  $G$  and by

And, since  $C$  by multiplying the numbers

therefore, as  $C$  is to  $D$ ,

For the same reason also,

as  $C$  is to  $D$ , so is  $F$  to  $G$

[vii 18]

Again, since  $C$  by multiplying the numbers  $E, F$  has made  $A, H$  respectively,

therefore, as  $E$  is to  $F$ , so is  $A$  to  $H$

[vii 17]

But as  $E$  is to  $F$ , so is  $C$  to  $D$

Therefore also as  $C$  is to  $D$ , so is  $A$  to  $H$

Again, since the numbers  $C, D$  by multiplying  $F$  have made  $H, K$  respectively,

therefore, as  $C$  is to  $D$ , so is  $H$  to  $K$ .

[vii 18]

Again since  $D$  by multiplying each of the numbers  $F, G$  has made  $A, B$  respectively,

therefore, as  $F$  is to  $G$ , so is  $K$  to  $B$

[vii 17]

But, as  $F$  is to  $G$ , so is  $C$  to  $D$ ,

therefore also as  $C$  is to  $D$ , so is  $A$  to  $H$ ,  $H$  to  $K$ , and  $K$  to  $B$

Therefore  $H, K$  are two mean proportionals between  $A, B$

I say next that  $A$  also has to  $B$  the ratio triplicate of that which  $C$  has to  $D$

For, since  $A, H, K, B$  are four numbers in proportion, therefore  $A$  has to  $B$  the ratio triplicate of that which  $A$  has to  $H$  [v Def 10]

But, as  $A$  is to  $H$ , so  $\equiv C$  to  $D$ ,

therefore  $A$  also has to  $B$  the ratio triplicate of that which  $C$  has to  $D$

Q E D

## PROPOSITION 13

If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional, and, if the original numbers by multiplying the products make certain numbers, the latter will also be proportional.

Let there be as many numbers as we please,  $A, B, C$ , in continued proportion, so that, as  $A$  is to  $B$ , so is  $B$  to  $C$ , let  $A, B, C$  by multiplying themselves make  $D, E, F$ , and by multiplying  $D, E, F$  let them make  $G, H, K$ ,

I say that  $D, E, F$  and  $G, H, K$  are in continued proportion

A	_____	G	_____
B	_____	H	_____
C	_____	K	_____
D	_____		
E	_____	M	_____
F	_____	N	_____
L	_____	P	_____
O	_____	Q	_____

For let  $A$  by multiplying  $B$  make  $L$ ,

and let the numbers  $A, B$  by multiplying  $L$  make  $M, N$  respectively

And again let  $B$  by multiplying  $C$  make  $O$ ,

and let the numbers  $B, C$  by multiplying  $O$  make  $P, Q$  respectively

Then, in manner similar to the foregoing, we can prove that

$D, L, E$  and  $G, M, N, H$  are continuously proportional in the ratio of  $A$  to  $B$ , and further  $E, O, F$  and  $H, P, Q, K$  are continuously proportional in the ratio of  $B$  to  $C$

Now, as  $A$  is to  $B$ , so is  $B$  to  $C$ ,

therefore  $D, L, E$  are also in the same ratio with  $E, O, F$ ,

and further  $G, M, N, H$  in the same ratio with  $H, P, Q, K$

And the multitude of  $D, L, E$  is equal to the multitude of  $E, O, F$  and that of  $G, M, N, H$  to that of  $H, P, Q, K$ ,

therefore, *ex aequali*,

as  $D$  is to  $E$ , so is  $E$  to  $F$ ,

as  $G$  is to  $H$ , so is  $H$  to  $K$

and,

[VII 14]

Q E D

## PROPOSITION 14

If a square measure a square, the side will also measure the side and, if the side measure the side, the square will also measure the square

Let  $A, B$  be square numbers, let  $C, D$  be their sides and let  $A$  measure  $B$ ,

I say that  $C$  also measures  $D$

For let  $C$  by multiplying  $D$  make  $E$ ,

therefore  $A, E, B$  are continuously proportional in the ratio of  $C$  to  $D$  [VIII 11]

And since  $A, E, B$  are continuously proportional, and  $A$  measures  $B$ ,

therefore  $A$  also measures  $E$

[VII 71]

And, as  $A$  is to  $E$ , so is  $C$  to  $D$ .

therefore also  $C$  measures  $D$

[511 Def 20]

Again, let  $C$  measure  $D$ .

I say that  $A$  also measures  $B$

For, with the same construction, we can in a similar manner prove that  $A$ ,  $E$ ,  $B$  are continuously proportional in the ratio of  $C$  to  $D$ .

And since, as  $C$  is to  $D$ , so is  $A$  to  $E$ ,

and  $C$  measures  $D$ .

therefore  $A$  also measures  $E$

[311 Def 20]

And  $A, E, B$  are continuously proportional,

therefore  $A$  also measures  $B$

Therefore etc

Q E D

### PROPOSITION 15

*If a cube number measure a cube number, the side will also measure the side, and, if the side measure the side, the cube will also measure the cube*

For let the cube number  $A$  measure the cube  $B$ .

and let  $C$  be the side of  $A$  and  $D$  of  $B$ :

I say that  $C$  measures  $D$

For let  $C'$

Now it is manifest that  $E, F,$   
 $\blacksquare$  and  $A, H, K, B$  are continuously proportional in the ratio  
of  $C$  to  $D$  [VIII 11. 12]

And since  $A, H, K, B$  are continuously proportional,

A—

**B**\_\_\_\_\_

C- \_\_\_\_\_ H- \_\_\_\_\_

D\_\_\_\_\_ K\_\_\_\_\_

E—

**G\_\_\_\_\_**

F—

and  $A$  measures  $B$ ,  
therefore it also measures  $H$

[viii 7]

And, as  $A$  is to  $H$ , so is  $C$  to  $D$ .

therefore  $C$  also measures  $D$

[VR Def 20]

Next, let  $C$  measure  $D$ .

I say that  $A$  will also measure  $B$

For, with the same construction, we can prove in a similar manner that  $A, H, K, B$  are continuously proportional in the ratio of  $C$  to  $D$ .

And, since  $C$  measures  $D$ ,

and, as C is to D, so is A to E.

therefore  $A$  also measures  $H$ .

[VI Def 20]

so that  $A$  measures  $B$  also

**Q E D**

### PROPOSITION 16

*If a square number do not measure a square number, neither will the side measure the side, and, if the side do not measure the side, neither will the square measure the square*

Let  $A, B$  be square numbers, and let  $C, D$  be their sides, and let  $A$  not measure  $B$ .

A \_\_\_\_\_ I say that neither does  $C$  measure  $D$   
 B \_\_\_\_\_ For, if  $C$  measures  $D$ ,  $A$  will also measure  $B$  [VIII 14]  
 C \_\_\_\_\_ But  $A$  does not measure  $B$ ,  
 D \_\_\_\_\_ therefore neither will  $C$  measure  $D$   
 Again let  $C$  not measure  $D$ ,  
 I say that neither will  $A$  measure  $B$   
 For if  $A$  measures  $B$   $C$  will also measure  $D$  [VIII 14]  
 But  $C$  does not measure  $D$ ,  
 therefore neither will  $A$  measure  $B$  Q E D

## PROPOSITION 17

*If a cube number do not measure a cube number, neither will the side measure the side, and if the side do not measure the side, neither will the cube measure the cube*

For let the cube number  $A$  not measure the cube number  $B$ ,  
 and let  $C$  be the side of  $A$ , and  $D$  of  $B$ ,  
 A \_\_\_\_\_ I say that  $C$  will not measure  $D$   
 B \_\_\_\_\_ For if  $C$  measures  $D$ ,  $A$  will also measure  $B$  [VIII 15]  
 C \_\_\_\_\_ But  $A$  does not measure  $B$ ,  
 D \_\_\_\_\_ therefore neither does  $C$  measure  $D$   
 Again let  $C$  not measure  $D$ ,  
 I say that neither will  $A$  measure  $B$   
 For if  $A$  measures  $B$ ,  $C$  will also measure  $D$  [VIII 15]  
 But  $C$  does not measure  $D$ ,  
 therefore neither will  $A$  measure  $B$  Q E D

## PROPOSITION 18

*Between two similar plane numbers there is one mean proportional number and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side*

Let  $A$ ,  $B$  be two similar plane numbers and let the numbers  $C$ ,  $D$  be the sides of  $A$ , and  $E$   $F$  of  $B$

A _____	C _____
B _____	D _____
	E _____
G _____	F _____

Now since similar plane numbers are those which have their sides proportional [VII Def 21]

therefore as  $C$  is to  $D$  so is  $E$  to  $F$

I say then that between  $A$ ,  $B$  there is one mean proportional number and  $A$  has to  $B$  the ratio duplicate of that which  $C$  has to  $E$  or  $D$  to  $F$ , that is of that which the corresponding side has to the corresponding side

Now since as  $C$  is to  $D$  so is  $E$  to  $F$

therefore alternately as  $C$  is to  $E$  so is  $D$  to  $F$  [VII 13]

And since  $A$  is plane and  $C$ ,  $D$  are its sides

therefore  $D$  by multiplying  $C$  has made  $A$

For the same reason also



*E* by multiplying *F* has made *B*

Now let *D* by multiplying *E* make *G*

Then, since *D* by multiplying *C* has made *A*, and by multiplying *E* has made *G*,

therefore, as *C* is to *E*, so is *A* to *G* [VII 17]

But, as *C* is to *E*, so is *D* to *F*,

therefore also, as *D* is to *F*, so is *A* to *G*

Again, since *E* by multiplying *D* has made *G*, and by multiplying *F* has made *B*,

therefore, as *D* is to *F*, so is *G* to *B* [VII 17]

But it was also proved that,

as *D* is to *F*, so is *A* to *G*,

therefore also, as *A* is to *G*, so is *G* to *B*

Therefore *A*, *G*, *B* are in continued proportion

Therefore between *A*, *B* there is one mean proportional number

I say next that *A* also has to *B* the ratio duplicate of that which the corresponding side has to the corresponding side, that is, of that which *C* has to *E* or *D* to *F*

For, since *A*, *G*, *B* are in continued proportion,

*A* has to *B* the ratio duplicate of that which it has to *G* [v Def 9]

And as *A* is to *G*, so is *C* to *E*, and so is *D* to *F*.

Therefore *A* also has to *B* the ratio duplicate of that which *C* has to *E* or *D* to *F*

# PROPOSITION 19

*Between two similar solid numbers there fall two mean proportional numbers, and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side*

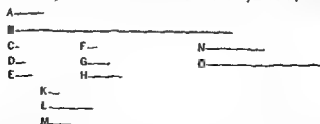
Let *A*, *B* be two similar solid numbers, and let *C*, *D*, *E* be the sides of *A*, and *F*, *G*, *H* of *B*

Now, since similar solid numbers are those which have their sides proportional,

therefore, as *C* is to *D*, so is *F* to *G*,

and, as *D* is to *E*, so is *G* to *H*

I say that between *A*, *B* there fall two mean proportional numbers, and *A* has to *B* the ratio triplicate of that which *C* has to *F*, *D* to *G*, and also *E* to *H*



For let *C* by multiplying *D* make *K*, and let *F* by multiplying *G* make *L*

Now, since *C*, *D* are in the same ratio with *F*, *G*,

and *K* is the product of *C*, *D*, and *L* the product of *F*, *G*, *K*, *L* are similar plane numbers,

[VII Def 21]

therefore between  $K, L$  there is one mean proportional number [VIII 18]

Let it be  $M$

Therefore  $M$  is the product of  $D, F$ , as was proved in the theorem preceding this [VIII 18]

Now, since  $D$  by multiplying  $C$  has made  $K$ , and by multiplying  $F$  has made  $M$ ,

therefore, as  $C$  is to  $F$ , so is  $K$  to  $M$  [VII 17]

But, as  $K$  is to  $M$ , so is  $M$  to  $L$

Therefore  $K, M, L$  are continuously proportional in the ratio of  $C$  to  $F$

And since, as  $C$  is to  $D$ , so is  $F$  to  $G$ ,

alternately therefore, as  $C$  is to  $F$ , so is  $D$  to  $G$  [VII 13]

For the same reason also,

as  $D$  is to  $G$ , so is  $E$  to  $H$

Therefore  $K, M, L$  are continuously proportional in the ratio of  $C$  to  $F$ , in the ratio of  $D$  to  $G$ , and also in the ratio of  $E$  to  $H$

Next, let  $E, H$  by multiplying  $M$  make  $N, O$  respectively

Now, since  $A$  is a solid number, and  $C, D, E$  are its sides,

therefore  $E$  by multiplying the product of  $C, D$  has made  $A$

But the product of  $C, D$  is  $K$ ,

therefore  $E$  by multiplying  $K$  has made  $A$

For the same reason also

$H$  by multiplying  $L$  has made  $B$

Now, since  $E$  by multiplying  $K$  has made  $A$ , and further also by multiplying  $M$  has made  $N$ ,

therefore as  $K$  is to  $M$ , so is  $A$  to  $N$  [VII 17]

But, as  $K$  is to  $M$ , so is  $C$  to  $F$ ,  $D$  to  $G$ , and also  $E$  to  $H$ ,

therefore also, as  $C$  is to  $F$ ,  $D$  to  $G$ , and  $E$  to  $H$ , so is  $A$  to  $N$

Again, since  $E, H$  by multiplying  $M$  have made  $N, O$  respectively,

therefore, as  $E$  is to  $H$ , so is  $N$  to  $O$  [VII 18]

But, as  $E$  is to  $H$ , so is  $C$  to  $F$  and  $D$  to  $G$ ,

therefore also, as  $C$  is to  $F$ ,  $D$  to  $G$ , and  $E$  to  $H$ , so is  $A$  to  $N$  and  $N$  to  $O$

Again, since  $H$  by multiplying  $M$  has made  $O$ , and further also by multiplying  $L$  has made  $B$ ,

therefore, as  $M$  is to  $L$ , so is  $O$  to  $B$  [VII 17]

But as  $M$  is to  $L$ , so is  $C$  to  $F$ ,  $D$  to  $G$ , and  $E$  to  $H$

Therefore also as  $C$  is to  $F$ ,  $D$  to  $G$ , and  $E$  to  $H$ , so not only is  $O$  to  $B$ , but also  $A$  to  $N$  and  $N$  to  $O$

Therefore  $A, N, O, B$  are continuously proportional in the aforesaid ratios of the sides

I say that  $A$  also has to  $B$  the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number  $C$  has to  $F$  or  $D$  to  $G$ , and also  $E$  to  $H$

For, since  $A, N, O, B$  are four numbers in continued proportion, therefore  $A$  has to  $B$  the ratio triplicate of that which  $A$  has to  $N$  [v Def 10]

But as  $A$  is to  $N$ , so it was proved that  $C$  is to  $F$ ,  $D$  to  $G$ , and also  $E$  to  $H$

Therefore  $A$  also has to  $B$  the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number  $C$  has to  $F$ ,  $D$  to  $G$ , and also  $E$  to  $H$

## PROPOSITION 20

*If one mean proportional number fall between two numbers, the numbers will be similar plane numbers*

For let one mean proportional number  $C$  fall between the two numbers  $A, B$ ,

I say that  $A, B$  are similar plane numbers

Let  $D, E$ , the least numbers of those which have the same ratio with  $A, C$ , be taken [vii 33]

therefore  $D$  measures  $A$  the same number of times that  $E$  measures  $C$  [vii 20]

Now, as many times as  $D$  measures  $A$ , so many units let there be in  $F$ ,

therefore  $F$  by multiplying  $D$  has made  $A$ ,

so that  $A$  is plane and  $D, F$  are its sides

Again, since  $D, E$  are the least of the numbers which have the same ratio with  $C, B$ ,

therefore  $D$  measures  $C$  the same number of times that  $E$  measures  $B$  [vii 20]



As many times then, as  $E$  measures  $B$ , so many units let there be in  $G$ ,

therefore  $E$  measures  $B$  according to the units in  $G$ ,

therefore  $G$  by multiplying  $E$  has made  $B$

Therefore  $B$  is plane, and  $E, G$  are its sides

Therefore  $A, B$  are similar plane numbers

I

I

made  $C$ , made  $A$ , and by multiplying  $E$  has

therefore as  $D$  is to  $E$  so is  $A$  to  $C$  that is,  $C$  to  $B$  [vii 17]

Again, since  $E$  by multiplying  $F, G$  has made  $C, B$  respectively,

therefore, as  $F$  is to  $G$  so is  $C$  to  $B$  [vii 17]

But, as  $C$  is to  $B$ , so is  $D$  to  $E$ ,

therefore also, as  $D$  is to  $E$  so is  $F$  to  $G$

And alternately, as  $D$  is to  $F$  so is  $E$  to  $G$  [vii 13]

Therefore  $A, B$  are similar plane numbers, for their sides are proportional  
Q E D

## PROPOSITION 21

*If two mean proportional numbers fall between two numbers the numbers are similar solid numbers*

For let two mean proportional numbers  $C, D$  fall between the two numbers  $A, B$ ,

I say that  $A, B$  are similar solid numbers

For let three numbers  $E, F, G$ , the least of those which have the same ratio with  $A, C, D$ , be taken, [vii 33 or viii 2]

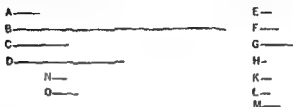
therefore the extremes of them  $E, G$  are prime to one another [VIII 3]

Now, since one mean proportional number  $F$  has fallen between  $E, G$ ,

therefore  $E, G$  are similar plane numbers [VIII 20]

Let, then,  $H, K$  be the sides of  $E$ , and  $L, M$  of  $G$

Therefore it is manifest from the theorem before this that  $E, F, G$  are continuously proportional in the ratio of  $H$  to  $L$  and that of  $K$  to  $M$



Now, since  $E, F, G$  are the least of the numbers which have the same ratio with  $A, C, D$ ,

and the multitude of the numbers  $E, F, G$  is equal to the multitude of the numbers  $A, C, D$ ,

therefore, *ex aequali*, as  $E$  is to  $G$ , so is  $A$  to  $D$  [VII 14]

But  $E, G$  are prime,

primes are also least, [VII 21]

and the least measure those which have the same ratio with them the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore  $E$  measures  $A$  the same number of times that  $G$  measures  $D$

Now, as many times as  $E$  measures  $A$ , so many units let there be in  $N$

Therefore  $N$  by multiplying  $E$  has made  $A$

But  $E$  is the product of  $H, K$ ,

therefore  $N$  by multiplying the product of  $H, K$  has made  $A$

Therefore  $A$  is solid, and  $H, K, N$  are its sides

Again, since  $E, F, G$  are the least of the numbers which have the same ratio as  $C, D, B$ ,

therefore  $E$  measures  $C$  the same number of times that  $G$  measures  $B$

Now as many times as  $E$  measures  $C$ , so many units let there be in  $O$

Therefore  $G$  measures  $B$  according to the units in  $O$ ,

therefore  $O$  by multiplying  $G$  has made  $B$

But  $G$  is the product of  $L, M$ ,

therefore  $O$  by multiplying the product of  $L, M$  has made  $B$

Therefore  $B$  is solid and  $L, M, O$  are its sides,

therefore  $A, B$  are solid

I say that they are also similar

For, since  $N, O$  by multiplying  $E$  have made  $A, C$ ,

therefore as  $N$  is to  $O$  so is  $A$  to  $C$ , that is  $E$  to  $F$  [VII 18]

But, as  $E$  is to  $F$ , so is  $H$  to  $L$  and  $K$  to  $M$ ,

therefore also as  $H$  is to  $L$  so is  $K$  to  $M$  and  $N$  to  $O$

And  $H, K, N$  are the sides of  $A$  and  $O, L, M$  the sides of  $B$

Therefore  $A, B$  are similar solid numbers

Q E D

## PROPOSITION 22

*If three numbers be in continued proportion, and the first be square, the third will also be square*

Let  $A, B, C$  be three numbers in continued proportion, and let  $A$  the first be square,

I say that  $C$  the third is also square

For, since between  $A, C$  there is one mean proportional number,  $B$ ,

therefore  $A, C$  are similar plane numbers

[VIII 20]

But  $A$  is square,

therefore  $C$  is also square

Q E D

## PROPOSITION 23

*If four numbers be in continued proportion, and the first be cube, the fourth will also be cube*

Let  $A, B, C, D$  be four numbers in continued proportion, and let  $A$  be cube,

I say that  $D$  is also cube

For since between  $A, D$  there are two mean proportional numbers  $B, C$

therefore  $A, D$  are similar solid numbers

[VIII 21]

But  $A$  is cube,

therefore  $D$  is also cube

Q E D

## PROPOSITION 24

*If two numbers have to one another the ratio which a square number has to a square number, and the first be square, the second will also be square*

For let the two numbers  $A, B$  have to one another the ratio which the square number  $C$  has to the square number  $D$  and let  $A$  be square,

I say that  $B$  is also square

For since  $C, D$  are square

$C, D$  are similar plane numbers

Therefore one mean proportional number falls between  $C, D$

[VIII 18]

And as  $C$  is to  $D$ , so is  $A$  to  $B$ ,

therefore one mean proportional number falls between  $A, B$  also

[VIII 8]

And  $A$  is square,

therefore  $B$  is also square

[VIII 22]

Q E D

## PROPOSITION 25

*If two numbers have to one another the ratio which a cube number has to a cube number, and the first be cube the second will also be cube*

For let the two numbers  $A, B$  have to one another the ratio which the cube number  $C$  has to the cube number  $D$  and let  $A$  be cube,

I say that  $B$  is also cube

For since  $C, D$  are cube

$C, D$  are similar solid numbers

Therefore two mean proportional numbers fall between  $C, D$  [VIII 19]

And, as many numbers as fall between  $C, D$  in continued proportion, so many will also fall between those which have the same ratio with them, [VIII 8] so that two mean proportional numbers fall between  $A, B$  also

A —————  
B —————  
C —————  
D —————

E —————  
F —————

Let  $E, F$  so fall

Since, then, the four numbers  $A, E, F, B$  are in continued proportion,  
and  $A$  is cube,  
therefore  $B$  is also cube

[VIII 23]

Q E D

## PROPOSITION 26

*Similar plane numbers have to one another the ratio which a square number has to a square number*

Let  $A, B$  be similar plane numbers,

I say that  $A$  has to  $B$  the ratio which a square number has to a square number

A ————— E —————  
C —————  
D ————— E ————— F —————  
F —————

$C, B$ , be taken,

[VII 33 or VIII 2]

therefore the extremes of them  $D, F$  are square [VIII 2, Por]

And since, as  $D$  is to  $F$ , so is  $A$  to  $B$ ,

and  $D, F$  are square

therefore  $A$  has to  $B$  the ratio which a square number has to a square number

Q E D

## PROPOSITION 27

*Similar solid numbers have to one another the ratio which a cube number has to a cube number*

Let  $A, B$  be similar solid numbers,

I say that  $A$  has to  $B$  the ratio which a cube number has to a cube number

A ————— E —————  
B ————— D —————  
E ————— F ————— G ————— H —————

For, since  $A, B$  are similar solid numbers  
therefore two mean proportional numbers fall between  $A, B$

[VIII 10]

Let  $C, D$  so fall,  
 and let  $E, F, G, H$ , the least numbers of those which have the same ratio with  
 $A, C, D, B$ , and equal with them in multitude, be taken, [VII 33 or VIII 2]  
 therefore the extremes of them  $E, H$  are cube [VIII 2, Por]

And, as  $E$  is to  $H$ , so is  $A$  to  $B$ ,  
 therefore  $A$  also has to  $B$  the ratio which a cube number has to a cube number

—

— Q E D

## BOOK NINE

### PROPOSITION 1

*If two similar plane numbers by multiplying one another make some number the*

<p>B _____</p> <p>C _____</p> <p>D _____</p>	<p>For let <math>A</math> by multiplying itself make <math>D</math> . . .</p> <p>Therefore <math>D</math> is square</p> <p>Since then <math>A</math> by multiplying itself has made</p> <p><math>D</math>, and by multiplying <math>B</math> has made <math>C</math>,</p> <p>therefore, as <math>A</math> is to <math>B</math>, so is <math>D</math> to <math>C</math> [VII 17]</p> <p>And, since <math>A, B</math> are similar plane numbers,</p> <p>therefore one mean proportional number falls between them</p>
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. . . . . [VIII 8]

ratio,

so that one mean proportional number falls between  $D, C$  also

And  $D$  is square,

therefore  $C$  is also square [VIII 22]

Q E D

### PROPOSITION 2

*If two numbers by multiplying one another make a square number, they are similar plane numbers*

Let  $A, B$  be two numbers, and let  $A$  by multiplying  $B$  make the square number  $C$ ,

<p>A _____</p> <p>B _____</p> <p>C _____</p> <p>D _____</p>	<p>I say that <math>A, B</math> are similar plane numbers</p> <p>For let <math>A</math> by multiplying itself make <math>D</math>,</p> <p>therefore <math>D</math> is square</p> <p>Now, since <math>A</math> by multiplying itself has</p> <p>made <math>D</math>, and by multiplying <math>B</math> has made <math>C</math></p> <p>therefore as <math>A</math> is to <math>B</math>, so is <math>D</math> to <math>C</math> [VII 17]</p>
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And, since  $D$  is square, and  $C$  is so also,

therefore  $D, C$  are similar plane numbers

Therefore one mean proportional number falls between  $D, C$  [VIII 18]

And, as  $D$  is to  $C$ , so is  $A$  to  $B$ ,

therefore one mean proportional number falls between  $A, B$  also [VIII 8]

But, if one mean proportional number fall between two numbers they are similar plane numbers, [VIII 20]

therefore  $A, B$  are similar plane numbers Q E D



## PROPOSITION 3

*If a cube number by multiplying itself make some number, the product will be cube*

For let the cube number  $A$  by multiplying itself make  $B$ ,

I say that  $B$  is cube

For let  $C$ , the side of  $A$ , be taken and let  $C$  by multiplying itself make  $D$

$A$  —————  
 $B$  —————  
 $C$  —————  
 $D$  —————

units in it,

therefore, as the unit is to  $C$ , so is  $C$  to  $D$  [VII Def 20]

Again, since  $C$  by multiplying  $D$  has made  $A$ ,

therefore  $D$  measures  $A$  according to the units in  $C$

But the unit also measures  $C$  according to the units in it,

therefore, as the unit is to  $C$ , so is  $D$  to  $A$

But as the unit is to  $C$ , so is  $C$  to  $D$ ,

therefore also as the unit is to  $C$ , so is  $C$  to  $D$  and  $D$  to  $A$

Therefore between the unit and the number  $A$  two mean proportional numbers  $C, D$  have fallen in continued proportion

Again, since  $A$  by multiplying itself has made  $B$ ,

therefore  $A$  measures  $B$  according to the units in itself

But the unit also measures  $A$  according to the units in it,

therefore as the unit is to  $A$ , so is  $A$  to  $B$  [VII Def 20]

But between the unit and  $A$  two mean proportional numbers have fallen, therefore two mean proportional numbers will also fall between  $A, B$  [VIII 8]

But if two mean proportional numbers fall between two numbers and the first be cube the second will also be cube [VIII 23]

And  $A$  is cube,

therefore  $B$  is also cube

Q E D

## PROPOSITION 4

*If a cube number by multiplying a cube number make some number, the product will be cube*

For let the cube number  $A$  by multiplying the cube number  $B$  make  $C$ ,

I say that  $C$  is cube

For let  $A$  by multiplying itself make  $D$ ,

therefore  $D$  is cube [IX 3]

And, since  $A$  by multiplying itself has made  $D$ , and by multiplying  $B$  has made  $C$

therefore as  $A$  is to  $B$  so is  $D$  to  $C$

[VII 17]

And, since  $A, B$  are cube numbers

$A, B$  are similar solid numbers

Therefore two mean proportional numbers fall between  $A, B$ , [VIII 19]  
 so that two mean proportional numbers will fall between  $D, C$  also [VIII 8]

And  $D$  is cube,

therefore  $C$  is also cube

[VIII 23]

Q E D

## PROPOSITION 5

*If a cube number by multiplying any number make a cube number, the multiplied number will also be cube*

For let the cube number  $A$  by multiplying any number  $B$  make the cube number  $C$ ,

$A$  \_\_\_\_\_  
 $B$  \_\_\_\_\_  
 $C$  \_\_\_\_\_  
 $D$  \_\_\_\_\_

I say that  $B$  is cube

For let  $A$  by multiplying itself make  $D$ ,  
 therefore  $D$  is cube [ix 3]

Now, since  $A$  by multiplying itself has made  $D$  and by multiplying  $B$  has made  $C$ ,

therefore, as  $A$  is to  $B$ , so is  $D$  to  $C$  [vii 17]

And since  $D$ ,  $C$  are cube

they are similar solid numbers

Therefore two mean proportional numbers fall between  $D$ ,  $C$  [viii 19]

And as  $D$  is to  $C$ , so is  $A$  to  $B$ ,

therefore two mean proportional numbers fall between  $A$ ,  $B$  also [viii 8]

And  $A$  is cube,

therefore  $B$  is also cube [viii 23]

## PROPOSITION 6

*If a number by multiplying itself make a cube number, it will itself also be cube*

For let the number  $A$  by multiplying itself make the cube number  $B$ ,

I say that  $A$  is also cube

$A$  \_\_\_\_\_

For let  $A$  by multiplying  $B$  make  $C$

$B$  \_\_\_\_\_

Since then  $A$  by multiplying itself has made  $B$ , and

$C$  \_\_\_\_\_

by multiplying  $B$  has made  $C$ ,

therefore  $C$  is cube

And since  $A$  by multiplying itself has made  $B$

therefore  $A$  measures  $B$  according to the units in itself

But the unit also measures  $A$  according to the units in it

Therefore, as the unit is to  $A$ , so is  $A$  to  $B$  [vii Def 20]

And since  $A$  by multiplying  $B$  has made  $C$ ,

therefore  $B$  measures  $C$  according to the units in  $A$

But the unit also measures  $A$  according to the units in it

Therefore as the unit is to  $A$ , so is  $B$  to  $C$  [vii Def 20]

But as the unit is to  $A$ , so is  $A$  to  $B$ ,

therefore also as  $A$  is to  $B$ , so is  $B$  to  $C$

And since  $B$ ,  $C$  are cube,

they are similar solid numbers

Therefore there are two mean proportional numbers between  $B$ ,  $C$  [viii 19]

And as  $B$  is to  $C$  so is  $A$  to  $B$

Therefore there are two mean proportional numbers between  $A$ ,  $B$  also

[viii 8]

And  $B$  is cube,

therefore  $A$  is also cube

[cf viii 23]

Q E D

## PROPOSITION 7

If a composite number by multiplying any number make some number, the product will be solid

For, let the composite number  $A$  by multiplying any number  $B$  make  $C$ ,

I say that  $C$  is solid

For, since  $A$  is composite, it will be measured by some number

[VII Def 13]

Let it be measured by  $D$ ,  
and, as many times as  $D$  measures  $A$ ,  
so many units let there be in  $E$



Since, then,  $D$  measures  $A$  according to the units in  $E$ ,

therefore  $E$  by multiplying  $D$  has made  $A$  [VII Def 15]

And, since  $A$  by multiplying  $B$  has made  $C$ ,

and  $A$  is the product of  $D$ ,  $E$ ,

therefore the product of  $D$ ,  $E$  by multiplying  $B$  has made  $C$

Therefore  $C$  is solid, and  $D$ ,  $E$ ,  $B$  are its sides

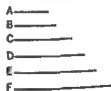
Q E D

## PROPOSITION 8

If as many numbers as we please beginning from an unit be in continued proportion the third from the unit will be square, as will also those which successively leave out one, the fourth will be cube, as will also all those which leave out two, and the seventh will be at once cube and square, as will also those which leave out five

Let there be as many numbers as we please,  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ , beginning from an unit and in continued proportion,

I say that  $B$ , the third from the unit, is square as are also all those which leave out one,  $C$ , the fourth, is cube, as are also all those which leave out two, and  $F$ , the seventh is at once cube and square, as are also all those which leave out five



For since as the unit is to  $A$ , so is  $A$  to  $B$ ,

therefore the unit measures the number  $A$  the same number of times that  $A$  measures  $B$  [VII Def 20]

But the unit measures the number  $A$  according to the units in it,

therefore  $A$  also measures  $B$  according to the units in  $A$

Therefore  $A$  by multiplying itself has made  $B$ ,

therefore  $B$  is square

And since  $B$ ,  $C$ ,  $D$  are in continued proportion and  $B$  is square,

therefore  $D$  is also square [VIII 22]

For the same reason

$F$  is also square

Similarly we can prove that all those which leave out one are square

I say next that  $C$  the fourth from the unit, is cube as are also all those which leave out two

For since, as the unit is to  $A$ , so is  $B$  to  $C$

therefore the unit measures the number  $A$  the same number of times that  $B$  measures  $C$

But the unit measures the number  $A$  according to the units in  $A$ ,

therefore  $B$  also measures  $C$  according to the units in  $A$

Therefore  $A$  by multiplying  $B$  has made  $C$

Since then  $A$  by multiplying itself has made  $B$ , and by multiplying  $B$  has made  $C$ ,

therefore  $C$  is cube

And, since  $C, D, E, F$  are in continued proportion, and  $C$  is cube,

therefore  $F$  is also cube [VIII 23]

But it was also proved square,

therefore the seventh from the unit is both cube and square

Similarly we can prove that all the numbers which leave out five are also both cube and square Q E D

### PROPOSITION 9

*If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be square, all the rest will also be square And, if the number after the unit be cube, all the rest will also be cube*

Let there be as many numbers as we please,  $A, B, C, D, E, F$ , beginning from an unit and in continued proportion, and let  $A$ , the number after the unit, be square,

I say that all the rest will also be square

Now it has been proved that  $B$ , the third from the unit, is square, as are also all those which leave out one, [IX 8]

I say that all the rest are also square

For, since  $A, B, C$  are in continued proportion,

and  $A$  is square,

therefore  $C$  is also square [VIII 22]

Again, since  $B, C, D$  are in continued proportion,

and  $B$  is square,

$D$  is also square [VIII 22]

Similarly we can prove that all the rest are also square

Next, let  $A$  be cube,

I say that all the rest are also cube

Now it has been proved that  $C$ , the fourth from the unit, is cube, as also are all those which leave out two, [IX 8]

I say that all the rest are also cube

For, since, as the unit is to  $A$ , so is  $A$  to  $B$ ,

therefore the unit measures  $A$  the same number of times as  $A$  measures  $B$

But the unit measures  $A$  according to the units in it,

therefore  $A$  also measures  $B$  according to the units in itself,

therefore  $A$  by multiplying itself has made  $B$

And  $A$  is cube

But, if a cube number by multiplying itself make some number, the product

is cube [IX 3]

Therefore  $B$  is also cube

And, since the four numbers  $A, B, C, D$  are in continued proportion

and  $A$  is cube,

$D$  also is cube [VIII 23]

For the same reason

$E$  is also cube, and similarly all the rest are cube. Q. E. D.

### PROPOSITION 10

after the unit be not cube, neither will any other be cube except the fourth from the unit and all those which leave out two.

Let there be as many numbers as we please,  $A, B, C, D, E, F$ , beginning from an unit and in continued proportion,

and let  $A$ , the number after the unit, not be square;

I say that neither will any other be square except the third from the unit <and those which leave out one>.

For, if possible, let  $C$  be square

But  $B \equiv$  also square;

[ix. 8]

[therefore  $B, C$  have to one another the ratio which a square number has to a square number].

And, as  $B$  is to  $C$ , so is  $A$  to  $B$ ;

therefore  $A, B$  have to one another the ratio which a square number has to a square number;

[so that  $A, B$  are similar plane numbers]

And  $B \equiv$  square,

therefore  $A$  is also square;

which is contrary to the hypothesis.

Therefore  $C$  is not square

Similarly we can prove that neither is any other of the numbers square except the third from the unit and those which leave out one

Next, let  $A$  not be cube

I say that neither will any other be cube except the fourth from the unit and those which leave out two.

For, if possible, let  $D$  be cube

Now  $C$  is also cube, for it is fourth from the unit

[ix. 6]

And, as  $C$  is to  $D$ , so is  $B$  to  $C$ ,

therefore  $B$  also has to  $C$  the ratio which a cube has to a cube

And  $C$  is cube,

therefore  $B$  is also cube

[viii. 25]

And since, as the unit  $\equiv$  to  $A$ , so is  $A$  to  $B$ ,

and the unit measures  $A$  according to the units in it,

therefore  $A$  also measures  $B$  according to the units in itself;

therefore  $A$  by multiplying itself has made the cube number  $B$

But, if a number by multiplying itself make a cube number, it is also itself cube

[ix. 6]

Therefore  $A$  is also cube

which is contrary to the hypothesis

Therefore  $D$  is not cube

Similarly we can prove that neither is any other of the numbers cube except the fourth from the unit and those which leave out two

Q. E. D.

## PROPOSITION 11

*If as many numbers as we please beginning from an unit be in continued proportion the less measures the greater according to some one of the numbers which have place among the proportional numbers*

Let there be as many numbers as we please,  $B, C, D, E$ , beginning from the unit  $A$  and in continued proportion,

$A$  ——— I say that  $B$ , the least of the numbers  $B, C, D, E$ ,  
measures  $E$  according to some one of the numbers  $C, D$

$B$  ——— For since, as the unit  $A$  is to  $B$ , so is  $D$  to  $E$ ,  
 $C$  ——— therefore the unit  $A$  measures the number  $B$  the same  
 $D$  ——— number of times as  $D$  measures  $E$ ,

$E$  ——— therefore, alternately, the unit  $A$  measures  $D$  the same  
number of times as  $B$  measures  $E$  [VII 15]

But the unit  $A$  measures  $D$  according to the units in it,  
therefore  $B$  also measures  $E$  according to the units in  $D$ ,  
so that  $B$  the less measures  $E$  the greater according to some number of those  
which have place among the proportional numbers —

ber before it —

Q E D

## PROPOSITION 12

*If as many numbers as we please beginning from an unit be in continued proportion, by however many prime numbers the last is measured, the next to the unit will also be measured by the same*

Let there be as many numbers as we please,  $A, B, C, D$ , beginning from an unit, and in continued proportion,

I say that, by however many prime numbers  $D$  is measured,  $A$  will also be measured by the same

For let  $D$  be measured by any prime number  $E$ ,

$A$ ———	$F$ ———	I say that $E$ measures $A$
$B$ ———	$G$ ———	For suppose it does not,
$C$ ———	$H$ ———	now $E$ is prime, and any prime
$D$ ———		number is prime to any which it
$E$ ———		does not measure, [XII 29]

therefore  $E, A$  are prime to one another

And since  $E$  measures  $D$  let it measure it according to  $F$ ,

therefore  $E$  by multiplying  $F$  has made  $D$

Again since  $A$  measures  $D$  according to the units in  $C$  [IX 11 and Por]

therefore  $A$  by multiplying  $C$  has made  $D$

But, further,  $E$  has also by multiplying  $F$  made  $D$ ,

therefore the product of  $A, C$  is equal to the product of  $E, F$

Therefore, as  $A$  is to  $E$ , so is  $F$  to  $C$  [VII 19]

But  $A, E$  are prime,

primes are also least [VII 21]

and the least measure those which have the same ratio the same number of  
times, the antecedent the antecedent and the consequent the consequent,

[VII 20]

therefore  $E$  measures  $C$

Let it measure it according to  $G$ ,

therefore  $E$  by multiplying  $G$  has made  $C$

But, further, by the theorem before this,

$A$  has also by multiplying  $B$  made  $C$  [IX 11 and Por]

Therefore the product of  $A$ ,  $B$  is equal to the product of  $E$ ,  $G$

Therefore, as  $A$  is to  $E$ , so is  $G$  to  $B$  [VII 19]

But  $A$ ,  $E$  are prime,

primes are also least, [VII 21]

and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent [VII 20]

therefore  $E$  measures  $B$

Let it measure it according to  $H$ ,

therefore  $E$  by multiplying  $H$  has made  $B$

But, further,  $A$  has also by multiplying itself made  $B$ , [IX 8]

therefore the product of  $E$ ,  $H$  is equal to the square on  $A$

Therefore, as  $E$  is to  $A$ , so is  $A$  to  $H$  [VII 19]

But  $A$ ,  $E$  are prime,

primes are also least, [VII 21]

and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore  $E$  measures  $A$ , as antecedent antecedent

But, again it also does not measure it

which is impossible

Therefore  $E$ ,  $A$  are not prime to one another

Therefore they are composite to one another

But numbers composite to one another are measured by some number [VII Def 14]

And, since  $E$  is by hypothesis prime,

and the prime is not measured by any number other than itself,

therefore  $E$  measures  $A$ ,  $E$ ,

so that  $E$  measures  $A$

[But it also measures  $D$ ,

therefore  $E$  measures  $A$ ,  $D$ ]

Similarly we can prove that, by however many prime numbers  $D$  is measured,  $A$  will also be measured by the same  $Q \ E \ D$

### PROPOSITION 13

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be prime, the greatest will not be measured by any except those which have a place among the proportional numbers

Let there be as many numbers as we please,  $A$ ,  $B$ ,  $C$ ,  $D$ , beginning from an unit and in continued proportion, and let  $A$  the number after the unit, be prime,

I say that  $D$ , the greatest of them, will not be measured by any other number except  $A$ ,  $B$ ,  $C$

For, if possible, let it be measured by  $E$ , and let  $E$  not be the same with any of the numbers  $A$ ,  $B$ ,  $C$

It is then manifest that  $E$  is not prime

For, if  $E$  is prime and measures  $D$ ,

it will also measure  $A$  [ix 12], which is prime, though it is not the same with it  
which is impossible

$A$  —————

$E$  ———

Therefore  $E$  is not prime

$B$  —————

$F$  —————

Therefore it is composite

$C$  —————

$G$  —————

But any composite num-

$D$  —————

$H$  —————

ber is measured by some  
prime number, [vii 31]

therefore  $E$  is measured by some prime number

I say next that it will not be measured by any other prime except  $A$

For, if  $E$  is measured by another,

and  $E$  measures  $D$ ,

that other will also measure  $D$ ,

so that it will also measure  $A$  [ix 12], which is prime, though it is not the same  
with it

which is impossible

Therefore  $A$  measures  $E$

And since  $E$  measures  $D$ , let it measure it according to  $F$

I say that  $F$  is not the same with any of the numbers  $A, B, C$

For if  $F$  is the same with one of the numbers  $A, B, C$ ,

and measures  $D$  according to  $E$ ,

therefore one of the numbers  $A, B, C$  also measures  $D$  according to  $E$

But one of the numbers  $A, B, C$  measures  $D$  according to some one of the  
numbers  $A, B, C$ , [ix 11]

therefore  $E$  is also the same with one of the numbers  $A, B, C$

which is contrary to the hypothesis

Therefore  $F$  is not the same as any one of the numbers  $A, B, C$

Similarly we can prove that  $F$  is measured by  $A$ , by proving again that  $F$  is  
not prime

$F$  —————

therefore  $F$  is not prime

Therefore it is composite

But any composite number is measured by some prime number [vii 31]

therefore  $F$  is measured by some prime number

I say next that it will not be measured by any other prime except  $A$

For, if any other prime number measures  $F$

and  $F$  measures  $D$ ,

that other will also measure  $D$ ,

so that it will also measure  $A$  [ix 12] which is prime though it is not the same  
with it

which is impossible

Therefore  $A$  measures  $F$

And since  $E$  measures  $D$  according to  $F$

therefore  $E$  by multiplying  $F$  has made  $D$

But further  $A$  has also by multiplying  $C$  made  $D$ ,

therefore the product of  $A, C$  is equal to the product of  $E, F$  [ix 11]



Therefore, proportionally, as  $A$  is to  $E$ , so is  $F$  to  $C$ .

[vii 19]

But  $A$  measures  $E$ ,

therefore  $F$  also measures  $C$

Let it measure it according to  $G$

Similarly, then we can prove that  $G$  is not the same with any of the numbers  $A, B$ , and that it is measured by  $A$

And since  $F$  measures  $C$  according to  $G$

therefore  $F$  by multiplying  $G$  has made  $C$

But, further,  $A$  has also by multiplying  $B$  made  $C$ ,

[ix 11]

therefore the product of  $A, B$  is equal to the product of  $F, G$

Therefore, proportionally, as  $A$  is to  $F$ , so is  $G$  to  $B$

[vii 19]

But  $A$  measures  $F$ ,

therefore  $G$  also measures  $B$

Let it measure it according to  $H$

Similarly then we can prove that  $H$  is not the same with  $A$

And since  $G$  measures  $B$  according to  $H$ ,

therefore  $G$  by multiplying  $H$  has made  $B$

But further  $A$  has also by multiplying itself made  $B$ ,

[ix 5]

therefore the product of  $H, G$  is equal to the square on  $A$

Therefore, as  $H$  is to  $A$ , so is  $A$  to  $G$

[vi 19]

But  $A$  measures  $G$ ,

therefore  $H$  also measures  $A$ , which is prime, though it is not the same with it  
which is absurd

Therefore  $D$  the greatest will not be measured by any other number except  
 $A, B, C$  Q E D

#### PROPOSITION 14

If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it

For let the number  $A$  be the least that is measured by the prime numbers  $B, C, D$ ,

I say that  $A$  will not be measured by any other prime number except  $B, C, D$

For if possible let it be measured by the prime number  $E$  and let  $E$  not be the same with any one of the numbers  $B, C, D$

$A$  —————

$B$  —

$E$  —————

$C$  —

$F$  —————

$D$  —

Now since  $E$  measures  $A$  let it measure it according to  $F$ ,

therefore  $E$  by multiplying  $F$  has made  $A$

And  $A$  is measured by the prime numbers  $B, C, D$

But if two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers, [vii 30]

therefore  $B, C, D$  will measure one of the numbers  $E, F$

Now they will not measure  $E$ ,

for  $E$  is prime and not the same with any one of the numbers  $B, C, D$

Therefore they will measure  $F$  which is less than  $A$

which is impossible for  $A$  is by hypothesis the least number measured by  $B, C, D$

Therefore no prime number will measure  $A$  except  $B, C, D$

Q E D

## PROPOSITION 15

If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number

Let  $A, B, C$ , three numbers in continued proportion, be the least of those which have the same ratio with them,  
 $A$  ———  $B$  ———  
 $C$  ———  
 $D$  —  $E$  —  $F$

I say that any two of the numbers  $A, B, C$  whatever added together are prime to the remaining number, namely  $A, B$  to  $C$ ,  $B, C$  to  $A$ , and further,  $A, C$  to  $B$ .

For let two numbers  $DE, EF$ , the least of those which have the same ratio with  $A, B, C$ , be taken [VIII 2]

It is then manifest that  $DE$  by multiplying itself has made  $A$ , and by multiplying  $EF$  has made  $B$ , and, further,  $EF$  by multiplying itself has made  $C$  [VIII 2]

Now, since  $DE, EF$  are least, they are prime to one another [VII 22]

But, if two numbers be prime to one another, their sum is also prime to each, [VII 28]

therefore  $DF$  is also prime to each of the numbers  $DE, EF$

But, further,  $DE$  is also prime to  $EF$ , therefore  $DF, DE$  are prime to  $EF$

But, if two numbers be prime to any number their product is also prime to the other, [VII 24]

so that the product of  $FD, DE$  is prime to  $EF$ ,

hence the product of  $FD, DE$  is also prime to the square on  $EF$  [VII 25]

But the product of  $FD, DE$  is the square on  $DE$  together with the product of  $DE, EF$ , [II 3]

therefore the square on  $DE$  together with the product of  $DE, EF$  is prime to the square on  $EF$

And the square on  $DE$  is  $A$

the product of  $DE, EF$  is  $B$ ,

and the square on  $EF$  is  $C$ ,

therefore  $A, B$  added together are prime to  $C$

Similarly we can prove that  $B, C$  added together are prime to  $A$

I say — — — — —

For

the squares on  $DE, EF$  together with twice the product of  $DE, EF$  are [VII 24, 25]

equal to the square on  $DF$ , [II 4]

therefore the squares on  $DE, EF$  together with twice the product of  $DE, EF$  are prime to the product of  $DE, EF$

*Separando*, the squares on  $DE, EF$  together with once the product of  $DE, EF$  are prime to the product of  $DE, EF$

Therefore *separando* again the squares on  $DE, EF$  are prime to the product of  $DE, EF$

And the square on  $DE$  is  $A$ ,

the product of  $DE, EF$  is  $B$ ,

and the square on  $EF$  is  $C$

Therefore  $A, C$  added together are prime to  $B$

Q E D

### PROPOSITION 16

*If two numbers be prime to one another, the second will not be to any other number as the first is to the second*

For let the two numbers  $A, B$  be prime to one another,

I say that  $B$  is not to any other number as  $A$  is to  $B$

For, if possible, as  $A$  is to  $B$ , so let  $B$  be to  $C$

Now  $A, B$  are prime,

primes are also least,

[VII 21]

and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent,

therefore  $A$  measures  $B$  as antecedent antecedent

But it also measures itself,

therefore  $A$  measures  $A, B$  which are prime to one another  
which is absurd

Therefore  $B$  will not be to  $C$ , as  $A$  is to  $B$ .

Q E D

### PROPOSITION 17

*If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the last will not be to any other number as the first to the second*

For let there be as many numbers as we please,  $A, B, C, D$ , in continued proportion

and let the extremes of them,  $A, D$ , be prime to one another,

$A$  —  $B$  —

$C$  —

I say that  $D$  is not to any other number as  $A$  is to  $B$

$D$  —

$E$  —

For, if possible as  $A$  is to  $B$ , so let  $D$  be to  $E$ ,

therefore alternately, as  $A$  is to  $D$ , so is  $B$  to  $E$

[VII 13]

But  $A, D$  are prime

primes are also least,

[VII 21]

and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent

[VII 20]

Therefore  $A$  measures  $B$

And as  $A$  is to  $B$ , so is  $B$  to  $C$

Therefore  $B$  also measures  $C$ ,

so that  $A$  also measures  $C$ .

And since, as  $B$  is to  $C$ , so is  $C$  to  $D$ ,

and  $B$  measures  $C$ ,

therefore  $C$  also measures  $D$

But  $A$  measured  $C$ ,

so that  $A$  also measures  $D$

But it also measures itself,

therefore  $A$  measures  $A, D$  which are prime to one another  
which is impossible

Therefore  $D$  will not be to any other number as  $A$  is to  $B$

Q E D

## PROPOSITION 18

*Given two numbers, to investigate whether it is possible to find a third proportional to them*

Let  $A, B$  be the given two numbers, and let it be required to investigate whether it is possible to find a third proportional to them

Now  $A, B$  are either prime to one another or not

And if they are prime to one another, it has been proved that it is impossible to find a third proportional to them [ix 16]

Next, let  $A, B$  not be prime to one another,

and let  $B$  by multiplying itself make  $C$ .

Then  $A$  either measures  $C$  or does not measure it

*F*

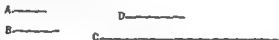
*I*

therefore the product of  $A, D$  is equal to the square on  $B$

Therefore, as  $A$  is to  $B$ , so is  $B$  to  $D$ , [vii 19]

therefore a third proportional number  $D$  has been found to  $A, B$

Next, let  $A$  not measure  $C$ ,



I say that it is impossible to find a third proportional number to  $A, B$ .

For, if possible, let  $D$ , such third proportional have been found

Therefore the product of  $A, D$  is equal to the square on  $B$

But the square on  $B$  is  $C$ ,

therefore the product of  $A, D$  is equal to  $C$

Hence  $A$  by multiplying  $D$  has made  $C$ ,

therefore  $A$  measures  $C$  according to  $D$

But, by hypothesis it also does not measure it

which is absurd

Therefore it is not possible to find a third proportional number to  $A, B$  when  $A$  does not measure  $C$  Q E D

## PROPOSITION 19

*Given three numbers, to investigate when it is possible to find a fourth proportional to them*

*A* \_\_\_\_\_

*B* \_\_\_\_\_

*C* \_\_\_\_\_

Let  $A, B, C$  be the given three numbers and let it be required to investigate when it is possible to find a fourth proportional to them

[The Greek text of this proposition is corrupt. However, analogously to Proposition 18 the condition that a fourth proportional to  $A, B, C$  exists is that  $A$  measure the product of  $B$  and  $C$ ]

## PROPOSITION 20

*Prime numbers are more than any assigned multitude of prime numbers*

Let  $A, B, C$  be the assigned prime numbers,

I say that there are more prime numbers than  $A, B, C$

For let the least number measured by  $A, B, C$  be taken,

and let it be  $DE$ ,

let the unit  $DF$  be added to  $DE$

Then  $EF$  is either prime or not

First, let it be prime,

then the prime numbers  $A, B, C, EF$  have been found which are more than  $A, B, C$

Next, let  $EF$  not be prime,

therefore it is measured by some prime number

[VII 31]

Let it be measured by the prime number  $G$

I say that  $G$  is not the same with any of the numbers  $A, B, C$ .

For, if possible, let it be so

Now  $A, B, C$  measure  $DE$ ,

therefore  $G$  also will measure  $DE$

But it also measures  $EF$

Therefore  $G$ , being a number, will measure the remainder, the unit  $DF$  which is absurd

Therefore  $G$  is not the same with any one of the numbers  $A, B, C$

And by hypothesis it is prime

Therefore the prime numbers  $A, B, C, G$  have been found which are more than the assigned multitude of  $A, B, C$

Q E D

#### PROPOSITION 21

If as many even numbers as we please be added together, the whole is even

For let as many even numbers as we please,  $AB, BC, CD, DE$ , be added together

I say that the whole  $AE$  is even

For, since each of the numbers  $AB,$

$BC, CD, DE$  is even it has a half part,



[VII Def 6]

so that the whole  $AE$  also has a half part

But an even number is that which is divisible into two equal parts, [id]

therefore  $AE$  is even

Q E D

#### PROPOSITION 22

If as many odd numbers as we please be added together, and their multitude be even, the whole will be even

For let as many odd numbers as we please,  $AB, BC, CD, DE$ , even in multitude, be added together,

I say that the whole  $AE$  is even

For, since each of the numbers

$AB, BC, CD, DE$  is odd if an unit

be subtracted from each each of

the remainders will be even,



[VII Def 7]

so that the sum of them will be even

[IX 21]

But the multitude of the units is also even

Therefore the whole  $AE$  is also even

[IX 21]

Q E D

## PROPOSITION 23

*If as many odd numbers as we please be added together, and their multitude be odd the whole will also be odd*

For let  $\equiv$  many odd numbers as we please  $AB$   $BC$   $CD$  the multitude of which is odd be added together

$A$   $B$   $C$   $E$   $D$  I say that the whole  $AD$   $\equiv$  also odd

Let the unit  $DE$  be subtracted from  $CD$

therefore the remainder  $CE$  is even [VII Def 7]

But  $CA$  is also even [IX 22]

therefore the whole  $AE$  is also even [IX 21]

And  $DE$   $\equiv$  an unit

Therefore  $AD$  is odd [VII Def 7]

Q E D

1

## PROPOSITION 24

*If from an even number an even number be subtracted the remainder will be even*

For from the even number  $AB$  let the even number  $BC$  be subtracted

$A$   $C$   $B$  I say that the remainder  $CA$  is even

For since  $AB$  is even it has a half part [VII Def 6]

For the same reason  $BC$  also has a half part

so that the remainder  $[CA$  also has a half part and]  $AC$   $\equiv$  therefore even

Q E D

## PROPOSITION 25

*If from an even number an odd number be subtracted the remainder will be odd*

For from the even number  $AB$  let the odd number  $BC$  be subtracted

$A$   $C$   $D$   $B$  I say that the remainder  $CA$  is odd

For let the unit  $CD$  be subtracted from  $BC$

therefore  $DB$  is even [VII Def 7]

But  $AB$  is also even

therefore the remainder  $AD$   $\equiv$  also even [IX 24]

And  $CD$   $\equiv$  an unit

therefore  $CA$  is odd [VII Def 7]

Q E D

## PROPOSITION 26

*If from an odd number an odd number be subtracted the remainder will be even*

For from the odd number  $AB$  let the odd number  $BC$  be subtracted

$A$   $C$   $D$   $B$  I say that the remainder  $CA$   $\equiv$  even

For since  $AB$  is odd let the unit  $BD$  be subtracted

therefore the remainder  $AD$  is even [VII Def 7]

For the same reason  $CD$   $\equiv$  also even [VII Def 7]

so that the remainder  $CA$  is also even [IX 24]

Q E D

## PROPOSITION 27

*If from an odd number an even number be subtracted the remainder will be odd*

For from the odd number  $AB$  let the even number  $BC$  be subtracted

I say that the remainder  $CA$  is odd

Let the unit  $AD$  be subtracted,

therefore  $DB$  is even [VII Def 7]

$A \quad D \quad C \quad B$

But  $BC$  is also even

therefore the remainder  $CD$  is even

[IX 24]

Therefore  $CA$  is odd

[VII Def 7]

Q E D

### PROPOSITION 28

If an odd number by multiplying an even number make some number, the product will be even

For let the odd number  $A$  by multiplying the even number  $B$  make  $C$ ,

I say that  $C$  is even

$A \quad \text{---}$

$B \quad \text{---}$

For since  $A$  by multiplying  $B$  has made  $C$ ,  
therefore  $C$  is made up of as many numbers equal to  $B$

$C \quad \text{---}$

as there are units in  $A$

[VII Def 15]

And  $B$  is even,

therefore  $C$  is made up of even numbers

But if as many even numbers as we please be added together, the whole is even

[IX 21]

Therefore  $C$  is even

Q E D

### PROPOSITION 29

If an odd number by multiplying an odd number make some number, the product will be odd

For let the odd number  $A$  by multiplying the odd number  $B$  make  $C$ ,

I say that  $C$  is odd

$A \quad \text{---}$

$B \quad \text{---}$

For since  $A$  by multiplying  $B$  has made  $C$ ,  
therefore  $C$  is made up of as many numbers equal to  $B$

$C \quad \text{---}$

as there are units in  $A$

[VII Def 15]

And each of the numbers  $A$   $B$  is odd,

therefore  $C$  is made up of odd numbers the multitude of which is odd

[IX 23]

Thus  $C$  is odd

Q E D

### PROPOSITION 30

If an odd number measure an even number, it will also measure the half of it

For let the odd number  $A$  measure the even number  $B$ ,

I say that it will also measure the half of it

$A \quad \text{---}$

For since  $A$  measures  $B$

$B \quad \text{---}$

let it measure it according to  $C$ ,

$C \quad \text{---}$

I say that  $C$  is not odd

For if possible let it be so

Then since  $A$  measures  $B$  according to  $C$ ,

therefore  $A$  by multiplying  $C$  has made  $B$

Therefore  $B$  is made up of odd numbers the multitude of which is odd

[IX 23]

Therefore  $B$  is odd

which is absurd for by hypothesis it is even

Therefore  $C$  is not odd,

therefore  $C$  is even

Thus  $A$  measures  $B$  an even number of times

For this reason then it also measures the half of it

Q E D

### PROPOSITION 31

*If an odd number be prime to any number it will also be prime to the double of it*

For let the odd number  $A$  be prime to any number  $B$ ,

and let  $C$  be double of  $B$ ,

A \_\_\_\_\_

B \_\_\_\_\_

C \_\_\_\_\_

D \_\_\_\_\_

I say that  $A$  is prime to  $C$

For if they are not prime to one another  
some number will measure them

Let a number measure them and let it be  $D$

Now  $A$  is odd

therefore  $D$  is also odd

And since  $D$  which is odd measures  $C$

and  $C$  is even

therefore  $\{D\}$  will measure the half of  $C$  also

[IX 30]

But  $B$  is half of  $C$

therefore  $D$  measures  $B$

But it also measures  $A$

therefore  $D$  measures  $A$   $B$  which are prime to one another  
which is impossible

Therefore  $A$  cannot but be prime to  $C$

Therefore  $A$   $C$  are prime to one another

Q E D

### PROPOSITION 32

*Each of the numbers which are continually doubled beginning from a dyad is even times even only*

For let as many numbers as we please  $B$   $C$   $D$  have been continually doubled beginning from the dyad  $A$

A \_\_\_\_\_

B \_\_\_\_\_

C \_\_\_\_\_

D \_\_\_\_\_

I say that  $B$   $C$   $D$  are even times even only

Now that each of the numbers  $B$   $C$   $D$  is  
even times even is manifest for it is  
doubled from a dyad

I say that it is also even times even only

For let an unit be set out

Since then as many numbers as we please beginning from an unit are in continued proportion

and the number  $A$  after the unit is prime

therefore  $D$  the greatest of the numbers  $A$   $B$   $C$   $D$  will not be measured by  
any other number except 1  $B$   $C$

[IX 13]

And each of the numbers  $A$   $B$   $C$  is even

therefore  $D$  is even times even only

[VII Def 8]

Similarly we can prove that each of the numbers  $B$   $C$  is even times even  
only

Q F D

### PROPOSITION 33

*If a number have its half odd it is even times odd only*

For let the number  $A$  have its half odd



I say that  $A$  is even-times odd only

Now that it is even-times odd is manifest, for the half of it, being odd, measures it an even number of times [VII Def 3]

I say next that it is also even-times odd only

For, if  $A$  is even times even also,  
it will be measured by an even number according to an even number,  
so that the half of it will also be measured by an even number though it is odd  
[VII Def 3]  
which is absurd

Therefore  $A$  is even-times odd only Q E D

### PROPOSITION 34

If a number neither be one of its  
half odd, it is both

For let the number  $A$  neither be doubled from a dyad, nor have  
its half odd,

I say that  $A$  is both even times even and even-times odd A

Now that  $A$  is even times even is manifest,  
for it has not its half odd [VII Def 3]

I say next that it is also even-times odd

For, if we bisect  $A$ , then bisect its half, and do this continually, we shall  
come upon some odd number which will measure  $A$  according to an even number

For if not we shall come upon a dyad,  
and  $A$  will be among those which are doubled from a dyad  
which is contrary to the hypothesis

Thus  $A$  is even times odd

But it was also proved even times even

Therefore  $A$  is both even-times even and even-times odd Q E D

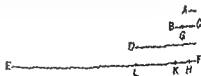
### PROPOSITION 35

If as many numbers as we please be in continued proportion, and there be subtracted from the second and the last numbers equal to the first then, as the excess of the second is to the first, so will the excess of the last be to all those before it

Let there be as many numbers as we please in continued proportion,  $A, BC,$

to  $A$

I say that as  $GC$  is to  $A$  so is  $EH$   
to  $A, BC, D$



For let  $FA$  be made equal to  $BC$ , and  $FL$  equal to  $D$

Then since  $FA$  is equal to  $BC$ ,

and of these the part  $FH$  is equal to the part  $BG$ ,

therefore the remainder  $HK$  is equal to the remainder  $GC$

And since, as  $EF$  is to  $D$  so is  $D$  to  $BC$ , and  $BC$  to  $A$ ,

while  $D$  is equal to  $FL$ ,  $BC$  to  $FK$  and  $A$  to  $FH$ ,

therefore as  $EF$  is to  $FL$  so is  $LF$  to  $FK$  and  $FK$  to  $FH$

Separando, as  $EL$  is to  $LF$  so is  $LK$  to  $FA$ , and  $KH$  to  $FH$  [VII 11, 13]

Therefore also, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, {vii 12}

therefore, as  $KH$  is to  $FH$ , so are  $EL$ ,  $LK$ ,  $KH$  to  $LF$ ,  $FK$ ,  $HF$

But  $KH$  is equal to  $CG$ ,  $FH$  to  $A$ , and  $LF$ ,  $FK$ ,  $HF$  to  $D$ ,  $BC$ ,  $A$ ,

therefore, as  $CG$  is to  $A$ , so is  $EH$  to  $D$ ,  $BC$ ,  $A$

Therefore, as the excess of the second is to the first, so is the excess of the last to all those before it. Q E D

## PROPOSITION 36

If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect

For let as many numbers as we please,  $A$ ,  $B$ ,  $C$ ,  $D$ , beginning from an unit be set out in double proportion, until the sum of all becomes prime,

let  $E$  be equal to the sum and let  $E$  by multiplying  $D$  make  $FG$ ,

I say that  $FG$  is perfect

For, however many  $A$ ,  $B$ ,  $C$ ,  $D$  are in multitude, let so many  $E$ ,  $HK$ ,  $L$ ,  $M$  be taken in double proportion beginning from  $E$ ,

therefore, ex aequali, as  $A$  is to  $D$ , so is  $E$  to  $M$  {vii 14}

Therefore the product of  $E$ ,  $D$  is equal to the product of  $A$ ,  $M$  {vii 19}

And the product of  $E$ ,  $D$  is  $FG$ ,

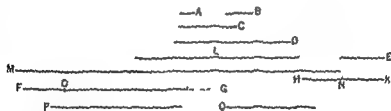
therefore the product of  $A$ ,  $M$  is also  $FG$

Therefore  $A$  by multiplying  $M$  has made  $FG$ ,

therefore  $M$  measures  $FG$  according to the units in  $A$ .

And  $A$  is a dyad,

therefore  $FG$  is double of  $M$



But  $M$ ,  $L$ ,  $HK$ ,  $E$  are continuously double of each other, therefore  $E$ ,  $HK$ ,  $L$ ,  $M$ ,  $FG$  are continuously proportional in double proportion

Now let there be subtracted from the second  $HK$  and the last  $FG$  the numbers  $HN$ ,  $FO$ , each equal to the first  $E$ ,

therefore, as the excess of the second is to the first, so is the excess of the last to all those before it {ix 35}

Therefore, as  $NK$  is to  $E$ , so is  $OG$  to  $M$ ,  $L$ ,  $HK$ ,  $E$

And  $NK$  is equal to  $E$ ,

therefore  $OG$  is also equal to  $M$ ,  $L$ ,  $HK$ ,  $E$

But  $FO$  is also equal to  $E$

and  $E$  is equal to  $A$ ,  $B$ ,  $C$ ,  $D$  and the unit

Therefore the whole  $FG$  is equal to  $E$ ,  $HK$ ,  $L$ ,  $M$  and  $A$ ,  $B$ ,  $C$ ,  $D$  and the unit,

and it is measured by them

I say also that  $FG$  will not be measured by any other number except  $A, B, C, D, E, HK, L, M$  and the unit

For, if possible, let some number  $P$  measure  $FG$ ,  
and let  $P$  not be the same with any of the numbers  $A, B, C, D, E, HK, L, M$   
And, as many times as  $P$  measures  $FG$ , so many units let there be in  $Q$ ,  
therefore  $Q$  by multiplying  $P$  has made  $FG$

But, further,  $E$  has also by multiplying  $D$  made  $FG$ ,  
therefore, as  $E$  is to  $Q$ , so is  $P$  to  $D$  [VII 19]

And, since  $A, B, C, D$  are continuously proportional beginning from an unit  
therefore  $D$  will not be measured by any other number except  $A, B, C$  [IX 13]

And, by hypothesis,  $P$  is not the same with any of the numbers  $A, B, C$ ,  
therefore  $P$  will not measure  $D$

But, as  $P$  is to  $D$ , so is  $E$  to  $Q$ ,  
therefore neither does  $E$  measure  $Q$  [VII Def 20]

And  $E$  is prime,  
and any prime number is prime to any number which it does not measure  
[VII 29]

Therefore  $E, Q$  are prime to one another

But primes are also least, [VII 21]  
and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent,  
[VII 20]

and, as  $E$  is to  $Q$ , so is  $P$  to  $D$ ,  
therefore  $E$  measures  $P$  the same number of times that  $Q$  measures  $D$   
But  $D$  is not measured by any other number except  $A, B, C$ ,  
therefore  $Q$  is the same with one of the numbers  $A, B, C$

Let it be the same with  $B$

And however many  $B, C, D$  are in multitude, let so many  $E, HK, L$  be taken beginning from  $E$

Now  $E, HK, L$  are in the same ratio with  $B, C, D$ ,  
therefore *ex aequali*, as  $B$  is to  $D$ , so is  $E$  to  $L$  [VII 14]

Therefore the product of  $B, L$  is equal to the product of  $D, E$  [VII 19]

But the product of  $D, E$  is equal to the product of  $Q, P$ ,

therefore the product of  $Q, P$  is also equal to the product of  $B, L$   
Therefore as  $Q$  is to  $B$ , so is  $L$  to  $P$  [VII 19]

And  $Q$  is the same with  $B$ ,

therefore  $L$  is also the same with  $P$

which is impossible for by hypothesis  $P$  is not the same with any of the numbers set out

Therefore no number will measure  $FG$  except  $A, B, C, D, E, HK, L, M$  and the unit

And  $FG$  was proved equal to  $A, B, C, D, E, HK, L, M$  and the unit,  
and a perfect number is that which is equal to its own parts, [VII Def 22]  
therefore  $FG$  is perfect Q E D



and let this process be repeated continually until the divisions in  $AB$  are equal in multitude with the divisions in  $DE$

Let then  $AK$   $AH$   $HB$  be divisions which are equal in multitude with  $DF$   $FG$   $GE$

Now since  $DE$  is greater than  $AB$

and from  $DE$  there has been subtracted  $EG$  less than its half

and from  $AB$   $BH$  greater than its half

therefore the remainder  $GD$  is greater than the remainder  $HA$

And since  $GD$  is greater than  $HA$

and there has been subtracted from  $GD$  the half  $GF$ ,

and from  $HA$   $HK$  greater than its half

therefore the remainder  $DF$  is greater than the remainder  $AK$

But  $DF$  is equal to  $C$ ,

therefore  $C$  is also greater than  $AK$

Therefore  $AK$  is less than  $C$

Therefore there is left of the magnitude  $AB$  the magnitude  $AK$  which is less than the lesser magnitude set out namely  $C$  Q E D

And the theorem can be similarly proved even if the parts subtracted be halves

### PROPOSITION 2

If when the less of two unequal magnitudes is continually subtracted in turn from the greater that which is left never measures the one before it the magnitudes will be incommensurable

For there being two unequal magnitudes  $AB$   $CD$  and  $AB$  being the less when the less is continually subtracted in turn from the greater let that which is left over never measure the one before it,

I say that the magnitudes  $AB$   $CD$  are incommensurable

For if they are commensurable some magnitude will measure them

Let a magnitude measure them if possible and let it be  $E$



let  $AB$  measuring  $FD$  leave  $CF$  less than itself

let  $CF$  measuring  $BG$  leave  $AG$  less than itself

and let this process be repeated continually until there is left some magnitude which is less than  $E$

Suppose this done and let there be left  $AG$  less than  $E$

Then since  $E$  measures  $AB$

while  $AB$  measures  $DF$

therefore  $E$  will also measure  $FD$

But it measures the whole  $CD$  also

therefore it will also measure the remainder  $CF$

But  $CF$  measures  $BG$

therefore  $E$  also measures  $BG$

But it measures the whole  $AB$  also

therefore it will also measure the remainder  $AG$  the greater the less which is impossible

Therefore no magnitude will measure the magnitudes  $AB, CD$ ,  
 therefore the magnitudes  $AB, CD$  are incommensurable [x Def 1]  
 Therefore etc Q E D

## PROPOSITION 3

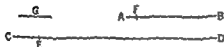
Given two commensurable magnitudes, to find their greatest common measure

Let the two given commensurable magnitudes be  $AB, CD$  of which  $AB$  is the less,  
 then

ure of  $AB, CD$

And it is manifest that it is also the greatest,  
 for a greater magnitude than the magnitude  $AB$  will not measure  $AB$   
 Next, let  $AB$  not measure  $CD$

Then if the less be continually subtracted in turn from the greater, that which is left over will sometime measure the one before it, because  $AB, CD$  are not incommensurable, [cf x 2]



Since, then,  $AF$  measures  $CE$ ,

while  $CE$  measures  $FB$

therefore  $AF$  will also measure  $FB$

But it measures itself also,

therefore  $AF$  will also measure the whole  $AB$

But  $AB$  measures  $DE$ ,

therefore  $AF$  will also measure  $ED$

But it measures  $CE$  also,

therefore it also measures the whole  $CD$

Therefore  $AF$  is a common measure of  $AB, CD$

I say next that it is also the greatest

For, if not, there will be some magnitude greater than  $AF$  which will measure  $AB, CD$

Let it be  $G$

Since then  $G$  measures  $AB$

while  $AB$  measures  $ED$

therefore  $G$  will also measure  $ED$

But it measures the whole  $CD$  also,

therefore  $G$  will also measure the remainder  $CE$

But  $CE$  measures  $FB$ ,

therefore  $G$  will also measure  $FB$

But it measures the whole  $AB$  also

and it will therefore measure the remainder  $AF$  the greater the less which is impossible

Therefore no magnitude greater than  $AF$  will measure  $AB, CD$ ,  
therefore  $AF$  is the greatest common measure of  $AB, CD$

Therefore the greatest common measure of the two given commensurable magnitudes  $AB, CD$  has been found Q E D

PORISM From this it is manifest that, if a magnitude measure two magnitudes, it will also measure their greatest common measure

### PROPOSITION 4

*Given three commensurable magnitudes, to find their greatest common measure*

Let  $A, B, C$  be the three given commensurable magnitudes;  
thus it is required to find the greatest common measure of  $A, B, C$

Let the greatest common measure of the two magnitudes  $A, B$  be taken and let it be  $D$ , [x 3]  
then  $D$  either measures  $C$  or does not measure it

First, let it measure it

Since then  $D$  measures  $C$

while it also measures  $A, B$ ,

therefore  $D$  is a common measure of  $A, B, C$

And it is manifest that it is also the greatest,  
for a greater magnitude than the magnitude  $D$  does not measure  $A, B$

Next let  $D$  not measure  $C$

I say first that  $C, D$  are commensurable

For since  $A, B, C$  are commensurable,

some magnitude will measure them,

and this will of course measure  $A, B$  also

so that it will also measure the greatest common measure of  $A, B$ , namely  $D$   
[x 3 For]

But it also measures  $C$

so that the said magnitude will measure  $C, D$ ,

therefore  $C, D$  are commensurable

Now let their greatest common measure be taken and let it be  $E$  [x 3]

Since then  $E$  measures  $D$ ,

while  $D$  measures  $A, B$ ,

therefore  $E$  will also measure  $A, B$

But it measures  $C$  also

therefore  $E$  measures  $A, B, C$ ,

therefore  $E$  is a common measure of  $A, B, C$

I say next that it is also the greatest

For if possible let there be some magnitude  $F$  greater than  $E$ , and let it measure  $A, B, C$

Now since  $F$  measures  $A, B, C$

it will also measure  $A, B$

∴  $F$  measures  $A, B, C$  = [x 3, Por]

But it measures  $C$  also

therefore  $F$  measures  $C, D$

therefore  $F$  will also measure the greatest common measure of  $C, D$   
[x 3, Por]

But that is  $E$ ,

therefore  $F$  will measure  $E$ , the greater the less

which is impossible

Therefore no magnitude greater than the magnitude  $E$  will measure  $A, B, C$ ,  
therefore  $E$  is the greatest common measure of  $A, B, C$  if  $D$  do not measure  $C$ ,

and, if it measure it,  $D$  is itself the greatest common measure

Therefore the greatest common measure of the three given commensurable magnitudes has been found

PORISM From this it is manifest that, if a magnitude measure three magnitudes it will also measure their greatest common measure

Similarly too, with more magnitudes, the greatest common measure can be found, and the porism can be extended Q E D

### PROPOSITION 5

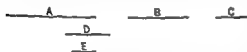
*Commensurable magnitudes have to one another the ratio which a number has to a number*

Let  $A, B$  be commensurable magnitudes,

I say that  $A$  has to  $B$  the ratio which a number has to a number

For, since  $A, B$  are commensurable, some magnitude will measure them

Let it measure them, and let it be  $C$



And, as many times as  $C$  measures  $A$ , so many units let there be in  $D$ ,

and, as many times as  $C$  measures  $B$ , so many units let there be in  $E$

Since then  $C$  measures  $A$  according to the units in  $D$ ,

while the unit also measures  $D$  according to the units in it,

therefore the unit measures the number  $D$  the same number of times as the magnitude  $C$  measures  $A$ ,

therefore as  $C$ , is to  $A$ , so is the unit to  $D$ , [VII Def 20]

therefore, inversely, as  $A$  is to  $C$ , so is  $D$  to the unit [cf v 7, Por]

Again, since  $C$  measures  $B$  according to the units in  $E$ ,

while the unit also measures  $E$  according to the units in it,

therefore the unit measures  $E$  the same number of times as  $C$  measures  $B$ ,

therefore, as  $C$  is to  $B$ , so is the unit to  $E$

But it was also proved that,

as  $A$  is to  $C$ , so is  $D$  to the unit,

therefore, *ex aequali*,

as  $A$  is to  $B$ , so is the number  $D$  to  $E$  [v 22]

Therefore the commensurable magnitudes  $A, B$  have to one another the ratio which the number  $D$  has to the number  $E$  Q E D

### PROPOSITION 6

*If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable*

For let the two magnitudes  $A, B$  have to one another the ratio which the number  $D$  has to the number  $E$ ,

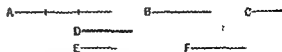
I say that the magnitudes  $A, B$  are commensurable



For let  $A$  be divided into as many equal parts as there are units in  $D$ ,  
and let  $C$  be equal to one of them;

and let  $F$  be made up of as  
many magnitudes equal to  $C$   
as there are units in  $E$

Since then there are in  $A$   
as many magnitudes equal to  
 $C$  as there are units in  $D$ ,



whatever part the unit is of  $D$ , the same part is  $C$  of  $A$  also;  
therefore, as  $C$  is to  $A$ , so is the unit to  $D$ . [vii Def 20]

But the unit measures the number  $D$ ,  
therefore  $C$  also measures  $A$ .

And since, as  $C$  is to  $A$ , so is the unit to  $D$ ,  
therefore, inversely, as  $A$  is to  $C$ , so is the number  $D$  to the unit  
[cf v 7, Por]

Again since there are in  $F$  as many magnitudes equal to  $C$  as there are units  
in  $E$ ,

therefore, as  $C$  is to  $F$ , so is the unit to  $E$ . [vii Def. 20]

But it was also proved that,  
as  $A$  is to  $C$ , so is  $D$  to the unit;  
therefore, *ex aequali*, as  $A$  is to  $F$ , so is  $D$  to  $E$ . [v 22]

But, as  $D$  is to  $E$ , so is  $A$  to  $B$ ,  
therefore also, as  $A$  is to  $B$ , so is it to  $F$  also [v. 11]

Therefore  $A$  has the same ratio to each of the magnitudes  $B$ ,  $F$ ;  
therefore  $B$  is equal to  $F$  [v 9]

But  $C$  measures  $F$ ,  
therefore it measures  $B$  also.

Further it measures  $A$  also,  
therefore  $C$  measures  $A$ ,  $B$

Therefore  $A$  is commensurable with  $B$

Therefore etc  
PORTSM From this it is manifest that if there be two numbers, as  $D$ ,  $E$ , and  
a straight line as  $A$ , it is possible to make a straight line [ $F$ ] such that the  
given straight line is to it as the number  $D$  is to the number  $E$

And, if a mean proportional be also taken between  $A$ ,  $F$ , as  $B$ ,  
as  $A$  is to  $F$  so will the square on  $A$  be to the square on  $B$ , that is, as the first  
is to the third, so is the figure on the first to that which is similar and similarly  
described on the second [vi 19, Por]

But, as  $A$  is to  $F$  so is the number  $D$  to the number  $E$ ,  
therefore it has been contrived that as the number  $D$  is to the number  $E$ , so  
also is the figure on the straight line  $A$  to the figure on the straight line  $B$

Q E D

### PROPOSITION 7

*Incommensurable magnitudes have not to one another the ratio which a number has  
to a number*

Let  $A$ ,  $B$  be incommensurable magnitudes,  
I say that  $A$  has not to  $B$  the ratio which a number has to a number  
For, if  $A$  has to  $B$  the ratio which a number has to a number,  $A$  will be com-

measurable with  $B$

[x 6]

But it is not;

$\frac{A}{B}$  therefore  $A$  has not to  $B$  the ratio which a number has to a number

Therefore etc

Q E D

### PROPOSITION 8

*If two magnitudes have not to one another the ratio which a number has to a number, the magnitudes will be incommensurable*

For let the two magnitudes  $A, B$  not have to one another the ratio which a number has to a number,

I say that the magnitudes  $A, B$  are incommensurable

$\frac{A}{B}$  For, if they are commensurable  $A$  will have to  $B$  the ratio which a number has to a number

[x 5]

But it has not,

therefore the magnitudes  $A, B$  are incommensurable

Therefore etc

Q E D

### PROPOSITION 9

*The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number, and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number, and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either*

For let  $A, B$  be commensurable in length,

$\frac{A}{C}$   
 $\frac{D}{B}$

I say that the square on  $A$  has to the square on  $B$  the ratio which a square number has to a square number

For, since  $A$  is commensurable in length with  $B$ ,

therefore  $A$  has to  $B$  the ratio which a number has to a number [x 5]

Let it have to it the ratio which  $C$  has to  $D$

Since then, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,

while the ratio of the square on  $A$  to the square on  $B$  is duplicate of the ratio of  $A$  to  $B$ ,

for similar figures are in the duplicate ratio of their corresponding sides, [vi 20, Por.]

and the ratio of the square on  $C$  to the square on  $D$  is duplicate of the ratio of  $C$  to  $D$ ,

for between two square numbers there is one mean proportional number and the square number has to the square number the ratio duplicate of that which the side has to the side, [viii 11]

therefore also as the square on  $A$  is to the square on  $B$ , so is the square on  $C$  to the square on  $D$

Next as the square on  $A$  is to the square on  $B$ , so let the square on  $C$  be to the square on  $D$ ,

I say that  $A$  is commensurable in length with  $B$

For since, as the square on  $A$  is to the square on  $B$ , so is the square on  $C$  to the square on  $D$ ,  
while the ratio of the square on  $A$  to the square on  $B$  is duplicate of the ratio of  $A$  to  $B$ ,  
and the ratio of the square on  $C$  to the square on  $D$  is duplicate of the ratio of  $C$  to  $D$ ,

therefore also as  $A$  is to  $B$ , so is  $C$  to  $D$

Therefore  $A$  has to  $B$  the ratio which the number  $C$  has to the number  $D$ ,  
therefore  $A$  is commensurable in length with  $B$  [x 6]

Next, let  $A$  be incommensurable in length with  $B$ ;

I say that the square on  $A$  has not to the square on  $B$  the ratio which a square number has to a square number

For, if the square on  $A$  has to the square on  $B$  the ratio which a square number has to a square number,  $A$  will be commensurable with  $B$

But it is not,

therefore the square on  $A$  has not to the square on  $B$  the ratio which a square number has to a square number

Again, let the square on  $A$  not have to the square on  $B$  the ratio which a square number has to a square number,

I say that  $A$  is incommensurable in length with  $B$

For, if  $A$  is commensurable with  $B$ , the square on  $A$  will have to the square on  $B$  the ratio which a square number has to a square number

But it has not,

therefore  $A$  is not commensurable in length with  $B$

Therefore etc

**PORISM** And it is manifest from what has been proved that straight lines commensurable in length are always commensurable in square also, but those commensurable in square are not always commensurable in length also

[LEMMA It has been proved in the arithmetical books that similar plane numbers have to one another the ratio which a square number has to a square number [VIII 26]

and that, if two numbers have to one another the ratio which a square number has to a square number, they are similar plane numbers (Converse of VIII 26)

And to a mean fact with --

For, if they have, they will be similar plane numbers which is contrary to the hypothesis

Therefore numbers which are not similar plane numbers have not to one another the ratio which a square number has to a square number ]

#### PROPOSITION 10

To find two straight lines incommensurable, the one in length only, and the other in square also, with an assigned straight line

Let  $A$  be the assigned straight line,  
thus it is required to find two straight lines incommensurable, the one in length only, and the other in square also with  $A$

Let two numbers  $B$ ,  $C$  be set out which have not to one another the ratio

which a square number has to a square number, that is which are not similar plane numbers,

and let it be contrived that,  
 A \_\_\_\_\_ as B is to C, so is the square on A to the square on D  
 D \_\_\_\_\_ —for we have learnt how to do thus— [x 6, Por]  
 E \_\_\_\_\_ therefore the square on A is commensurable with the  
 B \_\_\_\_\_ square on D [x 6]  
 C \_\_\_\_\_ And, since B has not to C the ratio which a square num-

ber has to a square number,  
 therefore neither has the square on A to the square on D the ratio which a

square number has to a square number,  
 therefore A is incommensurable in length with D [x 9]

Let E be taken a mean proportional between A, D,  
 therefore as A is to D, so is the square on A to the square on E [v Def 9]

But A is incommensurable in length with D,  
 therefore the square on A is also incommensurable with the square on E,  
 [x 11]

therefore A is incommensurable in square with E  
 Q E D

## PROPOSITION 11

Let A, B, C, D be four magnitudes in proportion so that as A is to B, so is C to D,

and let A be commensurable with B,  
 I say that C will also be commensurable

A \_\_\_\_\_ B \_\_\_\_\_ with D  
 C \_\_\_\_\_ D \_\_\_\_\_

For, since A is commensurable with B,  
 therefore A has to B the ratio which a number has to a number [x 5]

And as A is to B so is C to D,  
 therefore C also has to D the ratio which a number has to a number,  
 therefore C is commensurable with D [x 6]

Next let A be incommensurable with B,  
 I say that C will also be incommensurable with D

For since A is incommensurable with B  
 therefore A has not to B the ratio which a number has to a number [x 7]

And as A is to B so is C to D,  
 therefore neither has C to D the ratio which a number has to a number,  
 therefore C is incommensurable with D [x 8]

Therefore etc Q E D

## PROPOSITION 12

Magnitudes commensurable with the same magnitude are commensurable with one another also

For let each of the magnitudes A, B be commensurable with C,

I say that  $A$  is also commensurable with  $B$

For, since  $A$  is commensurable with  $C$ ,  
therefore  $A$  has to  $C$  the ratio  
which a number has to a  
number [x 5]

Let it have the ratio which  
 $D$  has to  $E$

Again since  $C$  is commensurable with  $B$ ,

therefore  $C$  has to  $B$  the ratio which a number has to a number [x 5]

Let it have the ratio which  $F$  has to  $G$

And given any number of ratios we please, namely the ratio which  $D$  has to  $E$  and that which  $F$  has to  $G$

let the numbers  $H, K, L$  be taken continuously in the given ratios, [cf viii 4]

so that as  $D$  is to  $E$ , so is  $H$  to  $K$ ,

and, as  $F$  is to  $G$ , so is  $K$  to  $L$

Since, then, as  $A$  is to  $C$ , so is  $D$  to  $E$ ,

while, as  $D$  is to  $E$ , so is  $H$  to  $K$ ,

therefore also, as  $A$  is to  $C$ , so is  $H$  to  $K$ . [v 11]

Again since as  $C$  is to  $B$ , so is  $F$  to  $G$ ,

while as  $F$  is to  $G$ , so is  $K$  to  $L$ ,

therefore also as  $C$  is to  $B$ , so is  $K$  to  $L$ . [v 11]

But also as  $A$  is to  $C$ , so is  $H$  to  $K$ ;

therefore *ex aequali* as  $A$  is to  $B$ , so is  $H$  to  $L$ . [v 22]

Therefore  $A$  has to  $B$  the ratio which a number has to a number,

therefore  $A$  is commensurable with  $B$ . [x 6]

Therefore etc

Q E D

### PROPOSITION 13

If two magnitudes be commensurable and the one of them be incommensurable with any magnitude the remaining one will also be incommensurable with the same

Let  $A, B$  be two commensurable magnitudes and let one of them,  $A$ , be incommensurable with any other magnitude  $C$ ,

I say that the remaining one  $B$ , will also be incommensurable with  $C$

For, if  $B$  is commensurable with  $C$ ,

while  $A$  is also commensurable with  $B$ ,

$A$  is also commensurable with  $C$

But it is also incommensurable with it

which is impossible

Therefore  $B$  is not commensurable with  $C$ ,

therefore it is incommensurable with it

Therefore etc

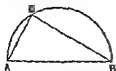
Q E D

### LEMMA

Given two unequal straight lines to find by what square the square on the greater is greater than the square on the less

Let  $AB, C$  be the given two unequal straight lines and let  $AB$  be the greater of them,

thus it is required to find by what square the square on  $AB$  is greater than the square on  $C$



Let the semicircle  $ADB$  be described on  $AB$ ,  
and let  $AD$  be fitted into it equal to  $C$ , [iv 11]  
let  $DB$  be joined

It is then manifest that the angle  $ADB$  is  
right, [iii 31]

and that the square on  $AB$  is greater than the  
square on  $AD$ , that is,  $C$ , by the square on  $DB$  [i 47]

Similarly also, if two straight lines be given, the straight line the square on which is equal to the sum of the squares on them is found in this manner

Let  $AD$ ,  $DB$  be the given two straight lines, and let it be required to find the straight line the square on which is equal to the sum of the squares on them

Let them be placed so as to contain a right angle, that formed by  $AD$ ,  $DB$ ;  
and let  $AB$  be joined

It is again manifest that the straight line the square on which is equal to the  
sum of the squares on  $AD$ ,  $DB$  is  $AB$  [i 47]

Q E D

#### PROPOSITION 14

If four straight lines be proportional, and the square on the first be greater than the square on the second by the square on a straight line commensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line commensurable with the third

And, if the square on the first be greater than the square on the second by the square on a straight line incommensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line incommensurable with the third

Let  $A$ ,  $B$ ,  $C$ ,  $D$  be four straight lines in proportion, so that, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,



and let the square on  $A$  be greater than the square on  $B$   
by the square on  $E$ ,  
and let the square on  $C$  be greater than the square on  $D$   
by the square on  $F$ ,

I say that, if  $A$  is commensurable with  $E$ ,  $C$  is also commensurable with  $F$ ,  
and, if  $A$  is incommensurable with  $E$ ,  $C$  is also incommensurable with  $F$

For since, as  $A$  is to  $B$ , so is  $C$  to  $D$ ,  
therefore also, as the square on  $A$  is to the square on  $B$ ,  
so is the square on  $C$  to the square on  $D$  [vi 22]

But the squares on  $E$ ,  $B$  are equal to the square on  $A$ ,  
and the squares on  $D$ ,  $F$  are equal to the square on  $C$

Therefore, as the squares on  $E$ ,  $B$  are to the square on  $B$ , so are the squares  
on  $D$ ,  $F$  to the square on  $D$ ,

therefore, *separando* as the square on  $E$  is to the square on  $B$ , so is the square  
on  $F$  to the square on  $D$ , [v 17]

therefore also as  $E$  is to  $B$ , so is  $F$  to  $D$ , [vi 22]

therefore, *inversely*, as  $B$  is to  $E$ , so is  $D$  to  $F$

But, as  $A$  is to  $B$ , so also is  $C$  to  $D$ ,

therefore, *ex aequali*, as  $A$  is to  $E$ , so is  $C$  to  $F$  [v 22]

Therefore, if  $A$  is commensurable with  $E$ ,  $C$  is also commensurable with  $F$ ,  
and if  $A$  is incommensurable with  $E$ ,  $C$  is also incommensurable with  $F$  [x 11]

Therefore etc

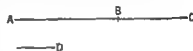
Q E D

### PROPOSITION 15

*If two commensurable magnitudes be added together, the whole will also be commensurable with each of them, and, if the whole be commensurable with one of them, the original magnitudes will also be commensurable*

For let the two commensurable magnitudes  $AB$ ,  $BC$  be added together,

I say that the whole  $AC$  is also commensurable with each of the magnitudes  $AB$ ,  $BC$



For, since  $AB$ ,  $BC$  are commensurable, some magnitude will measure them

Let it measure them, and let it be  $D$

Since then  $D$  measures  $AB$ ,  $BC$ , it will also measure the whole  $AC$

But it measures  $AB$ ,  $BC$  also,

therefore  $D$  measures  $AB$ ,  $BC$ ,  $AC$ ,

therefore  $AC$  is commensurable with each of the magnitudes  $AB$ ,  $BC$

[x Def 1]

Next, let  $AC$  be commensurable with  $AB$ ,

I say that  $AB$ ,  $BC$  are also commensurable

For, since  $AC$ ,  $AB$  are commensurable, some magnitude will measure them

Let it measure them, and let it be  $D$

Since then  $D$  measures  $CA$ ,  $AB$ , it will also measure the remainder  $BC$

But it measures  $AB$  also,

therefore  $D$  will measure  $AB$ ,  $BC$ ,

therefore  $AB$ ,  $BC$  are commensurable

[x Def 1]

Therefore etc

Q E D

### PROPOSITION 16

*If two incommensurable magnitudes be added together, the whole will also be incommensurable with each of them, and if the whole be incommensurable with one of them, the original magnitudes will also be incommensurable*

For let the two incommensurable magnitudes  $AB$ ,  $BC$  be added together, I say that the whole  $AC$  is also incommensurable with each of the magnitudes  $AB$ ,  $BC$

For, if  $CA$ ,  $AB$  are not incommensurable, some magnitude will measure them

Let it measure them, if possible, and let it be  $D$

Since then  $D$  measures  $CA$ ,  $AB$ ,

therefore it will also measure the remainder  $BC$

But it measures  $AB$  also,

therefore  $D$  measures  $AB$ ,  $BC$

Therefore  $AB$ ,  $BC$  are commensurable,

but they were also by hypothesis incommensurable

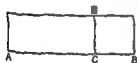
which is impossible



Therefore no magnitude will measure  $CA, AB$ ,  
 therefore  $CA, AB$  are incommensurable [X Def 1]  
 Similarly we can prove that  $AC, CB$  are also incommensurable  
 Therefore  $AC$  is incommensurable with each of the magnitudes  $AB, BC$ .  
 Next, let  $AC$  be incommensurable with one of the magnitudes  $AB, BC$   
 First, let it be incommensurable with  $AB$ ,  
 I say that  $AB, BC$  are also incommensurable  
 For, if they are commensurable, some magnitude will measure them  
 Let it measure them, and let it be  $D$   
 Since, then,  $D$  measures  $AB, BC$ ,  
 therefore it will also measure the whole  $AC$   
 But it measures  $AB$  also,  
 therefore  $D$  measures  $CA, AB$   
 Therefore  $CA, AB$  are commensurable,  
 but they were also by hypothesis, incommensurable  
 which is impossible  
 Therefore no magnitude will measure  $AB, BC$ ,  
 therefore  $AB, BC$  are incommensurable [X Def 1]  
 Therefore etc Q E D

## LEMMA

For let there be applied to the straight line  $AB$  the parallelogram  $AD$  deficient by the square figure  $DB$ ;



I say that  $AD$  is equal to the rectangle contained by  $AC, CB$

This is indeed at once manifest,

for, since  $DB$  is a square,

$DC$  is equal to  $CB$ ,

and  $AD$  is the rectangle  $AC, CD$ , that is the rectangle  $AC, CB$

Therefore etc

Q E D

## PROPOSITION 17

If there be two unequal straight lines, and to the greater there be applied to a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into parts which are commensurable in length then the square on the greater will be greater than the square on the less by the square on a straight line commensurable with the greater

And, if the square on the greater be greater than the square on the less by the square on a straight line commensurable with the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, it will divide it into parts which are commensurable in length

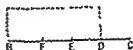
Let  $A B C$  be two unequal straight lines. of which  $BC$  is the greater.



the square on

and let  $EF$  be made equal to  $DE$ Therefore the remainder  $DC$  is equal to  $BF$ 

And since the straight line  $BC$  has been cut into equal parts at  $E$ , and into unequal parts at  $D$ , therefore the rectangle contained by  $BD, DC$ , together with the square on  $ED$ , is equal to the square on  $EC$ ,



[II 5]

And the same is true of their quadruples, therefore four times the rectangle  $BD, DC$ , together with four times the square on  $DE$ , is equal to four times the square on  $EC$

But the square on  $A$  is equal to four times the rectangle  $BD, DC$ , and the square on  $DF$  is equal to four times the square on  $DE$ , for  $DF$  is double of  $DE$

And the square on  $BC$  is equal to four times the square on  $EC$ , for again  $BC$  is double of  $CE$

Therefore the squares on  $A, DF$  are equal to the square on  $BC$ , so that the square on  $BC$  is greater than the square on  $A$  by the square on  $DF$ . It is to be proved that  $BC$  is also commensurable with  $DF$

with  $CD$  [x 13] $CD$  is equal to  $BF$ 

[x 6]

Therefore  $BC$  is also commensurable in length with  $BF, CD$ , [x 12] so that  $BC$  is also commensurable in length with the remainder  $FD$ , [x 15] therefore the square on  $BC$  is greater than the square on  $A$  by the square on a straight line commensurable with  $BC$

Next let the square on  $BC$  be greater than the square on  $A$  by the square on a straight line commensurable with  $BC$ ,

let a parallelogram be applied to  $BC$  equal to the fourth part of the square on  $A$  and deficient by a square figure and let it be the rectangle  $BD, DC$

It is to be proved that  $BD$  is commensurable in length with  $DC$

With the same construction we can prove similarly that the square on  $BC$  is greater than the square on  $A$  by the square on  $FD$

But the square on  $BC$  is greater than the square on  $A$  by the square on a straight line commensurable with  $BC$

Therefore  $BC$  is commensurable in length with  $FD$ ,

so that  $BC$  is also commensurable in length with the remainder, the sum of  $BF, DC$  [x 15]

But the sum of  $BF, DC$  is commensurable with  $DC$ , [x 6]

so that  $BC$  is also commensurable in length with  $CD$ , [x 12]

and therefore *separando*,  $BD$  is commensurable in length with  $DC$  [x 15]

Therefore etc

Q E D

## PROPOSITION 18

If there be two unequal straight lines and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square

figure, and if it divide it into parts which are incommensurable, the square on the greater will be greater than the square on the less by the square on a straight line incommensurable with the greater

And, if the square on the greater be greater than the square on the less by the square on a straight line incommensurable with the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient

Let this be the rectangle  $BD, DC$ , [cf Lemma before x 17]  
 and let  $BD$  be incommensurable in length with  $DC$ ,  
 I say that the square on  $BC$  is greater than the square on  $A$  by  
 the square on a straight line incommensurable with  $BC$   
 For, with the same construction as before, we can prove simi-  
 larly that the square on  $BC$  is greater than the square on  $A$  by  
 the square on  $FD$

It is to be proved that  $BC$  is incommensurable in length with  $DF$   
 Since  $BD$  is incommensurable in length with  $DC$ ,

therefore  $BC$  is also incommensurable in length with  $CD$  [x 16]

But  $DC$  is commensurable with the sum of  $BF, DC$ , [x 6]

therefore  $BC$  is also incommensurable with the sum of  $BF, DC$ , [x 13]

so that  $BC$  is also incommensurable in length with the remainder  $FD$  [x 16]

And the square on  $BC$  is greater than the square on  $A$  by the square on  $FD$ ,  
 therefore the square on  $BC$  is greater than the square on  $A$  by the square on a  
 straight line incommensurable with  $BC$

Again, let the square on  $BC$  be greater than the square on  $A$  by the square on  
 a straight line commensurable with  $BC$ , and let there be applied to  $BC$  a

It is to be proved that  $BD$  is incommensurable in length with  $DC$

For, with the same construction, we can prove similarly that the square on  
 $BC$  is greater than the square on  $A$  by the square on  $FD$

But the square on  $BC$  is greater than the square on  $A$  by the square on a  
 straight line incommensurable with  $BC$ ,

therefore  $BC$  is incommensurable in length with  $FD$ ,

so that  $BC$  is also incommensurable with the remainder, the sum of  $BF, DC$

But the sum of  $BF, DC$  is commensurable in length with  $DC$ , [x 6]

therefore  $BC$  is also incommensurable in length with  $DC$ , [x 13]

so that, *separando*,  $BD$  is also incommensurable in length with  $DC$  [x 16]

Therefore etc

Q E D

#### LEMMA

Since it has been proved that straight lines commensurable in length are al-  
 ways commensurable in square also while those commensurable in square are  
 not always commensurable in length also, but can of course be either com-  
 mensurable or incommensurable in length, it is manifest that, if any straight  
 line be commensurable in length with a given rational straight line, it is called

rational and commensurable with the other not only in length but in square also, since straight lines commensurable in length are always commensurable in square also

### PROPOSITION 19

*The rectangle contained by rational straight lines commensurable in length is rational*

For let the rectangle  $AC$  be contained by the rational straight lines  $AB$ ,  $BC$  commensurable in length;

I say that  $AC$  is rational

For on  $AB$  let the square  $AD$  be described,

therefore  $AD$  is rational [x Def 4]

And, since  $AB$  is commensurable in length with  $BC$ ,

while  $AB$  is equal to  $BD$ ,

therefore  $BD$  is commensurable in length with  $BC$

And, as  $BD$  is to  $BC$ , so is  $DA$  to  $AC$ . [vi 1]

Therefore  $DA$  is commensurable with  $AC$  [x 11]

But  $DA$  is rational,

therefore  $AC$  is also rational

Therefore etc

[x. Def 4]  
Q. E. D

### PROPOSITION 20

*If a rational area be applied to a rational straight line, it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied*

For let the rational area  $AC$  be applied to  $AB$ , a straight line once more rational in any of the aforesaid cases

I say that  $BC$  is rational

For on  $AB$  let

therefore  $AD$  is rational [x Def 4]

But  $AC$  is also rational,

therefore  $DA$  is commensurable with  $AC$

And, as  $DA$  is to  $AC$ , so is  $DB$  to  $BC$  [vi 1]

Therefore  $DB$  is commensurable with  $BC$  [x 11]

But  $AB$  is rational,

therefore  $BC$  is also rational and commensurable in length with  $AB$ .

Therefore etc

Q. E. D

### PROPOSITION 21

*The rectangle contained by rational straight lines commensurable in square only is irrational, and the side of the square equal to it is irrational. Let the latter be called medial*

For let the rectangle  $AC$  be contained by the rational straight lines  $AB$ ,  $BC$



For on  $AB$  let the square  $AD$  be described,  
therefore  $AD$  is rational [x Def 4]

And, since  $AB$  is incommensurable in length with  $BC$ ,  
for by hypothesis they are commensurable in square only,  
while  $AB$  is equal to  $BD$ ,

therefore  $DB$  is also incommensurable in length with  $BC$

And as  $DB$  is to  $BC$ , so is  $AD$  to  $AC$ , [vi 1]

therefore  $DA$  is incommensurable with  $AC$  [x 11]

But  $DA$  is rational,

therefore  $AC$  is irrational,

so that the side of the square equal to  $AC$  is also irrational [x Def 4]

And let the latter be called *medial*

Q E D

### LEMMA

If there be two straight lines, then, as the first is to the second, so is the square on the first to the rectangle contained by the two straight lines



Let  $FE$ ,  $EG$  be two straight lines

I say that, as  $FE$  is to  $EG$ , so is the square on  $FE$  to the rectangle  $FE$ ,  $EG$

For on  $FE$  let the square  $DF$  be described,  
and let  $GD$  be completed

Since then, as  $FE$  is to  $EG$ , so is  $FD$  to  $DG$ , [vi 1]

and  $FD$  is the square on  $FE$ ,

and  $DG$  is the rectangle  $FE$ ,  $EG$

therefore

Similar

is to  $FD$ , so is  $GE$  to  $EF$

Q E D

### PROPOSITION 22

The square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied



Let  $A$  be medial and  $CB$  rational,

and let a rectangular area  $BD$  equal to the square on  $A$  be applied to  $BC$ , producing  $CD$  as breadth,

equal to a rectangular area contained by rational straight lines commensurable in square only [x 21]

Let the square on it be equal to  $GF$

But the square on it is also equal to  $BD$ ,

therefore  $BD$  is equal to  $GF$

But it is also equiangular with it,  
and in equal and equiangular parallelograms the sides about the equal angles  
are reciprocally proportional, [vi 14]

therefore, proportionally, as  $BC$  is to  $EG$ , so is  $EF$  to  $CD$

Therefore also as the square on  $BC$  is to the square on  $EG$ , so is the square  
on  $EF$  to the square on  $CD$  [vi 22]

But the square on  $CB$  is commensurable with the square on  $EG$ , for each of  
these straight lines is rational,

therefore the square on  $EF$  is also commensurable with the square on  $CD$  [x 11]

But the square on  $EF$  is rational,

therefore the square on  $CD$  is also rational, [x Def 4]

therefore  $CD$  is rational

And since  $EF$  is incommensurable in length with  $EG$ ,

for they are commensurable in square only,

and as  $EF$  is to  $EG$  so is the square on  $EF$  to the rectangle  $FE, EG$ , [Lemma]  
therefore the square on  $EF$  is incommensurable with the rectangle  $FE, EG$  [x 11]

But the square on  $CD$  is commensurable with the square on  $EF$ , for the  
straight lines are rational in square,  
and the rectangle  $DC, CB$  is commensurable with the rectangle  $FE, EG$  for  
they are equal to the square on  $A$ ,  
therefore the square on  $CD$  is also incommensurable with the rectangle  $DC, CB$  [x 13]

But, as the square on  $CD$  is to the rectangle  $DC, CB$ , so is  $DC$  to  $CB$ , [Lemmas]

therefore  $DC$  is incommensurable in length with  $CB$  [x 11]

Therefore  $CD$  is rational and incommensurable in length with  $CB$

■ E D

### PROPOSITION 23

*A straight line commensurable with a medial straight line is medial*

Let  $A$  be medial and let  $B$  be commensurable with  $A$ ,

I say that  $B$  is also medial

For let a rational straight line  $CD$  be set out  
and to  $CD$  let the rectangular area  $CE$  equal to the  
square on  $A$  be applied, producing  $ED$  as breadth,  
therefore  $ED$  is rational and incommensurable in  
length with  $CD$  [x 22]

And let the rectangular area  $CF$  equal to the square  
on  $B$  be applied to  $CD$  producing  $DF$  as breadth

Since, then  $A$  is commensurable with  $B$ ,  
the square on  $A$  is also commensurable with the  
square on  $B$

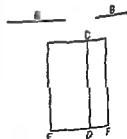
But  $EC$  is equal to the square on  $A$ ,

and  $CF$  is equal to the square on  $B$ ,

therefore  $EC$  is commensurable with  $CF$

And, as  $EC$  is to  $CF$  so is  $ED$  to  $DF$ ,

therefore  $ED$  is commensurable in length with  $DF$



[vi 1]  
[x 11]

But  $ED$  is rational and incommensurable in length with  $DC$ ,  
therefore  $DF$  is also rational [x Def 3] and incommensurable in length with  $DC$  [x 13]

therefore the side of the square equal to the rectangle  $CD, DF$  is medial

And  $B$  is the side of the square equal to the rectangle  $CD, DF$ ,

therefore  $B$  is medial

Q E D

**PORISM** From this it is manifest that an area commensurable with a medial area is medial

[And in the same way as was explained in the case of rationals [Lemma following x 18] it follows, as regards medials, that a straight line commensurable in length with a medial straight line is called *medial* and commensurable

called, in this case too, medial and commensurable in length and in square, but, if in square only, they are called *medial straight lines commensurable in square only*]

#### PROPOSITION 24



I say that  $AC$  is medial

For on  $AB$  let the square  $AD$  be described,

therefore  $AD$  is medial

And, since  $AB$  is commensurable in length with  $BC$ ,

while  $AB$  is equal to  $BD$

therefore  $DB$  is also commensurable in length with  $BC$ ,

so that  $DA$  is also commensurable with  $AC$  [vi 1 x 11]

But  $DA$  is medial,

therefore  $AC$  is also medial

[x 23, Por]

Q E D

#### PROPOSITION 25

The rectangle contained by medial straight lines commensurable in square only is either rational or medial

For let the rectangle  $AC$  be contained by the medial straight lines  $AB, BC$  which are commensurable in square only,

I say that  $AC$  is either rational or medial

For on  $AB, BC$  let the squares  $AD, BE$  be described,

therefore each of the squares  $AD, BE$  is medial

Let a rational straight line  $FG$  be set out

to  $FG$  let there be applied the rectangular parallelogram  $GH$  equal to  $AD$ , producing  $FH$  as breadth

to  $HM$  let there be applied the rectangular parallelogram  $MA$  equal to  $AC$ , producing  $HK$  as breadth,

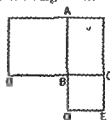
and further to  $KN$  let there be similarly applied  $NL$  equal to  $BE$ , producing  $KL$  as breadth,

therefore  $FH, HK, KL$  are in a straight line

Since then each of the squares  $AD, BE$  is medial,

and  $AD$  is equal to  $GH$ , and  $BE$  to  $NL$ , therefore each of the rectangles  $GH, NL$  is also medial

And they are applied to the rational straight line  $FG$ ,



And since  $AD$  is commensurable with  $BE$ ,

therefore  $GH$  is also commensurable with  $NL$

And as  $GH$  is to  $NL$ , so is  $FH$  to  $KL$ ,

therefore  $FH$  is commensurable in length with  $KL$

Therefore  $FH, KL$  are rational straight lines commensurable in length,

therefore the rectangle  $FH, KL$  is rational

And, since  $DB$  is equal to  $BA$ , and  $OB$  to  $BC$ ,

therefore as  $DB$  is to  $BC$ , so is  $AB$  to  $BO$

But, as  $DB$  is to  $BC$ , so is  $DA$  to  $AC$ ,

and as  $AB$  is to  $BO$ , so is  $AC$  to  $CO$ ,

therefore, as  $DA$  is to  $AC$ , so is  $AC$  to  $CO$

But  $AD$  is equal to  $GH$ ,  $AC$  to  $MK$  and  $CO$  to  $NL$ ,

therefore as  $GH$  is to  $MK$ , so is  $MK$  to  $NL$ ,

therefore also as  $FH$  is to  $HK$ , so is  $HK$  to  $KL$ ,

therefore the rectangle  $FH, KL$  is equal to the square on  $HK$

But the rectangle  $FH, KL$  is rational,

therefore the square on  $HK$  is also rational

Therefore  $HK$  is rational

And if it is commensurable in length with  $FG$ ,

$HN$  is rational,

but if it is incommensurable in length with  $FG$ ,

$KH, HM$  are rational straight lines commensurable in square only, and therefore  $HN$  is medial

Therefore  $HN$  is either rational or medial

But  $HA$  is equal to  $AC$ ,

therefore  $AC$  is either rational or medial

Therefore etc

Q E D

### PROPOSITION 26

A medial area does not exceed a medial area by a rational area

For, if possible, let the medial area  $AB$  exceed the medial area  $AC$  by the rational area  $DB$ ,

and let a rational straight line  $EF$  be set out,

to  $EF$  let there be applied the rectangular parallelogram  $FH$  equal to  $AB$ , producing  $EH$  as breadth,

and let the rectangle  $FG$  equal to  $AC$  be subtracted,

therefore the remainder  $BD$  is equal to the remainder  $KH$

But  $DB$  is rational,

therefore  $KH$  is also rational

Since, then each of the rectangles  $AB$ ,  $AC$  is medial,

and  $AB$  is equal to  $FH$ , and  $AC$  to  $FG$ ,  
therefore each of the rectangles  $FH$ ,  $FG$  is also medial

And they are applied to the rational straight line  $EF$ ,

therefore each of the straight lines  $HE$ ,  $EG$  is rational and incommensurable in length with  $EF$  [x 22]

And since  $[DB$  is rational and is equal to  $KH$ ,

therefore]  $KH$  is [also] rational,

and it is applied to the rational straight line  $EF$ ,

therefore  $GH$  is rational and commensurable in length with  $EF$  [x 20]

But  $EG$  is also rational, and is incommensurable in length with  $EF$ ,

therefore  $EG$  is incommensurable in length with  $GH$  [x 13]

And as  $EG$  is to  $GH$ , so is the square on  $EG$  to the rectangle  $EG$ ,  $GH$ ,  
therefore the square on  $EG$  is incommensurable with the rectangle  $EG$ ,  $GH$  [x 11]

But the squares on  $EG$ ,  $GH$  are commensurable with the square on  $EG$ , for both are rational,

and twice the rectangle  $EG$ ,  $GH$  is commensurable with the rectangle  $EG$ ,  $GH$ , for it is double of it, [x 6]

therefore the squares on  $EG$ ,  $GH$  are incommensurable with twice the rectangle  $EG$ ,  $GH$ , [x 13]

therefore also the sum of the squares on  $EG$ ,  $GH$  and twice the rectangle  $EG$ ,  $GH$ , that is, the square on  $EH$  [ii 4] is incommensurable with the squares on  $EG$ ,  $GH$  [x 16]

But the squares on  $EG$ ,  $GH$  are rational,

therefore the square on  $EH$  is irrational [x Def 4]

Therefore  $EH$  is irrational

But it is also rational

which is impossible

Therefore etc

Q E D

### PROPOSITION 27

To find medial straight lines commensurable in square only which contain a rational rectangle

Let two rational straight lines  $A$ ,  $B$  commensurable in square only be set out,

let  $C$  be taken a mean proportional between  $A$ ,  $B$ , [vi 13]

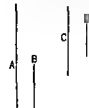
and let it be contrived that,

as  $A$  is to  $B$ , so is  $C$  to  $D$  [vi 12]

Then since  $A$ ,  $B$  are rational and commensurable in square only,

the rectangle  $A$ ,  $B$ , that is the square on  $C$

[vi 17] is medial [x 21]





Therefore  $C$  is medial

[x 21]

[x 11]

And  $C$  is medial,

therefore  $D$  is also medial

[x 23 addition]

Therefore  $C$   $D$  are medial and commensurable in square only

I say that they also contain a rational rectangle

For since as  $A$  is to  $B$  so is  $C$  to  $D$ ,

therefore alternately as  $A$  is to  $C$  so is  $B$  to  $D$

[v 16]

But as  $A$  is to  $C$  so is  $C$  to  $B$ ,

therefore also as  $C$  is to  $B$  so is  $B$  to  $D$ ,

therefore the rectangle  $C$   $D$  is equal to the square on  $B$

But the square on  $B$  is rational,

therefore the rectangle  $C$   $D$  is also rational

Therefore medial straight lines commensurable in square only have been found which contain a rational rectangle

Q E D

### PROPOSITION 28

To find medial straight lines commensurable in square only which contain a medial rectangle

Let the rational straight lines  $A$   $B$   $C$  commensurable in square only be set out,

let  $D$  be taken a mean proportional between  $A$ ,  $B$

[vi 13]

and let it be contrived that

as  $B$  is to  $C$ , so is  $D$  to  $E$

[vi 12]

Since  $A$   $B$  are rational straight lines commensurable in square only

therefore the rectangle  $A$   $B$  that is the square on  $D$  [vi 17] is medial

al

[x 21]

Therefore  $D$  is medial

[x 21]

And since  $B$   $C$  are commensurable in square only,

and as  $B$  is to  $C$  so is  $D$  to  $E$

therefore  $D$   $E$  are also commensurable in square only

[x 11]

But  $D$  is medial

therefore  $E$  is also medial

[x 23 addition]

Therefore  $D$   $E$  are medial straight lines commensurable in square only

I say next that they also contain a medial rectangle

For since as  $B$  is to  $C$  so is  $D$  to  $E$

therefore alternately as  $B$  is to  $D$  so is  $C$  to  $E$

[v 16]

But as  $B$  is to  $D$  so is  $D$  to  $A$

therefore also as  $D$  is to  $A$  so is  $C$  to  $E$

therefore the rectangle  $A$   $C$  is equal to the rectangle  $D$ ,  $E$

[vi 16]

But the rectangle  $A$   $C$  is medial

[x 21]

therefore the rectangle  $D$   $E$  is also medial

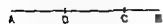
Therefore medial straight lines commensurable in square only have been found which contain a medial rectangle

Q E D

## LEMMA 1

To find two square numbers such that their sum is also square

Let two numbers  $AB$ ,  $BC$  be set out, and let them be either both even or both odd



Then since, whether an even number is subtracted from an even number, or an odd number from an odd number, the remainder is even, [ix 24, 26]  
therefore the remainder  $AC$  is even

Let  $AC$  be bisected at  $D$

Let  $AB$ ,  $BC$  also be either similar plane numbers, or square numbers which are themselves also similar plane numbers

Now the product of  $AB$ ,  $BC$  together with the square on  $CD$  is equal to the square on  $BD$  [ii 6]

And the product of  $AB$ ,  $BC$  is square, inasmuch as it was proved that, if two similar plane numbers by multiplying one another make some number, the product is square [ix 1]

Therefore two square numbers, the product of  $AB$ ,  $BC$ , and the square on  $CD$ , have been found which, when added together, make the square on  $BD$

And it is manifest that two square numbers, the square on  $BD$  and the square on  $CD$ , have again been found such that their difference, the product of  $AB$ ,  $BC$ , is a square, whenever  $AB$ ,  $BC$  are similar plane numbers

But when they are not similar plane numbers two square numbers the square on  $BD$  and the square on  $DC$ , have been found such that their difference, the product of  $AB$ ,  $BC$ , is not square Q E D

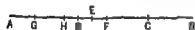
## LEMMA 2

To find two square numbers such that their sum is not square

For let the product of  $AB$ ,  $BC$ , as we said, be square,

and  $CA$  even,

and let  $CA$  be bisected by  $D$



It is then manifest that the square product of  $AB$ ,  $BC$  together with the square on  $CD$  is equal to the square on  $BD$

[See Lemma 1]

Let the unit  $DE$  be subtracted,  
therefore the product of  $AB$ ,  $BC$  together with the square on  $CE$  is less than the square on  $BD$

I say then that the square product of  $AB$ ,  $BC$  together with the square on  $CE$  will not be square

For, if it is square it is either equal to the square on  $BE$ , or less than the square on  $BE$ , but cannot any more be greater, lest the unit be divided

First if possible, let the product of  $AB$ ,  $BC$  together with the square on  $CE$  be equal to the square on  $BE$

and let  $GA$  be double of the unit  $DE$

Since then the whole  $AC$  is double of the whole  $CD$ ,

and in them  $AG$  is double of  $DE$ ,

therefore the remainder  $GC$  is also double of the remainder  $EC$ ,

therefore  $GC$  is bisected by  $E$

Therefore the product of  $GB$ ,  $BC$  together with the square on  $CE$  is equal to the square on  $BE$  [11 6]

But the product of  $AB$ ,  $BC$  together with the square on  $CE$  is also, by hypothesis equal to the square on  $BE$ ,  
therefore the product of  $GB$ ,  $BC$  together with the square on  $CE$  is equal to the product of  $AB$ ,  $BC$  together with the square on  $CE$

And if the common square on  $CE$  be subtracted,  
it follows that  $AB$  is equal to  $GB$   
which is absurd

Therefore the product of  $AB$ ,  $BC$  together with the square on  $CE$  is not equal to the square on  $BE$

I say next that neither is it less than the square on  $BE$

For, if possible let it be equal to the square on  $BF$ ,  
and let  $HA$  be double of  $DF$

Now it will again follow that  $HC$  is double of  $CF$ ,  
so that  $CH$  has also been bisected at  $F$

and for this reason the product of  $HB$ ,  $BC$  together with the square on  $FC$  is equal to the square on  $BF$  [11 6]

But by hypothesis the product of  $AB$ ,  $BC$  together with the square on  $CE$  is also equal to the square on  $BF$

Thus the product of  $HB$ ,  $BC$  together with the square on  $CF$  will also be equal to the product of  $AB$ ,  $BC$  together with the square on  $CE$   
which is absurd

Therefore the product of  $AB$ ,  $BC$  together with the square on  $CE$  is not less than the square on  $BE$

And it was proved that neither is it equal to the square on  $BE$

Therefore the product of  $AB$ ,  $BC$  together with the square on  $CE$  is not square Q E D

### PROPOSITION 29

To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater

sum  
na 1]

and let it be contrived that

as  $DC$  is to  $CE$  so is the square on  $BA$  to the square on  $AF$  [x 6 Por]

Let  $FB$  be joined

Since as the square on  $BA$  is to the square on  $AF$   
so is  $DC$  to  $CE$

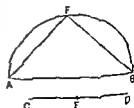
therefore the square on  $BA$  has to the square on  $AF$   
the ratio which the number  $DC$  has to the number  $CE$ ,

therefore the square on  $BA$  is commensurable with the square on  $AF$  [x 6]

But the square on  $AB$  is rational [x Def 4]

therefore the square on  $AF$  is also rational, [id]

therefore  $AF$  is also rational



And, since  $DC$  has not to  $CE$  the ratio which a square number has to a square number,  
neither has the square on  $BA$  to the square on  $AF$  the ratio which a square number has to a square number,

therefore  $AB$  is incommensurable in length with  $AF$  [v 9]

Therefore  $BA, AF$  are rational straight lines commensurable in square only

And since, as  $DC$  is to  $CE$ , so is the square on  $BA$  to the square on  $AF$ ,  
therefore, *conuertendo*, as  $CD$  is to  $DE$ , so is the square on  $AB$  to the square on  $BF$  [v 19, Por, III 31, I 47]

But  $CD$  has to  $DE$  the ratio which a square number has to a square number  
therefore also the square on  $AB$  has to the square on  $BF$  the ratio which a square number has to a square number,

therefore  $AB$  is commensurable in length with  $BF$ . [x 9]

And the square on  $AB$  is equal to the squares on  $AF, FB$ ;

therefore the square on  $AB$  is greater than the square on  $AF$  by the square on  $BF$  commensurable with  $AB$

Therefore there have been found two rational straight lines  $BA, AF$  commensurable in square only and such that the square on the greater  $AB$  is greater than the square on the less  $AF$  by the square on  $BF$  commensurable in length with  $AB$  Q E D

### PROPOSITION 30

To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater

Let there be set out a rational straight line  $AB$ ,

and two square numbers  $CE, ED$  such that their sum  $CD$  is not square, [Lemma 2]

let there be described on  $AB$  the semicircle  $AFB$ ,

let it be contrived that,

as  $DC$  is to  $CE$ , so is the square on  $BA$  to the square on  $AF$  [x 6, Por]

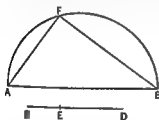
and let  $FB$  be joined

Then, in a similar manner to the preceding, we can prove that  $BA, AF$  are rational straight lines commensurable in square only

And since, as  $DC$  is to  $CE$ , so is the square on  $BA$  to the square on  $AF$ ,  
therefore, *conuertendo*, as  $CD$  is to  $DE$ , so is the square on  $AB$  to the square on  $BF$  [v 19, Por, III 31, I 47]

But  $CD$  has not to  $DE$  the ratio which a square number has to a square number,

therefore neither has the square on  $AB$  to the square on  $BF$  the ratio which a square number has to a square number,



## PROPOSITION 31

To find two medial straight lines commensurable in square only containing a rational rectangle, and such that the square on the less by the square on the greater is greater than the square on a straight line commensurable with the greater.

Let there be set out two straight lines  $A$ ,  $B$  commensurable in square only and such that the square on  $A$  is greater than the square on  $B$  by the square on a straight line commensurable with  $A$ . [x 29]  
the rectangle  $A$ ,  $B$  [x 21]

therefore the square on  $C$  is also medial,  
therefore  $C$  is also medial [x 21]

Let the rectangle  $C$ ,  $D$  be equal to the square on  $B$

Now the square on  $B$  is rational

therefore the rectangle  $C$ ,  $D$  is also rational

And since, as  $A$  is to  $B$ , so is the rectangle  $A$ ,  $B$  to the square on  $B$ ,

while the square on  $C$  is equal to the rectangle  $A$ ,  $B$ ,

and the rectangle  $C$ ,  $D$  is equal to the square on  $B$ ,

therefore as  $A$  is to  $B$  so is the square on  $C$  to the rectangle  $C$ ,  $D$

But, as the square on  $C$  is to the rectangle  $C$ ,  $D$ , so is  $C$  to  $D$ ,

therefore also as  $A$  is to  $B$  so is  $C$  to  $D$

But  $A$  is commensurable with  $B$  in square only,

therefore  $C$  is also commensurable with  $D$  in square only [x 11]

And  $C$  is medial,

therefore  $D$  is also medial [x 23, addition]

And since as  $A$  is to  $B$ , so is  $C$  to  $D$

and the square on  $A$  is greater than the square on  $B$  by the square on a straight line commensurable with  $A$ ,

therefore also the square on  $C$  is greater than the square on  $D$  by the square on a straight line commensurable with  $C$  [x 14]

Therefore two medial straight lines  $C$ ,  $D$  commensurable in square only and containing a rational rectangle have been found and the square on  $C$  is greater than the square on  $D$  by the square on a straight line commensurable in length with  $C$

Similarly also it can be proved that the square on  $C$  exceeds the square on  $D$  by the square on a straight line incommensurable with  $C$ , when the square on  $A$  is greater than the square on  $B$  by the square on a straight line incommensurable with  $A$  [x 30]

## PROPOSITION 32

To find two medial straight lines commensurable in square only containing a medial rectangle and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater

Let there be set out three rational straight lines  $A$ ,  $B$ ,  $C$  commensurable in square only and such that the square on  $A$  is greater than the square on  $C$  by the square on a straight line commensurable with  $A$ , [x 29]

and let the square on  $D$  be equal to the rectangle  $A$ ,  $B$

Therefore the square on  $D$  is medial

therefore  $D$  is also medial

[x 21]

Let the rectangle  $D, E$  be equal to the rectangle  $B, C$

$A$  \_\_\_\_\_

$B$  \_\_\_\_\_

$C$  \_\_\_\_\_

$D$  \_\_\_\_\_

$E$  \_\_\_\_\_

Then since, as the rectangle  $A, B$  is to the rectangle  $B, C$ , so  $\equiv A$  to  $C$ ,

while the square on  $D$  is equal to the rectangle  $A, B$ ,

and the rectangle  $D, E$  is equal to the rectangle  $B, C$ ,

therefore, as  $A$  is to  $C$ , so is the square on  $D$  to the rectangle  $D, E$

But, as the square on  $D$  is to the rectangle  $D, E$ , so  $\equiv D$  to  $E$ ,

therefore also, as  $A$  is to  $C$ , so is  $D$  to  $E$

But  $A \equiv$  commensurable with  $C$  in square only,

therefore  $D$  is also commensurable with  $E$  in square only [x 11]

But  $D \equiv$  medial,

therefore  $E$  is also medial

[x 23, addition]

And, since, as  $A$  is to  $C$ , so is  $D$  to  $E$ ,

while the square on  $A$  is greater than the square on  $C$  by the square on  $\equiv$  straight line commensurable with  $A$ ,

therefore also the square on  $D$  will be greater than the square on  $E$  by the square on a straight line commensurable with  $D$  [x 14]

I say next that the rectangle  $D, E$  is also medial

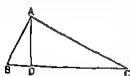
For, since the rectangle  $B, C$  is equal to the rectangle  $D, E$ , while the rectangle  $B, C \equiv$  medial [x 21]

the greater is greater than the square on the less by the square on a straight line commensurable with the greater

Similarly again it can be proved that the square on  $D$  is greater than the square on  $E$  by the square on  $\equiv$  straight line incommensurable with  $D$ , when the square on  $A$  is greater than the square on  $C$  by the square on a straight line incommensurable with  $A$  [x 30]

### LEMMA

Let  $ABC$  be a right-angled triangle having the angle  $A$  right, and let the perpendicular  $AD$  be drawn,



I say that the rectangle  $CB, BD$  is equal to the square on  $BA$ ,

the rectangle  $BC, CD$  equal to the square on  $CA$ ,

the rectangle  $BD, DC$  equal to the square on  $AD$ ,

and further, the rectangle  $BC, AD$  equal to the rectangle  $BA, AC$

And first that the rectangle  $CB, BD$  is equal to the square on  $BA$

For, since in a right-angled triangle  $AD$  has been drawn from the right angle perpendicular to the base, therefore the triangles  $ABD, ADC$  are similar both to the whole  $ABC$  and to one another [vi 8]

And since the triangle  $ABC$  is similar to the triangle  $ABD$ , therefore as  $CB$  is to  $BA$ , so is  $BA$  to  $BD$ , [vi 4]

therefore the rectangle  $CB, BD$  is equal to the square on  $AB$  [VI 17]

For the same reason the rectangle  $BC, CD$  is also equal to the square on  $AC$

And since, if in a right-angled triangle a perpendicular be drawn from the right angle to the base, the perpendicular so drawn is a mean proportional between the segments of the base, [VI 8 For]

therefore, as  $BD$  is to  $DA$ , so  $\square AD$  to  $DC$ ,

therefore the rectangle  $BD, DC$  is equal to the square on  $AD$  [VI 17]

I say that the rectangle  $BC, AD$  is also equal to the rectangle  $BA, AC$

For since, as we said,  $ABC$  is similar to  $ABD$ ,

therefore, as  $BC$  is to  $CA$ , so is  $BA$  to  $AD$  [VI 4]

Therefore the rectangle  $BC, AD$  is equal to the rectangle  $BA, AC$  [VI 16]

Q E D

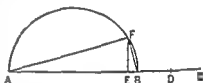
### PROPOSITION 33

To find two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial

Let there be set out two rational straight lines  $AB, BC$  commensurable in square only and such that the square on the greater  $AB$  is greater than the square on the less  $BC$  by the square on a straight line incommensurable with  $AB$ , [X 30]

let  $BC$  be bisected at  $D$ ,

let there be applied to  $AB$  a parallelogram equal to the square on either of the straight lines  $BD, DC$  and deficient by a square figure, and let it be the rectangle  $AE, EB$ , [VI 28]



let the semicircle  $AFB$  be described on  $AB$ ,

let  $EF$  be drawn at right angles to  $AB$ ,

and let  $AF, FB$  be joined

Then since  $AB, BC$  are unequal straight lines, and the square on  $AB$  is greater than the square on  $BC$  by the square on a straight line incommensurable with  $AB$ , while there has been applied to  $AB$  a parallelogram equal to the fourth part of the square on  $BC$ , that is to the square on half of it and deficient by a square figure making the rectangle  $AE, EB$ ,

therefore  $AE$  is incommensurable with  $EB$  [X 18]

And, as  $AE$  is to  $EB$ , so  $\square BA, AE$  to the rectangle  $AB, BE$ ,

while the rectangle  $BA, AE$  is equal to the square on  $AF$ ,

and the rectangle  $AB, BE$  to the square on  $FB$ ,

therefore the square on  $AF$  is incommensurable with the square on  $FB$ ,

therefore  $AF, FB$  are incommensurable in square

And, since  $AB$  is rational,

therefore the square on  $AB$  is also rational,

so that the sum of the squares on  $AF, FB$  is also rational [I 47]

And since again, the rectangle  $AE, EB$  is equal to the square on  $FF$ , and, by hypothesis, the rectangle  $AE, EB$  is also equal to the square on  $BD$ ,

therefore  $FE$  is equal to  $BD$ ,

therefore  $BC$  is double of  $FE$ ,

so that the rectangle  $AB, BC$  is also commensurable with the rectangle  $AB, EF$

But the rectangle  $AB, BC$  is medial, [x 21]  
 therefore the rectangle  $AB, EF$  is also medial [x 23, Por ]  
 But the rectangle  $AB, EF$  is equal to the rectangle  $AF, FB$ , [Lemma]  
 therefore the rectangle  $AF, FB$  is also medial

But it was also proved that the sum of the squares on these straight lines is rational

Therefore two straight lines  $AF, FB$  incommensurable in square have been found which make the sum of the squares on them rational, but the rectangle contained by them medial Q E D

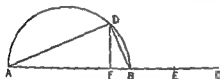
## PROPOSITION 34

To find two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational

Let there be set out two medial straight lines  $AB, BC$ , commensurable in square only, such that the rectangle which they contain is rational, and the square on  $AB$  is greater than the square on  $BC$  by the square on a straight line incommensurable with  $AB$ , [x 31, ad fin ]

let the semicircle  $ADB$  be described on  $AB$ ,

let  $BC$  be bisected at  $E$ ,  
 let there be applied to  $AB$  a parallelogram equal to the square on  $BE$  and deficient by a square figure, namely the rectangle  $AF, FB$ , [vi 23]



therefore  $AF$  is incommensurable in length with  $FB$  [x 18]

Let  $FD$  be drawn from  $F$  at right angles to  $AB$ ,  
 and let  $AD, DB$  be joined

Since  $AF$  is incommensurable in length with  $FB$   
 therefore the rectangle  $BA, AF$  is also incommensurable with the rectangle  $AB, BF$  [x. 11]

But the rectangle  $BA, AF$  is equal to the square on  $AD$ , and the rectangle  $AB, BF$  to the square on  $DB$ ,

therefore the square on  $AD$  is also incommensurable with the square on  $DB$

And, since the square on  $AB$  is medial,  
 therefore the sum of the squares on  $AD, DB$  is also medial [III 31, I 47]

And, since  $BC$  is double of  $DF$ ,  
 therefore the rectangle  $AB, BC$  is also double of the rectangle  $AB, FD$

But the rectangle  $AB, BC$  is rational,  
 therefore the rectangle  $AB, FD$  is also rational [v 6]

But the rectangle  $AB, FD$  is equal to the rectangle  $AD, DB$ , [Lemma]

contained by them rational

Q E D

## PROPOSITION 35

To find two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them



able with  $AB$ ,

[x 32, *ad fin*]

let the semicircle  $ADB$  be described on  $AB$ ,  
and let the rest of the construction be as above

Then, since  $AP$  is incommensurable  
in length with  $FB$ , [x 18]

$AD$  is also incommensurable in square  
with  $DB$  [x 11]

And, since the square on  $AB$  is  
medial,

therefore the sum of the squares on  
 $AD$ ,  $DB$  is also medial [III 31, I 47]

And, since the rectangle  $AF$ ,  $FB$  is equal to the square on each of the straight  
lines  $BE$ ,  $DF$ ,

therefore  $BE$  is equal to  $DF$ ; "

therefore  $BC$  is double of  $FD$ ,

so that the rectangle  $AB$ ,  $BC$  is also double of the rectangle  $AB$ ,  $FD$

But the rectangle  $AB$ ,  $BC$  is medial,

therefore the rectangle  $AB$ ,  $FD$  is also medial [x. 32, Por]

And it is equal to the rectangle  $AD$ ,  $DB$ , [Lemma after x 32]

therefore the rectangle  $AD$ ,  $DB$  is also medial

And, since  $AB$  is incommensurable in length with  $BC$ ,

while  $CB$  is commensurable with  $BE$ ,

therefore  $AB$  is also incommensurable in length with  $BE$ , [x 13]

so that the square on  $AB$  is also incommensurable with the rectangle  $AB$ ,  $BE$

[x 11]

But the squares on  $AD$ ,  $DB$  are equal to the square on  $AB$ , [I 47]  
and the rectangle  $AB$ ,  $FD$ , that is, the rectangle  $AD$ ,  $DB$ , is equal to the rectangle  $AB$ ,  $BE$ ,

therefore the sum of the squares on  $AD$ ,  $DB$  is incommensurable with the rectangle  $AD$ ,  $DB$

Therefore two straight lines  $AD$ ,  $DB$  incommensurable in square have been  
found which make the sum of the squares on them medial and the rectangle  
contained by them medial and moreover incommensurable with the sum of the  
squares on them

Q E D

### PROPOSITION 36

If two rational straight lines commensurable in square only be added together, the  
whole is irrational, and let it be called binomial

For let two rational straight lines  $AB$ ,  $BC$  commensurable in square only be  
added together,

I say that the whole  $AC$  is irrational

For, since  $AB$  is incommensurable in length with  $BC$ —for they are commensurable in square only—

and, as  $AB$  is to  $BC$ , so is the rectangle  $AB$ ,  $BC$  to the square on  $BC$ ,

therefore the rectangle  $AB$ ,  $BC$  is incommensurable with the square  
on  $BC$  [x 11]

But twice the rectangle  $AB, BC$  is commensurable with the rectangle  $AB, BC$  [x 6], and the squares on  $AB, BC$  are commensurable with the square on  $BC$ —for  $AB, BC$  are rational straight lines commensurable in square only — [x 15]

therefore twice the rectangle  $AB, BC$  is incommensurable with the squares on  $AB, BC$

And, componendo, the squares on  $AB, BC$  the square [x 16]

[x Def 4]

Q E D

## PROPOSITION 37

If two medial straight lines commensurable in square only and containing a rational rectangle be added together, the whole is irrational, and let it be called a first bimedial straight line

For let two medial straight lines  $AB, BC$  commensurable in square only and containing a rational rectangle be added together,

I say that the whole  $AC$  is irrational

For, since  $AB$  is incommensurable in length with  $BC$ ,

therefore the squares on  $AB, BC$  are also incommensurable with twice the rectangle  $AB, BC$ , [cf x 36, ll 9–20]

and, componendo, the squares on  $AB, BC$  together with twice the rectangle  $AB, BC$ , that is the square on  $AC$  [II 4], is incommensurable with the rectangle  $AB, BC$  [x 16]

But the rectangle  $AB, BC$  is rational for, by hypothesis  $AB, BC$  are straight lines containing a rational rectangle,

therefore the square on  $AC$  is irrational,

therefore  $AC$  is irrational

[x Def 4]

And let it be called a first bimedial straight line

Q E D

## PROPOSITION 38

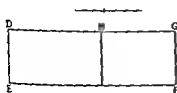
If two medial straight lines commensurable in square only and containing a medial rectangle be added together, the whole is irrational and is called a second bimedial straight line

For let two medial straight lines commensurable in square only and containing a medial rectangle be added together,

I say that  $AC$  is irrational

For let a rational straight line  $DE$  be set out and let the parallelogram  $DF$  equal to the square on  $AC$  be applied to  $DE$ , producing  $DG$  as breadth [I 44]

Then the rectangle  $DE, EG$  is equal to the square on  $AC$  [II 4]



the square on  $AC$

therefore the remainder  $HF$  is equal to twice the rectangle  $AB, BC$

And, since each of the straight lines  $AB, BC$  is medial,

therefore the squares on  $AB, BC$  are also medial

But, by hypothesis, twice the rectangle  $AB, BC$  is also medial

And  $EH$  is equal to the squares on  $AB, BC$ ,

while  $FH$  is equal to twice the rectangle  $AB, BC$ ,

in length with  $DE$

[x 22]

Since then  $AB$  is incommensurable in length with  $BC$ ,

and, as  $AB$  is to  $BC$ , so is the square on  $AB$  to the rectangle  $AB, BC$ ,

therefore the square on  $AB$  is incommensurable with the rectangle  $AB, BC$

[x 11]

But the sum of the squares on  $AB, BC$  is commensurable with the square on  $AB$ ,

[x 15]

and twice the rectangle  $AB, BC$  is commensurable with the rectangle  $AB, BC$

[x 6]

Therefore the sum of the squares on  $AB, BC$  is incommensurable with twice the rectangle  $AB, BC$

[x 13]

But  $EH$

Therefore

so that  $DH$  is also incommensurable in length with  $HG$  [vi 1, x 11]

Therefore  $DH, HG$  are rational straight lines commensurable in square only,

so that  $DG$  is irrational

[x 36]

But  $DE$  is rational,

and the rectangle contained by an irrational and a rational straight line is irrational,

[cf x 20]

therefore the area  $DF$  is irrational,

and the side of the square equal to it is irrational

[x Def 4]

But  $AC$  is the side of the square equal to  $DF$ ,

therefore  $AC$  is irrational

And let it be called a *second binomial* straight line

Q E D

### PROPOSITION 39

If two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial, be added together, the whole straight line is irrational and let it be called *major*

For let two straight lines  $AB, BC$  incommensurable in square and fulfilling the given conditions [x 33], be added together,

I say that  $AC$  is irrational



For, since the rectangle  $AB, BC$  is medial

twice the rectangle  $AB, BC$  is also medial [x 6 and 23, For]

But the sum of the squares on  $AB, BC$  is rational,

therefore twice the rectangle  $AB, BC$  is incommensurable with the sum of the squares on  $AB, BC$

so that the squares on  $AB, BC$  together with twice the rectangle  $AB, BC$  that

is, the square on  $AC$ , is also incommensurable with the sum of the squares on  $AB, BC$ , [x 16]

therefore the square on  $AC$  is irrational,

so that  $AC$  is also irrational

[x Def 4]

And let it be called *major*

Q E D

### PROPOSITION 40

If two straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational, be added together, the whole straight line is irrational, and let it be called the side of a rational plus a medial area

F

A

For, since the sum of the squares on  $AB, BC$  is medial, while twice the rectangle  $AB, BC$  is rational,

therefore the sum of the squares on  $AB, BC$  is incommensurable with twice the rectangle  $AB, BC$ ,

so that the square on  $AC$  is also incommensurable with twice the rectangle  $AB, BC$

[x 16]

But twice the rectangle  $AB, BC$  is rational,

therefore the square on  $AC$  is irrational

Therefore  $AC$  is irrational

[x Def 4]

And let it be called the *side of a rational plus a medial area*

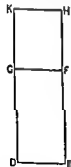
Q E D

### PROPOSITION 41

If two straight lines incommensurable in square which make the sum of the squares on them medial, and the rectangle contained by them medial and also incommensurable with the sum of the squares on them, be added together, the whole straight line is irrational and let it be called the side of the sum of two medial areas

For let two straight lines  $AB, BC$  incommensurable in square and satisfying the given conditions [x 35] be added together,

I say that  $AC$  is irrational



Let a rational straight line  $DE$  be set out, and let there be applied to  $DE$  the rectangle  $DF$  equal to the squares on  $AB, BC$ , and the rectangle  $GH$  equal to twice the rectangle  $AB, BC$ ,

therefore the whole  $DH$  is equal to the square on  $AC$  [II 4]

Now, since the sum of the squares on  $AB, BC$  is medial, and is equal to  $DF$ ,

therefore  $DF$  is also medial

And it is applied to the rational straight line  $DE$ , therefore  $DG$  is rational and incommensurable in length with  $DE$

[x 22]

For the same reason  $GK$  is also rational and incommensurable in length with  $GF$ , that is  $DE$

And, since the squares on  $AB, BC$  are incommensurable with twice the rectangle  $AB, BC$ ,

$DF$  is incommensurable with  $GH$ ,

so that  $DG$  is also incommensurable with  $GK$  [VI 1, r 11]

And they are rational,  
therefore  $DG, GK$  are rational straight lines commensurable in square only,  
therefore  $DK$  is irrational and what is called binomial [x 36]

But  $DE$  is rational,  
therefore  $DH$  is irrational, and the side of the square which is equal to it is irrational [x Def 4]

But  $AC$  is the side of the square equal to  $HD$ ,  
therefore  $AC$  is irrational

And let it be called the *side of the sum of two medial areas* Q E D

### LEMMA

And that the aforesaid irrational straight lines are divided only in one way into the straight lines of which they are the sum and which produce the types in question we will now prove after premising the following lemma

Let the straight line  $AB$  be set out, let the whole be cut into unequal parts at each of the points  $C, D$ ,

and let  $AC$  be supposed greater than  $DB$ ,

I say that the squares on  $AC, CB$  are greater than the squares on  $AD, DB$

For let  $AB$  be bisected at  $E$

Then since  $AC$  is greater than  $DB$ ,

let  $DC$  be subtracted from each,

therefore the remainder  $AD$  is greater than the remainder  $CB$

But  $AE$  is equal to  $EB$ ,

therefore  $DE$  is less than  $EC$

therefore the points  $C, D$  are not equidistant from the point of bisection

And since the rectangle  $AC, CB$  together with the square on  $EC$  is equal to the square on  $EB$  [ii 5]

and further the rectangle  $AD, DB$  together with the square on  $DE$  is equal to the square on  $EB$  [id]

therefore the rectangle  $AC, CB$  together with the square on  $EC$  is equal to the rectangle  $AD, DB$  together with the square on  $DE$

therefore the rectangle  $AC, CB$  is also less than the rectangle  $AD, DB$

therefore the remainder the rectangle  $AC, CB$ , is also less than the rectangle  $AD, DB$

so that twice the rectangle  $AC, CB$  is also less than twice the rectangle  $AD, DB$

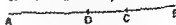
Therefore also the remainder the sum of the squares on  $AC, CB$ , is greater than the sum of the squares on  $AD, DB$  Q E D

### PROPOSITION 42

A binomial straight line is divided into its terms at one point only

Let  $AB$  be a binomial straight line divided into its terms at  $C$ ,  
therefore  $AC, CB$  are rational straight lines commensurable in square only

I say that  $AB$  is not divided at another point into two rational straight lines commensurable in square only



For, if possible, let it be divided at  $D$  also so that  $AD, DB$  are also rational straight lines commensurable in square only

It is then manifest that  $AC$  is not the same with  $DB$

For, if possible, let it be so

Then  $AD$  will also be the same as  $CB$ ,  
and, as  $AC$  is to  $CB$ , so will  $BD$  be to  $DA$ ,

thus  $AB$  will be divided at  $D$  also in the same way as by the division at  $C$   
which is contrary to the hypothesis

Therefore  $AC$  is not the same with  $DB$

For this reason also the points  $C, D$  are not equidistant from the point of bisection

Therefore that by which the squares on  $AC, CB$  differ from the squares on  $AD, DB$  is also that by which twice the rectangle  $AD, DB$  differs from twice the rectangle  $AC, CB$ ,

because both the squares on  $AC, CB$  together with twice the rectangle  $AC, CB$ , and the squares on  $AD, DB$  together with twice the rectangle  $AD, DB$ , are equal to the square on  $AB$  [II 4]

But the squares on  $AC, CB$  differ from the squares on  $AD, DB$  by a rational area

for both are rational,

therefore twice the rectangle  $AD, DB$  also differs from twice the rectangle  $AC, CB$  by a rational area, though they are medial [X 21]

which is absurd, for a medial area does not exceed a medial by a rational area [X 20]

Therefore a binomial straight line is not divided at different points,  
therefore it is divided at one point only Q E D

#### PROPOSITION 43

A first binomial straight line is divided at one point only

Let  $AB$  be a first binomial straight line divided at  $C$ , so that  $AC, CB$  are medial straight lines commensurable in square only and containing a rational rectangle; [X 37]



I say that  $AB$  is not so divided at another point

For, if possible, let it be divided at  $D$  also, so that  $AD, DB$  are also medial straight lines commensurable in square only and containing a rational rectangle

Since, then that by which twice the rectangle  $AD, DB$  differs from twice the rectangle  $AC, CB$  is that by which the squares on  $AC, CB$  differ from the squares on  $AD, DB$ ,

while twice the rectangle  $AD, DB$  differs from twice the rectangle  $AC, CB$  by a rational area—for both are rational—

therefore the squares on  $AC, CB$  also differ from the squares on  $AD, DB$  by a rational area, though they are medial

which is absurd

[X 20]

Therefore a first binomial straight line is not divided into its terms at different points,

therefore it is so divided at one point only

Q E D

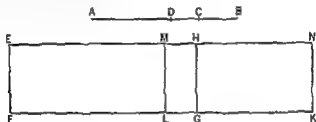
## PROPOSITION 44

*A second binomial straight line is divided at one point only*

Let  $AB$  be a second binomial straight line divided at  $C$ , so that  $AC$ ,  $CB$  are medial straight lines commensurable in square only and containing a medial rectangle, [x 38]

it is then manifest that  $C$  is not at the point of bisection, because the segments are not commensurable in length

I say that  $AB$  is not so divided at another point



For, if possible, let it be divided at  $D$  also so that  $AC$  is not the same with  $DB$ , but  $AC$  is supposed greater, it is then clear that the squares on  $AD$ ,  $DB$  are also, as we proved above [Lemma], less than the squares on  $AC$ ,  $CB$ , and suppose that  $AD$ ,  $DB$  are medial straight lines commensurable in square only and containing a medial rectangle

Now let a rational straight line  $LF$  be set out, let there be applied to  $EF$  the rectangular parallelogram  $EK$  equal to the square on  $AB$ ,

and let  $EC$  be the square on  $AC$ , and  $CB$  the square on  $CB$ , [x 4]

therefore the remainder  $EM$  is equal to twice the rectangle  $AC$ ,  $CB$ , [x 17]

Again let the square on  $AD$  be applied to  $EF$ , and let  $AL$  be the square on  $AD$ , [x 17]

were proved less than the squares on  $AC$ ,  $CB$  [Lemma], therefore the remainder  $ML$  is also equal to twice the rectangle  $AD$ ,  $DB$

Now, since the squares on  $AC$ ,  $CB$  are medial therefore  $EG$  is medial

And it is applied to the rational straight line  $EF$ , therefore  $EH$  is rational and incommensurable in length with  $EF$  [x 22]

For the same reason  $HN$  is also rational and incommensurable in length with  $EF$

And, since  $AC$ ,  $CB$  are medial straight lines commensurable in square only, therefore  $AC$  is incommensurable in length with  $CB$

But, as  $AC$  is to  $CB$ , so is the square on  $AC$  to the rectangle  $AC$ ,  $CB$ , therefore the square on  $AC$  is incommensurable with the rectangle  $AC$ ,  $CB$ , [x 11]

But the squares on  $AC$ ,  $CB$  are commensurable with the square on  $AC$ , for  $AC$ ,  $CB$  are commensurable in square [x 15]

And twice the rectangle  $AC$ ,  $CB$  is commensurable with the rectangle  $AC$ ,  $CB$ , [x 6]

Therefore the squares on  $AC$ ,  $CB$  are also incommensurable with twice the rectangle  $AC$ ,  $CB$ , [x 13]

But  $EG$  is equal to the squares on  $AC, CB$ ,  
 and  $HK$  = equal to twice the rectangle  $AC, CB$ ;  
 therefore  $EG$  is incommensurable with  $HK$ ,  
 so that  $EH$  is also incommensurable in length with  $HN$  [vi 1, x 11]  
 And they are rational,

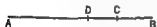
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commensurable in square only,  
 and  $EN$  will be = binomial straight line divided at different points,  $H$  and  $M$   
 And  $EH$  is not the same with  $MN$   
 For the squares on  $AC, CB$  are greater than the squares on  $AD, DB$   
 But the squares on  $AD, DB$  are greater than twice the rectangle  $AD, DB$ ,  
 therefore also the squares on  $AC, CB$ , that is,  $EG$ , are much greater than twice  
 the rectangle  $AD, DB$ , that is,  $MK$ ,  
 so that  $EH$  is also greater than  $MN$   
 Therefore  $EH$  is not the same with  $MN$  Q E D

#### PROPOSITION 45

*A major straight line is divided at one and the same point only*

Let  $AB$  be a major straight line divided at  $C$ , so that  $AC, CB$  are incommensurable in square and make the sum of the squares on  $AC, CB$  rational, but the rectangle  $AC, CB$  medial,



I say that  $AB$  is not so divided at another point

For, if possible, let it be divided at  $D$  also, so that  $AD, DB$  are also incommensurable in square and make the sum of the squares on  $AD, DB$  rational, but the rectangle contained by them medial

Then, since that by which the squares on  $AC, CB$  differ from the squares on  $AD, DB$  is also that by which twice the rectangle  $AD, DB$  differs from twice the rectangle  $AC, CB$ ,

while the squares on  $AC, CB$  exceed the squares on  $AD, DB$  by a rational area—for both are rational—

therefore twice the rectangle  $AD, DB$  also exceeds twice the rectangle  $AC, CB$  by a rational area, though they are medial

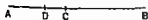
which is impossible [x 26]

Therefore a major straight line is not divided at different points,  
 therefore it is only divided at one and the same point Q E D

#### PROPOSITION 46

*The side of a rational plus a medial area is divided at one point only*

Let  $AB$  be the side of a rational plus a medial area divided at  $C$ , so that  $AC, CB$  are incommensurable in square and make the sum of the squares on  $AC, CB$  medial, but twice the rectangle  $AC, CB$  rational, [x 40]



I say that  $AB$  is not so divided at another point

For, if possible, let it be divided at  $D$  also, so that  $AD, DB$  are also incommensurable in square and make the sum of the squares on  $AD, DB$  rational, but twice the rectangle  $AD, DB$  medial, which is impossible



measurable in square and make the sum of the squares on  $AD$ ,  $DB$  medial, but twice the rectangle  $AD$ ,  $DB$  rational

Since, then, that by which twice the rectangle  $AC$ ,  $CB$  differs from twice the rectangle  $AD$ ,  $DB$  is also that by which the squares on  $AD$ ,  $DB$  differ from the squares on  $AC$ ,  $CB$ , while twice the rectangle  $AC$ ,  $CB$  exceeds twice the rectangle  $AD$ ,  $DB$  by a rational area, therefore the squares on  $AD$ ,  $DB$  also exceed the squares on  $AC$ ,  $CB$  by a rational area, though they are medial.

Therefore the side of a rational plus a medial area is not divided at different points, which is impossible [x 26]

therefore it is divided at one point only

Q E D

## PROPOSITION 47

The side of the sum of two medial areas is divided at one point only

Let  $AB$  be divided at  $C$ , so that  $AC$ ,  $CB$  are incommensurable in square and make the sum of the squares on  $AC$ ,  $CB$  medial, and the rectangle  $AC$ ,  $C$  medial and also incommensurable with the sum of the squares on them,

I say that  $AB$  is not divided at another point so as to fulfil the given condition. For, if possible, let it be divided at  $D$ , so that again  $AC$  is of course not the same as  $BD$ , but  $AC$  is supposed greater,

let a rational straight line  $EF$  be set out,

and let there be applied to  $EF$  the rectangle  $EG$  equal to the squares on  $AC$ ,  $CB$ ,

and the rectangle  $HK$  equal to twice the rectangle  $AC$ ,  $CB$ ,

therefore the whole  $EK$  is equal to the square on  $AB$  [ii 4]

Again let  $EL$ , equal to the squares on  $AD$ ,  $DB$ , be applied to  $EF$ , therefore the remainder twice the rectangle  $AD$ ,  $DB$ , is equal to the remainder  $MA$

And since by hypothesis the sum of the squares on  $AC$ ,  $CB$  is medial, therefore  $EG$  is also medial

And it is applied to the rational straight line  $EF$ , therefore  $HE$  is rational and incommensurable in length with  $EF$ , [x 22]

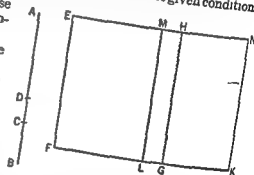
For the same reason

$HN$  is also rational and incommensurable in length with  $EF$

And since the sum of the squares on  $AC$ ,  $CB$  is incommensurable with twice the rectangle  $AC$ ,  $CB$ , therefore  $EG$  is also incommensurable with  $GN$ ,

so that  $EH$  is also incommensurable with  $HN$  [vi 1, x 11]

And they are rational, therefore  $EH$ ,  $HN$  are rational straight lines commensurable in square only, therefore  $EN$  is a binomial straight line divided at  $H$  [x 36]



Similarly we can prove that it is also divided at  $M$

And  $EH$  is not the same with  $MN$ ,

therefore a binomial has been divided at different points  
which is absurd

[x 42]

Therefore a side of the sum of two medial areas is not divided at different points,

therefore it is divided at one point only

Q E D

## DEFINITIONS II

1 Given a rational straight line and a binomial divided into its terms, such that the square on the greater term is greater than the square on the lesser by

2 but if the lesser term be commensurable in length with the rational straight line set out, let the whole be called a *second binomial*,

3 and if neither of the terms be commensurable in length with the rational straight line set out, let the whole be called a *third binomial*

4 Again, if the square on the greater term be greater than the square on the lesser by the square on a straight line incommensurable in length with the greater, then, if the greater term be commensurable in length with the rational straight line set out, let the whole be called a *fourth binomial*,

5 if the lesser, a *fifth binomial*,

6 and if neither, a *sixth binomial*

## PROPOSITION 48

To find the first binomial straight line

Let two numbers  $AC$ ,  $CB$  be set out such that the sum of them  $AB$  has to  $BC$  the ratio which a square number has to a square number, but has not to  $CA$  the ratio which a square number has to a square number, [Lemma 1 after x 28]  
let any rational straight line  $D$  be set out, and let  $EF$  be commensurable in length with  $D$



Therefore  $EF$  is also rational

Let it be contrived that,  
as the number  $BA$  is to  $AC$ , so is the square on  $EF$  to the square on  $FG$  [x 6, Por]

But  $AB$  has to  $AC$  the ratio which a number has to a number,  
therefore the square on  $EF$  also has to the square on  $FG$  the ratio which a number has to a number,  
so that the square on  $EF$  is commensurable with the square on  $FG$  [x 6]

And  $EF$  is rational,

therefore  $FG$  is also rational

And, since  $BA$  has not to  $AC$  the ratio which a square number has to a square number,  
neither, therefore, has the square on  $EF$  to the square on  $FG$  the ratio which a square number has to a square number

therefore  $EF$  is incommensurable in length with  $FG$  [x 9]

Therefore  $EF$ ,  $FG$  are rational straight lines commensurable in square only,  
therefore  $EG$  is binomial [x 36]

I say that it is also a first binomial straight line

For since, as the number  $BA$  is to  $AC$ , so is the square on  $EF$  to the square on  $FG$ ,

while  $BA$  is greater than  $AC$ ,

therefore the square on  $EF$  is also greater than the square on  $FG$

Let then the squares on  $FG$ ,  $H$  be equal to the square on  $EF$

Now since, as  $BA$  is to  $AC$ , so is the square on  $EF$  to the square on  $FG$ ,  
therefore, *convertendo*,

as  $AB$  is to  $BC$ , so is the square on  $EF$  to the square on  $H$  [v 19, Por]

But  $AB$  has to  $BC$  the ratio which a square number has to a square number,  
therefore the square on  $EF$  also has to the square on  $H$  the ratio which a square number has to a square number

Therefore  $EF$  is commensurable in length with  $H$ , [x 9]

therefore the square on  $EF$  is greater than the square on  $FG$  by the square on a straight line commensurable with  $EF$

And  $EF$ ,  $FG$  are rational, and  $EF$  is commensurable in length with  $D$

Therefore  $EF$  is a first binomial straight line

Q E D

#### PROPOSITION 49

To find the second binomial straight line

Let two numbers  $AC$ ,  $CB$  be set out such that the sum of them  $AB$  has to  $BC$  the ratio which a square number has to a square number, but has not to  $AC$  the ratio which a square number has to a square number,  
let a rational straight line  $D$  be set out, and let  $EF$  be commensurable in length with  $D$ ,

therefore  $EF$  is rational



Let it be contrived then that,

as the number  $CA$  is to  $AB$ , so also is the square on  $EF$  to the square on  $FG$ ,

[x 6 Por]

therefore the square on  $EF$  is commensurable with the square on  $FG$  [x 6]

Therefore  $FG$  is also rational

Now, since the number  $CA$  has not to  $AB$  the ratio which a square number has to a square number neither has the square on  $EF$  to the square on  $FG$  the ratio which a square number has to a square number

Therefore  $EF$  is incommensurable in length with  $FG$ , [x 9]

therefore  $EF$ ,  $FG$  are rational straight lines commensurable in square only,

therefore  $EG$  is binomial

[x 36]

It is next to be proved that it is also a second binomial straight line

For since inversely, as the number  $BA$  is to  $AC$ , so is the square on  $GF$  to the square on  $FE$ ,

while  $BA$  is greater than  $AC$ ,

therefore the square on  $GF$  is greater than the square on  $FE$

Let the squares on  $EF$ ,  $H$  be equal to the square on  $GF$ ,

therefore, *convertendo*, as  $AB$  is to  $BC$ , so is the square on  $FG$  to the square on  $H$  [v 19 Por]

But  $AB$  has to  $BC$  the ratio which a square number has to a square number,

therefore the square on  $FG$  also has to the square on  $H$  the ratio which a square

[x 9]  
the square on a

And  $FG$ ,  $FE$  are rational straight lines commensurable in square only, and  $EF$ , the lesser term, is commensurable in length with the rational straight line  $D$  set out

Therefore  $EG$  is a second binomial straight line

Q E D

### PROPOSITION 50

To find the third binomial straight line

Let two numbers  $AC$ ,  $CB$  be set out such that the sum of them  $AB$  has to  $BC$  the ratio which a square number has to a square number, but has not to  $AC$  the ratio which a square number has to a square number

Let any other number  $D$ , not square, be set out also, and let it not have to either of the numbers  $BA$ ,  $AC$  the ratio which a square number has to a square number



Let any rational straight line  $E$  be set out, and let it be contrived that, as  $D$  is to  $AB$ , so is the square on  $E$  to the square on  $FG$ ,  
therefore the square on  $E$  is commensurable with the square on  $FG$  [x 6, Por ]

And  $E$  is rational,

therefore  $FG$  is also rational

And, since  $D$  has not to  $AB$  the ratio which a square number has to a square number, neither has the square on  $E$  to the square on  $FG$  the ratio which a square number has to a square number,

therefore  $E$  is incommensurable in length with  $FG$  [x 9]

Next let it be contrived that, as the number  $BA$  is to  $AC$ , so is the square on  $FG$  to the square on  $GH$ , [x 6, Por ]

therefore the square on  $FG$  is commensurable with the square on  $GH$  [x 6]

But  $FG$  is rational,

therefore  $GH$  is also rational

And, since  $BA$  has not to  $AC$  the ratio which a square number has to a square number, neither has the square on  $FG$  to the square on  $HG$  the ratio which a square number has to a square number,

I say next that it is also a third binomial straight line

For since, as  $D$  is to  $AB$ , so is the square on  $E$  to the square on  $FG$ ,

and, as  $BA$  is to  $AC$  so is the square on  $FG$  to the square on  $GH$ ,

therefore, *ex aequali*, as  $D$  is to  $AC$ , so is the square on  $E$  to the square on  $GH$

[x 22]

But  $D$  has not to  $AC$  the ratio which a square number has to a square number,  
therefore neither has the square on  $E$  to the square on  $GH$  the ratio which a

[x 9]

therefore, *convertendo*, as  $AB$  is to  $BC$ , so is the square on  $EG$  to the square on  $K$  [v 19, Por]

But  $AB$  has to  $BC$  the ratio which a square number has to a square number,  
therefore the square on  $EG$  also has to the square on  $K$  the ratio which a square number has to a square number,

therefore  $EG$  is commensurable in length with  $K$  [x 9]

Therefore the square on  $EG$  is greater than the square on  $GH$  by the square

### PROPOSITION 51

To find the fourth binomial straight line

[x 9]

Let it be contrived that, as the number  $BA$  is to  $AC$ , so is the square on  $EF$  to the square on  $FG$ , [x 6, Por]  
therefore the square on  $EF$  is commensurable with the square on  $FG$ , [x 6]

therefore  $FG$  is also rational

Now, since  $BA$  has not to  $AC$  the ratio which a square number has to a square number,  
neither has the square on  $EF$  to the square on  $FG$  the ratio which a square number has to a square number



I say next that it is also a fourth binomial straight line

For since as  $BA$  is to  $AC$ , so is

therefore the square on  $E$

Let then the squares on  $FG$   $H$

therefore, *convertendo*, as the number  $AB$  is to  $BC$ , so is the square on  $EF$  to the square on  $H$  [v 19, Por]

But  $AB$  has not to  $BC$  the ratio which a square number has to a square number,  
therefore neither has the square on  $EF$  to the square on  $H$  the ratio which a square number has to a square number

Therefore  $EF$  is incommensurable in length with  $H$ , [x 9]  
 therefore the square on  $EF$  is greater than the square on  $GF$  by the square on  
 a straight line incommensurable with  $EF$

And  $EF$ ,  $FG$  are rational straight lines commensurable in square only, and  
 $EF$  is commensurable in length with  $D$

Therefore  $EG$  is a fourth binomial straight line

Q E D

## PROPOSITION 52

To find the fifth binomial straight line

Let two numbers  $AC$ ,  $CB$  be set out such that  $AB$  has not to either of them  
 the ratio which a square number has to a square number,

let any rational straight line  $D$  be set out,

and let  $EF$  be commensurable with  $D$ ,

therefore  $EF$  is rational

Let it be contrived that, as  $CA$  is to  $AB$ , so is the square on  $EF$   
 to the square on  $FG$  [x 6, Por]

But  $CA$  has not to  $AB$  the ratio which a square number  
 has to a square number,

therefore neither has the square on  $EF$  to the square on  $FG$  the  
 ratio which a square number has to a square number

Therefore  $EF$ ,  $FG$  are rational straight lines commensurable  
 in square only, [x 9]

therefore  $EG$  is binomial

[x 36]

I say next that it is also a fifth binomial straight line

For since, as  $CA$  is to  $AB$ , so is the square on  $EF$  to the square on  $FG$ ,  
 inversely, as  $BA$  is to  $AC$ , so is the square on  $FG$  to the square on  $FE$ ,

therefore the square on  $GF$  is greater than the square on  $FE$

Let then the squares on  $EF$ ,  $H$  be equal to the square on  $GF$ ,  
 therefore, *convertendo*, as the number  $AB$  is to  $BC$ , so is the square on  $GF$  to  
 the square on  $H$  [v 19, Por]

But  $AB$  has not to  $BC$  the ratio which a square number has to a square  
 number,

therefore neither has the square on  $FG$  to the square on  $H$  the ratio which a  
 square number has to a square number

Therefore  $FG$  is incommensurable in length with  $H$ , [x 9]  
 so that the square on  $FG$  is greater than the square on  $FE$  by the square on a  
 straight line incommensurable with  $FG$

And  $GF$ ,  $FE$  are rational straight lines commensurable in square only and  
 the lesser term  $EF$  is commensurable in length with the rational straight line  
 $D$  set out

Therefore  $EG$  is a fifth binomial straight line.

Q E D

## PROPOSITION 53

To find the sixth binomial straight line

Let two numbers  $AC$ ,  $CB$  be set out such that  $AB$  has not to either of them  
 the ratio which a square number has to a square number,

and let there also be another number  $D$  which is not square and which has not  
 to either of the numbers  $BA$ ,  $AC$  the ratio which a square number has to a  
 square number

Let any rational straight line  $E$  be set out,  
and let it be contrived that, as  $D$  is to  $AB$ , so is the square on  $E$  to the square on  $FG$ ,  
therefore the square on  $E$  is commensurable with the square on  $FG$  [x. 6]

And  $E$  is rational,

therefore  $FG$  is also rational

Now, since  $D$  has not to  $AB$  the ratio which a square number has to a square number,  
neither has the square on  $E$  to the square on  $FG$  the ratio which a square number has to a square number,  
therefore  $E$  is incommensurable in length with  $FG$  [x. 9]

Again, let it be contrived that, as  $BA$  is to  $AC$ , so is the square on  $FG$  to the square on  $GH$  [x. 6, Por.]

Therefore the square on  $FG$  is commensurable with the square on  $GH$  [x. 6]

Therefore the square on  $GH$  is rational,

therefore  $GH$  is rational

And, since  $BA$  has not to  $AC$  the ratio which a square number has to a square number,  
neither has the square on  $FG$  to the square on  $GH$  the ratio which a square number has to a square number,

therefore  $FG$  is incommensurable in length with  $GH$  [x. 9]

Therefore  $FG$ ,  $GH$  are rational straight lines commensurable in square only,  
therefore  $FH$  is binomial [x. 36]

It is next to be proved that it is also a sixth binomial straight line

For since, as  $D$  is to  $AB$ , so is the square on  $E$  to the square on  $FG$ ,  
and also, as  $BA$  is to  $AC$ , so is the square on  $FG$  to the square on  $GH$ ,  
therefore, *ex æquali*, as  $D$  is to  $AC$ , so is the square on  $E$  to the square on  $GH$ . [x. 22]

But  $D$  has not to  $AC$  the ratio which a square number has to a square number,

therefore neither has the square on  $E$  to the square on  $GH$  the ratio which a square number has to a square number,  
therefore  $E$  is incommensurable in length with  $GH$  [x. 9]

But 1

in length with  $E$

And since, as  $BA$  is to  $AC$ , so is the square on  $FG$  to the square on  $GH$ ,  
therefore the square on  $FG$  is greater than the square on  $GH$

Let then the squares on  $GH$ ,  $K$  be equal to the square on  $FG$ ,  
therefore, *convertendo*, as  $AB$  is to  $BC$ , so is the square on  $FG$  to the square on  $K$  [x. 19 Por.]

But  $AB$  has not to  $BC$  the ratio which a square number has to a square number,  
so that neither has the square on  $FG$  to the square on  $K$  the ratio which a square number has to a square number

[x. 9]  
by the square on

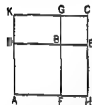
And  $FG$ ,  $GH$  are rational straight lines commensurable in square only, and neither of them is commensurable in length with the rational straight line  $E$  set out

Therefore  $FH$  is a sixth binomial straight line

Q E D

## LEMMA

Let there be two squares  $AB$ ,  $BC$ , and let them be placed so that  $DB$  is in a straight line with  $BE$ ,



between  $AB$ ,  $BC$ , and further that  $DC$  is a mean proportional between  $AC$ ,  $CB$

For, since  $DB$  is equal to  $BF$ , and  $BE$  to  $BG$ ,  
therefore the whole  $DE$  is equal to the whole  $FG$

But  $DE$  is equal to each of the straight lines  $AH$ ,  $KC$ ,  
and  $FG$  is equal to each of the straight lines  $AK$ ,  $HC$  [I. 34],

therefore each of the straight lines  $AH$ ,  $KC$  is also equal to each of the straight lines  $AK$ ,  $HC$

Therefore the parallelogram  $AC$  is equilateral

And it is also rectangular,

therefore  $AC$  is a square

And since, as  $FB$  is to  $BG$ , so is  $DB$  to  $BE$ ,

while, as  $FB$  is to  $BG$ , so is  $AB$  to  $DG$ ,

and, as  $DB$  is to  $BE$ , so is  $DG$  to  $BC$ ,

[VI. 1]

therefore also, as  $AB$  is to  $DG$ , so is  $DG$  to  $BC$

[V. 11]

Therefore  $DG$  is a mean proportional between  $AB$ ,  $BC$

I say next that  $DC$  is also a mean proportional between  $AC$ ,  $CB$

For since, as  $AD$  is to  $DK$ , so is  $KG$  to  $GC$ —

for they are equal respectively—

and, *componendo*, as  $AK$  is to  $KD$ , so is  $KC$  to  $CG$ ,

[V. 18]

while, as  $AK$  is to  $KD$ , so is  $AC$  to  $CD$ ,

and, as  $KC$  is to  $CG$ , so is  $DC$  to  $CB$ ,

[VI. 1]

therefore also as  $AC$  is to  $DC$ , so is  $DC$  to  $CB$

[V. 11]

Therefore  $DC$  is a mean proportional between  $AC$ ,  $CB$

Being what it was proposed to prove

## PROPOSITION 54

If an area be contained by a rational straight line and the first binomial, the “side of the area is the irrational straight line which is called binomial

For let the area  $AC$  be contained by the rational straight line  $AB$  and the first binomial  $AD$ ,

I say that the “side” of the area  $AC$  is the irrational straight line which is called binomial

For since  $AD$  is a first binomial straight line let it be divided into its terms at  $E$ ,

and let  $AE$  be the greater term

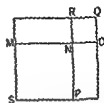
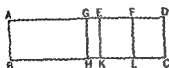
It is then manifest that  $AE$ ,  $ED$  are rational straight lines commensurable in square only,



the square on  $AE$  is greater than the square on  $ED$  by the square on a straight line commensurable with  $AE$ ,  
and  $AE$  is commensurable in length with the rational straight line  $AB$  set out.

[x Def 11]

Let  $ED$  be bisected at the point  $F$



Then, since the square on  $AE$  is greater than the square on  $ED$  by the square on a straight line commensurable with  $AE$ ,  
therefore, if there be applied to the greater  $AE$  a parallelogram equal to the fourth part of the square on the less that is to the square on  $EF$ , and deficient by a square figure it divides it into commensurable parts [x 17]

Let then the rectangle  $AG$   $GE$  equal to the square on  $EF$  be applied to  $AE$ ,  
therefore  $AG$  is commensurable in length with  $FG$

Let  $GH$ ,  $EK$ ,  $FL$  be drawn from  $G$ ,  $E$ ,  $F$  parallel to either of the straight lines  $AB$ ,  $CD$ ,

let the square  $SN$  be constructed equal to the parallelogram  $AH$ , and the square  $VQ$  equal to  $GK$ , [x 14]

and let them be placed so that  $MN$  is in a straight line with  $NO$ ,

therefore  $RN$  is also in a straight line with  $NP$

And let the parallelogram  $SQ$  be completed,

therefore  $SQ$  is a square [Lemma]

Now, since the rectangle  $AG$   $GE$  is equal to the square on  $EF$ ,

therefore as  $AG$  is to  $EF$ , so is  $FE$  to  $EG$ , [vi 17]

therefore also, as  $AH$  is to  $EL$ , so is  $EL$  to  $KG$ , [vi 11]

therefore  $EF$  is the mean proportional between  $AH$  and  $KG$

But  $AH$

But  $MR$  is also a mean proportional between the same  $SN$ ,  $NQ$ , [Lemma]

therefore  $EL$  is equal to  $MR$ ,

so that it is also equal to  $PO$

But  $AH$   $KG$  are also equal to  $SN$   $NQ$

therefore the whole  $AC$  is equal to the whole  $SQ$ , that is, to the square on  $MO$ ,

therefore  $MO$  is the 'side' of  $AC$

I say next that  $MO$  is binomial

For since  $AG$  is commensurable with  $GE$

therefore  $AE$  is also commensurable with each of the straight lines  $AG$ ,  $GE$  [x 15]

But  $AE$  is also by hypothesis commensurable with  $AB$ ,

therefore  $AG$   $GE$  are also commensurable with  $AB$  [x 12]

And  $AB$  is rational

therefore each of the straight lines  $AG$ ,  $GE$  is also rational,

therefore each of the rectangles  $AH$   $KG$  is rational, [x 19]

and  $AH$  is commensurable with  $GK$

But  $AH$  is equal to  $SN$ , and  $GK$  to  $NQ$ , therefore  $SN, NQ$ , that is, the squares on  $MN, NO$ , are rational and commensurable.

And, since  $AE$  is incommensurable in length with  $ED$ ,  
 while  $AE$  is commensurable with  $AG$ , and  $DE$  is commensurable with  $EF$ ,  
 therefore  $AG$  is also incommensurable with  $EF$ , [x 13]  
 so that  $AH$  is also incommensurable with  $EL$  [vi 1, x 11]

But  $AH$  is equal to  $SN$ , and  $EL$  to  $MR$ .

therefore  $SN$  is also incommensurable with  $MR$

But  $\square SN \equiv$  to  $MR$ , so is  $PN$  to  $NR$ , [VI 1]

therefore  $PN$  is incommensurable with  $NR$  [x 11]

But  $PN$  is equal to  $MN$ , and  $NR$  to  $NO$ .

therefore  $MN$  is incommensurable with  $NO$

And the square on  $MN$  is commensurable with the square on  $NO$ ,

and each is rational.

therefore  $MN$ ,  $NO$  are rational straight lines commensurable in square only

Therefore  $MO$  is binomial ( $\times 36$ ) and the "side" of  $AC$   $q \in D$

### PROPOSITION 55

*If an area be contained by a rational straight line and the second binomial the "side" of the area is the irrational straight line which is called a first binomial*

For let the area  $ABCD$  be contained by the rational straight line  $AB$  and the second binomial  $AD$ .

I say that the "side" of the area  $AC$  is a first bimedial straight line

For, since  $AD$  is a second binomial straight line, let it be divided into its

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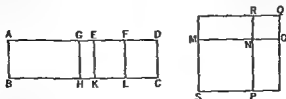
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5

and deficient by a square figure.

therefore  $AG$  is commensurable in length with  $GE$  [x 17]

Through  $G, E, F$  let  $GH, EK, FL$  be drawn parallel to  $AB, CD$ , let the square  $SN$  be constructed equal to the parallelogram  $AH$  and the square  $NQ$  equal to  $GA$ .



and let them be placed so that  $MN$  is in a straight line with  $NO$ ,

therefore,  $RN$  is also in a straight line with  $N$ .

Let the square  $SQ$  be completed

It is then manifest from what was proved before that  $MR$  is a mean proportional between  $SN$ ,  $NQ$  and is equal to  $EL$ , and that  $MO$  is the "side" of the area  $AC$

It is now to be proved that  $MO$  is a first bimedial straight line  
 Since  $AE$  is commensurable in length with  $ED$ ,

[x 13]

$G$ ,  $GE$  [x 15]

But  $AE$  is incommensurable in length with  $AB$ ,  
 therefore  $AG$ ,  $GE$  are also incommensurable with  $AB$  [x 13]

Therefore  $BA$ ,  $AG$  and  $BA$ ,  $GE$  are pairs of rational straight lines commensurable in square only,

so that each of the rectangles  $AH$ ,  $GK$  is medial [x 21]

Hence each of the squares  $SN$ ,  $NQ$  is medial

Therefore  $MN$ ,  $NO$  are also medial

And, since  $AG$  is commensurable in length with  $GE$ ,  
 $AH$  is also commensurable with  $GK$ , [vi 1, x 11]

that is,  $SN$  is commensurable with  $NQ$ ,

that is, the square on  $MN$  with the square on  $NO$

And, since  $AE$  is incommensurable in length with  $ED$ ,  
 while  $AE$  is commensurable with  $AG$ ,  
 and  $ED$  is commensurable with  $EF$ ,

therefore  $AG$  is incommensurable with  $EF$ , [x 13]

so that  $AH$  is also incommensurable with  $EL$ ,

that is,  $SN$  is incommensurable with  $MR$ ,

that is,  $PN$  with  $NR$ , [vi 1, x 11]

that is,  $MN$  is incommensurable in length with  $NO$

But  $MA$ ,  $NO$  were proved to be both medial and commensurable in square  
 therefore  $MN$ ,  $NO$  are medial straight lines commensurable in square only

I say next that they also contain a rational rectangle

For since  $DE$  is, by hypothesis, commensurable with each of the straight lines  $AB$ ,  $EF$ ,

therefore  $EF$  is also commensurable with  $EK$  [x 13]

And each of them is rational,

therefore  $EL$ , that is,  $MR$  is rational, [x 10]

Therefore  $MO$  is a first bimedial straight line

Q E D

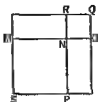
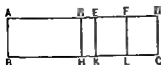
# PROPOSITION 56

If an area be contained by a rational straight line and the third binomial, the "side" of the area is the irrational straight line called a second bimedial

For let the area  $ABCD$  be contained by the rational straight line  $AB$  and the third binomial  $AD$  divided into its terms at  $E$ , of which terms  $AE$  is the greater,

I say that the "side" of the area  $AC$  is the irrational straight line called a second bimedial

For let the same construction be made as before  
Now, since  $AD$  is a third binomial straight line,



and neither of the terms  $AE$ ,  $ED$  is commensurable in length with  $AB$

[x Def 11 3]

Then, in manner similar to the foregoing, we shall prove that  $MO$  is the "side" of the area  $AC$ ,

and  $MN$ ,  $NO$  are medial straight lines commensurable in square only, so that  $MO$  is bimedral

It is next to be proved that it is also a second bimedral straight line

Since  $DE$  is incommensurable in length with  $AB$  that is, with  $EK$ , and  $DE$  is commensurable with  $EF$ ,

therefore  $EF$  is incommensurable in length with  $EK$  [x 13]

And they are rational,

therefore  $FE$ ,  $EK$  are rational straight lines commensurable in square only

Therefore  $EL$ , that is,  $MR$ , is medial [x 21]

And it is contained by  $MN$ ,  $NO$ ,

therefore the rectangle  $MN$ ,  $NO$  is medial

Therefore  $MO$  is a second bimedral straight line [x 36]

Q E D

#### PROPOSITION 57

If an area be contained by a rational straight line and the fourth binomial, the "side" of the area is the irrational straight line called major

For let the area  $AC$  be contained by the rational straight line  $AB$  and the fourth binomial  $AD$  divided into its terms at  $E$ , of which terms let  $AE$  be the greater,

I say that the "side" of the area  $AC$  is the irrational straight line called major

For, since  $AD$  is a fourth binomial straight line, therefore  $AE$ ,  $ED$  are rational straight lines commensurable in square only, the square on  $AE$  is greater than the square on  $ED$  by the square on a straight line incommensurable with  $AE$ ,

and  $AE$  is commensurable in length with  $AB$  [x Def 11 4]

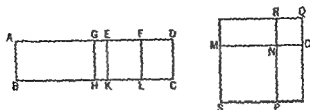
Let  $DE$  be bisected at  $F$ ,

and let there be applied to  $AE$  a parallelogram the rectangle  $AG$ ,  $GE$ , equal to the square on  $EF$ ,

therefore  $AG$  is incommensurable in length with  $GE$  [x 18]

Let  $GH$ ,  $EK$ ,  $FL$  be drawn parallel to  $AB$ ,

and let the rest of the construction be as before,  
it is then manifest that  $MO$  is the "side" of the area  $AC$



It is next to be proved that  $MO$  is the irrational straight line called major  
Since  $AG$  is incommensurable with  $EG$ ,  
 $AH$  is also incommensurable with  $GK$ , that is  $SN$  with  $NQ$ , [v 1, x 11]  
therefore  $MN$ ,  $NO$  are incommensurable in square

And, since  $AE$  is commensurable with  $AB$ ,  
 $AK$  is rational, [x 10]

and it is equal to the squares on  $MN$ ,  $NO$ ,  
therefore the sum of the squares on  $MN$ ,  $NO$  is also rational  
And, since  $DE$  is incommensurable in length with  $AB$ , that is, with  $EK$ ,  
while  $DE$  is commensurable with  $EF$ ,

therefore  $EF$  is incommensurable in length with  $EK$  [x 13]  
Therefore  $EK$ ,  $EF$  are rational straight lines commensurable in square only,  
therefore  $LE$ , that is,  $MR$ , is medial [x 21]

And it is contained by  $MN$ ,  $NO$ ,  
therefore the rectangle  $MN$ ,  $NO$  is medial

And the [sum] of the squares on  $MN$ ,  $NO$  is rational,  
and  $MN$ ,  $NO$  are incommensurable in square

But if two straight lines incommensurable in square and making the sum of  
the squares on them rational but the rectangle contained by them medial, be  
added together the whole is irrational and is called major [x 39]

Therefore  $MO$  is the irrational straight line called major and is the "side" of  
the area  $AC$  Q E D

### PROPOSITION 58

If an area be contained by a rational  
of the area is the irrational  
area

For let the area  $AC$  be contained by the rational straight line  $AB$  and the  
fifth binomial  $AD$  divided into its terms at  $E$ , so that  $AE$  is the greater term,  
I say that the "side" of the area  $AC$  is the irrational straight line called the  
side of a rational plus a medial area

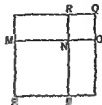
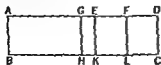
For let the same construction be made as before shown,

it is then manifest that  $MO$  is the "side" of the area  $AC$

It is then to be proved that  $MO$  is the side of a rational plus a medial area  
For, since  $AG$  is incommensurable with  $GE$ , [x 18]

therefore  $AH$  is also incommensurable with  $HE$ , [vi 1, x 11]  
that is the square on  $MA$  with the square on  $NO$ ,  
therefore  $MA$ ,  $NO$  are incommensurable in square

And, since  $AD$  is a fifth binomial straight line, and  $ED$  the lesser segment, therefore  $ED$  is commensurable in length with  $AB$  [X. Def. 11. 5]



But  $AE$  is incommensurable with  $ED$ ,

therefore  $AB$  is also incommensurable in length with  $AE$  [X. 13]

Therefore  $AA$ , that is the sum of the squares on  $MN$ ,  $NO$ , is medial [X. 21]

And, since  $DE$  is commensurable in length with  $AB$ , that is, with  $FA$ ,

while  $DE$  is commensurable with  $EF$ ,

therefore  $EF$  is also commensurable with  $EK$  [X. 12]

And  $EK$  is rational,

therefore  $EL$ , that is,  $MR$ , that is, the rectangle  $MN$ ,  $NO$ , is also rational

[X. 19]

Therefore  $MN$ ,  $NO$  are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational

Therefore  $MO$  is the side of a rational plus a medial area [X. 40] and is the "side" of the area  $AC$

Q.E.D.

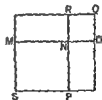
### PROPOSITION 59

If an area be contained by a rational straight line and the sixth binomial, the "side" of the area is the irrational straight line called the side of the sum of two medial areas

For let the area  $ABCD$  be contained by the rational straight line  $AB$  and the sixth binomial  $AD$ , divided into its terms at  $E$ , so that  $AE$  is the greater term,

I say that the "side" of  $AC$  is the side of the sum of two medial areas

Let the same construction be made as before shown



'side' of  $AC$  and that  $MN$  is incommen-

le in length with  $AB$ ,

therefore

therefore

A

1]

therefore  $FE$  is also incommensurable with  $EK$ , [x 13]

therefore  $FE$ ,  $EK$  are rational straight lines commensurable in square only,  
therefore  $EL$ , that is,  $MR$ , that is, the rectangle  $MN$ ,  $NO$ , is medial [x 21]

And, since  $AE$  is incommensurable with  $EF$ ,

$AK$  is also incommensurable with  $EL$  [vi 1, x 11]

But  $AK$  is the sum of the squares on  $MN$ ,  $NO$ ,

and  $EL$  is the rectangle  $MN$ ,  $NO$ ,

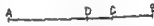
therefore the sum of the squares on  $MN$ ,  $NO$  is incommensurable with the rectangle  $MN$ ,  $NO$

And each of them is medial, and  $MN$ ,  $NO$  are incommensurable in square

Therefore  $MO$  is the side of the sum of two medial areas [x 41], and is the "side" of  $AC$  Q E D

### LEMMA

If a straight line be cut into unequal parts, the squares on the unequal parts are greater than twice the rectangle contained by the unequal parts

Let  $AB$  be a straight line, and let it be cut into unequal parts at  $C$ , and let  $AC$  be the greater, 

I say that the squares on  $AC$ ,  $CB$  are greater than twice the rectangle  $AC$ ,  $CB$

For let  $AB$  be bisected at  $D$

Since then, a straight line has been cut into equal parts at  $D$ , and into unequal parts at  $C$ ,

therefore the rectangle  $AC$ ,  $CB$  together with the square on  $CD$  is equal to the square on  $AD$ , [ii 5]

so that the rectangle  $AC$ ,  $CB$  is less than the square on  $AD$ ,

therefore twice the rectangle  $AC$ ,  $CB$  is less than double of the square on  $AD$

But the squares on  $AC$ ,  $CB$  are double of the squares on  $AD$ ,  $DC$ , [ii 9]

therefore the squares on  $AC$ ,  $CB$  are greater than twice the rectangle  $AC$ ,  $CB$  Q E D

### PROPOSITION 60

The square on the binomial straight line applied to a rational straight line produces as breadth the first binomial

Let  $AB$  be a binomial straight line divided into its terms at  $C$ , so that  $AC$  is the greater term,

let a rational straight line  $DE$  be set out

and let  $DEFG$  equal to the square on  $AB$  be applied to  $DE$  producing  $DG$  as its breadth,

I say that  $DC$  is as  $EG$  to  $AC$  as  $BC$  to  $AG$

eq

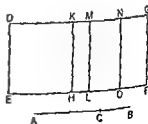
square on  $BC$

therefore the remainder, twice the rectangle  $AC$ ,  $CB$ , is equal to  $MF$

Let  $MG$  be bisected at  $N$  and let  $NO$  be drawn parallel [to  $ML$  or  $GF$ ]

Therefore each of the rectangles  $MO$ ,  $NO$  is equal to once the rectangle  $AC$ ,  $CB$

Now, since  $AB$  is a binomial divided into its terms at  $C$ ,



therefore  $AC$ ,  $CB$  are rational straight lines commensurable in square only, [x 36]  
 therefore the squares on  $AC$ ,  $CB$  are rational and commensurable with one another,

so that the sum of the squares on  $AC$ ,  $CB$  is also rational [x 15]  
 And it is equal to  $DL$ ,

therefore  $DL$  is rational

And it is applied to the rational straight line  $DE$ ,

therefore  $DM$  is rational and commensurable in length with  $DE$  [x 20]

Again, since  $AC$ ,  $CB$  are rational straight lines commensurable in square only,

therefore twice the rectangle  $AC$ ,  $CB$  is rational [x 21]

And it is applied to  $DE$

therefore  $ML$ , that is,  $DM$ , is commensurable in length with  $DE$ , [x 23]

and is commensurable in length with  $DE$ ,  
 therefore  $DM$  is incommensurable in length with  $MG$  [x 13]

And they are rational;

therefore  $DM$ ,  $MG$  are rational straight lines commensurable in square only,  
 therefore  $DG$  is binomial [x 36]

It is next to be proved that it is also a first binomial straight line

Since the rectangle  $AC$ ,  $CB$  is a mean proportional between the squares on  $AC$ ,  $CB$ , [cf Lemma after x 33]

therefore  $MO$  is also a mean proportional between  $DH$ ,  $KL$

Therefore, as  $DH$  is to  $MO$ , so is  $MO$  to  $KL$ ,

that is, as  $DK$  is to  $MN$ , so is  $MN$  to  $MK$ , [vi 1]

therefore the rectangle  $DK$ ,  $KM$  is equal to the square on  $MN$  [vi 17]

And, since the square on  $AC$  is commensurable with the square on  $CB$ ,

$DH$  is also commensurable with  $KL$ ,

so that  $DK$  is also commensurable with  $KM$  [vi 1 x 11]

And, since the squares on  $AC$ ,  $CB$  are greater than twice the rectangle  $AC$ ,  $CB$ , [Lemma]

therefore  $DL$  is also greater than  $MF$ ,

so that  $DM$  is also greater than  $MG$  [vi 1]

And the rectangle  $DK$ ,  $KM$  is equal to the square on  $MN$ , that is, to the fourth part of the square on  $MG$ ,

and  $DK$  is commensurable with  $KM$

But, if there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into commensurable parts, the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater, [x 17]

therefore the square on  $DM$  is greater than the square on  $MG$  by the square on a straight line commensurable with  $DM$

And  $DM$ ,  $MG$  are rational,

and  $DM$ , which is the greater term, is commensurable in length with the rational straight line  $DE$  set out

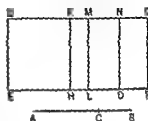
Therefore  $DG$  is a first binomial straight line [x Defn 11 1]



## PROPOSITION 61

*The square on the first binomial straight line applied to a rational straight line produces as breadth the second binomial*

Let  $AB$  be a first binomial straight line divided into its medials at  $C$ , of which medials  $AC$  is the greater,



its breadth,

I say that  $DG$  is a second binomial straight line

For let the same construction as before be made

Then since  $AB$  is a first binomial divided at  $C$ ,  
therefore  $AC, CB$  are medial straight lines commensurable in square only, and containing a rational rectangle, [x 37]  
so that the squares on  $AC, CB$  are also medial [x 21]

Therefore  $DL$  is medial [x 15 and 13, Por.]

And it has been applied to the rational straight line  $DE$ ,  
therefore  $MD$  is rational and incommensurable in length with  $DE$  [x 22]

Again since twice the rectangle  $AC, CB$  is rational,  $MF$  is also rational

And it is applied to the rational straight line  $ML$ ,  
therefore  $MG$  is also rational and commensurable in length with  $ML$ , that is,  $DE$ , [x 20]

therefore  $DM$  is incommensurable in length with  $MG$  [x 13]

And they are rational,  
therefore  $DM, MG$  are rational straight lines commensurable in square only,  
therefore  $DG$  is binomial [x 3b]

It is next to be proved that it is also a second binomial straight line

For, since the squares on  $AC, CB$  are greater than twice the rectangle  $AC, CB$ ,

therefore  $DL$  is also greater than  $MF$ ,  
so that  $DM$  is also greater than  $MG$  [vi 1]

And, since the square on  $AC$  is commensurable with the square on  $CB$ ,

$DH$  is also commensurable with  $KL$ ,  
so that  $DA$  is also commensurable with  $KM$  [vi 1, x 11]

And the rectangle  $DA, KM$  is equal to the square on  $MN$ ,  
the square on [x 17]

[x Def 11 2]

## PROPOSITION 62

*The square on the second binomial straight line applied to a rational straight line produces as breadth the third binomial*

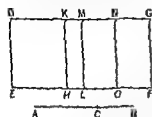
Let  $AB$  be a second binomial straight line divided into its medials at  $C$ , so that  $AC$  is the greater segment

let  $DE$  be any rational straight line,

and to  $DE$  let there be applied the parallelogram  $DF$  equal to the square on  $AB$  and producing  $DG$  as its breadth,

I say that  $DG$  is a third binomial straight line

Let the same construction be made as before shown



Then, since  $AB$  is a second binomial divided at  $C$ ,

therefore  $AC$ ,  $CB$  are medial straight lines commensurable in square only and containing a medial rectangle, [x 35]

so that the sum of the squares on  $AC$ ,  $CB$  is also medial [x 15 and 23 Por.]

And it is equal to  $DL$ ,

therefore  $DL$  is also medial

And it is applied to the rational straight line  $DE$ , therefore  $MD$  is also rational and incommensurable in length with  $DE$  [x 22]

For the same reason,

...

And, since  $AC$  is incommensurable in length with  $CB$ , and, as  $AC$  is to  $CB$ , so is the square on  $AC$  to the rectangle  $AC$ ,  $CB$ , therefore the square on  $AC$  is also incommensurable with the rectangle  $AC$ ,  $CB$  [x 11]

Hence the sum of the squares on  $AC$ ,  $CB$  is incommensurable with twice the rectangle  $AC$ ,  $CB$ , [x 12, 13]

that is,  $DL$  is incommensurable with  $MF$

so that  $DM$  is also incommensurable with  $MG$  [vi 1, x 11]

And they are rational,

therefore  $DG$  is binomial

[x 30]

It is to be proved that it is also a third binomial straight line

In manner similar to the foregoing we may conclude that  $DM$  is greater than  $MG$ ,

and that  $DK$  is commensurable with  $KM$

And the rectangle  $DK$ ,  $KM$  is equal to the square on  $MN$ ,

...

$DE$

Therefore  $DG$  is a third binomial straight line

[x Def II 3]

Q E D

# PROPOSITION 63

The ...

...

Let the same construction be made as before shown  
Then, since  $AB$  is a major straight line divided at  $C$ ,

$AC, CB$  are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial [x 39]

Since, then, the sum of the squares on  $AC, CB$  is rational, therefore  $DL$  is rational, therefore  $DM$  is also rational and commensurable in length with  $DE$  [x 20]

Again, since twice the rectangle  $AC, CB$ , that is  $MF$ , is medial, and it is applied to the rational straight line  $ML$ , therefore  $VG$  is also rational and incommensurable in length with  $DE$ , [x 22]

therefore  $DM$  is also incommensurable in length with  $MG$  [x 13]

Therefore  $DM, MG$  are rational straight lines commensurable in square only,

therefore  $DG$  is binomial [x 36]

It is to be proved that it is also a fourth binomial straight line

In manner similar to the foregoing we can prove that  $DM$  is greater than  $MG$ ,

and that the rectangle  $DK, KM$  is equal to the square on  $MN$

Since then the square on  $AC$  is incommensurable with the square on  $CB$ , therefore  $DH$  is also incommensurable with  $KL$ ,

so that  $DA$  is also incommensurable with  $KM$  [x 11, x 12]

But, if there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into incommensurable parts then the square on the greater will be greater than the square on the less by the square on a straight line incommensurable in length with the greater, [x 13] therefore the square on  $DM$  is greater than the square on  $MG$  by the square on a straight line incommensurable with  $DM$

And  $DM, MG$  are rational straight lines commensurable in square only, and  $DM$  is commensurable with the rational straight line  $DE$  set out

Therefore  $DG$  is a fourth binomial straight line [x Def II 4]

Q E D

# PROPOSITION 64

... straight

straight

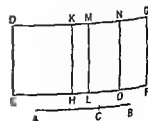
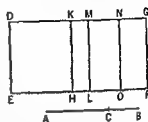
lines at  $C$ , so that  $AC$  is the greater, let a rational straight line  $DE$  be set out and let there be applied to  $DE$  the parallelogram  $DF$  equal to the square on  $AB$ , producing  $DG$  as its breadth,

I say that  $DG$  is a fifth binomial straight line

Let the same construction as before be made

Since then  $AB$  is the side of a rational plus a medial area, divided at  $C$ ,

therefore  $AC, CB$  are straight lines incommensurable in square which make



the sum of the squares on them medial, but the rectangle contained by them rational [x 40]

Since, then, the sum of the squares on  $AC$ ,  $CB$  is medial,  
therefore  $DL$  is medial,

so that  $DM$  is rational and incommensurable in length with  $DE$  [x 22]

Again, since twice the rectangle  $AC$ ,  $CB$ , that is  $MF$ , is rational,  
therefore  $MG$  is rational and commensurable with  $DE$  [x 20]

Therefore  $DM$  is incommensurable with  $MG$ , [x 13]

therefore  $DM$ ,  $MG$  are rational straight lines commensurable in square only,  
therefore  $DG$  is binomial [x 36]

I say next that it is also a fifth binomial straight line

For it can be proved similarly that the rectangle  $DK$ ,  $KM$  is equal to the square on  $IN$ ,

and that  $DK$  is incommensurable in length with  $KM$ ,  
therefore the square on  $DM$  is greater than the square on  $MG$  by the square on a straight line incommensurable with  $DM$  [x 18]

And  $DM$ ,  $MG$  are commensurable in square only, and the less  $MG$  is commensurable in length with  $DE$

Therefore  $DG$  is a fifth binomial

Q E D

#### PROPOSITION 65

*The square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial*

Let  $AB$  be the side of the sum of two medial areas, divided at  $C$ ,  
let  $DE$  be a rational straight line

and let there be applied to  $DE$  the parallelogram  $DF$  equal to the square on  $AB$ , producing  $DG$  as its breadth,

I say that  $DG$  is a sixth binomial straight line

For let the same construction be made as before

Then since  $AB$  is the side of the sum of two medial areas, divided at  $C$ ,

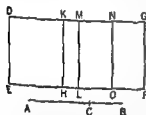
therefore  $AC$ ,  $CB$  are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial and moreover the sum of the squares on them incommensurable with the rectangle contained by them, [x 41]

so that, in accordance with what was before proved each of the rectangles  $DL$ ,  $MF$  is medial

And they are applied to the rational straight line  $DE$ ,

therefore each of the straight lines  $DM$ ,  $MG$  is rational and incommensurable in length with  $DE$  [x 22]

And, since the sum of the squares on  $AC$ ,  $CB$  is incommensurable with twice the rectangle  $AC$ ,  $CB$ ,



And they are applied to the rational straight line  $DE$ ,  
therefore each of the straight lines  $DM$ ,  $MG$  is rational and incommensurable in length with  $DE$  [x 22]

And, since the sum of the squares on  $AC$ ,  $CB$  is incommensurable with twice the rectangle  $AC$ ,  $CB$ ,

TI  
ther

I say next that it is also a sixth binomial straight line

Similarly again we can prove that the rectangle  $DK$ ,  $KM$  is equal to the square on  $MN$ ,

and  
by

the rational straight line  $DE$  set out

Therefore  $DC$  is a sixth binomial straight line

Q E D

### PROPOSITION 66

*A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order*

Let  $AB$  be binomial, and let  $CD$  be commensurable in length with  $AB$ ;

I say that  $CD$  is binomial and the same  
in order with  $AB$

For, since  $AB$  is binomial,  
let it be divided into its terms at  $E$ ,

and let  $AE$  be the greater term;

therefore  $AE$ ,  $EB$  are rational straight lines commensurable in square only [x 36]

Let it be contrived that,

as  $AB$  is to  $CD$ , so is  $AE$  to  $CF$ , [vi 12]

therefore also the remainder  $EB$  is to the remainder  $FD$  as  $AB$  is to  $CD$  [v 19]

But  $AB$  is commensurable in length with  $CD$ ,  
therefore  $AE$  is also commensurable with  $CF$ , and  $EB$  with  $FD$  [x 11]

And  $AE$ ,  $EB$  are rational,

therefore  $CF$ ,  $FD$  are also rational

And, as  $AE$  is to  $CF$ , so is  $EB$  to  $FD$  [v 11]

Therefore, alternately, as  $AE$  is to  $EB$ , so is  $CF$  to  $FD$  [v 16]

But  $AE$ ,  $EB$  are commensurable in square only,

therefore  $CF$ ,  $FD$  are also commensurable in square only. [x 11]

And they are rational,

therefore  $CD$  is binomial [x 36]

I say next that it is the same in order with  $AB$

For the square on  $AE$  is greater than the square on  $EB$  either by the square on a straight line commensurable with  $AE$  or by the square on a straight line incommensurable with it

If then the square on  $AE$  is greater than the square on  $EB$  by the square on a straight line commensurable with  $AE$ ,

the square on  $CF$  will also be greater than the square on  $FD$  by the square on a straight line commensurable with  $CF$  [x 13]

And, if  $AE$  is commensurable with the rational straight line set out,  $CF$  will also be commensurable with it, [x 12]

and for this reason each of the straight lines  $AB$ ,  $CD$  is a first binomial, that is, the same in order [x Def II 1]

But, if  $EB$  is commensurable with the rational straight line set out,  $FD$  is also commensurable with it, [x 12]

and for this reason again  $CD$  will be the same in order with  $AB$ ,  
for each of them will be a second binomial [x. Def II 2]

But, if neither of the straight lines  $AE$ ,  $EB$  is commensurable with the rational straight line set out, neither of the straight lines  $CF$ ,  $FD$  will be commensurable with it, [x 13]

and each of the straight lines  $AB$ ,  $CD$  is a fourth binomial [x Def 11 4]

But, if  $EB$  is so commensurable so is  $FD$  also  
and each of the straight lines  $AB$ ,  $CD$  will be a fifth binomial [x Def 11 5]

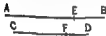
But, if neither of the straight lines  $AE$ ,  $EB$  is so commensurable, neither of the straight lines  $CF$ ,  $FD$  is commensurable with the rational straight line set out,

and each of the straight lines  $AB$ ,  $CD$  will be a sixth binomial [x Def 11 6]

Hence a straight line commensurable in length with a binomial straight line is binomial and the same in order q.e.d.

## PROPOSITION 67

*A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order*

Let  $AB$  be binomial, and let  $CD$  be commensurable in length with  $AB$ ,  

 I say that  $CD$  is binomial and the same in order with  $AB$   
 For, since  $AB$  is binomial,  
 let it be divided into its medials at  $E$ ,  
 therefore  $AE$ ,  $EB$  are medial straight lines commensurable in square only [x 37, 38]

And let it be contrived that  
 as  $AB$  is to  $CD$ , so is  $AE$  to  $CF$ ,  
 therefore also the remainder  $EB$  is to the remainder  $FD$  as  $AB$  is to  $CD$  [v 19]

But  $AB$  is commensurable in length with  $CD$ ,  
 therefore  $AE$ ,  $EB$  are also commensurable with  $CF$ ,  $FD$  respectively [x 11]

But  $AE$ ,  $EB$  are medial,  
 therefore  $CF$ ,  $FD$  are also medial [x 23]

And since, as  $AE$  is to  $EB$ , so is  $CF$  to  $FD$  [v 11]  
 and  $AE$ ,  $EB$  are commensurable in square only,  
 $CF$ ,  $FD$  are also commensurable in square only [x 11]

But they were also proved medial,  
 therefore  $CD$  is binomial

I say next that it is also the same in order with  $AB$   
 For since, as  $AE$  is to  $EB$ , so is  $CF$  to  $FD$ ,  
 therefore also, as the square on  $AE$  is to the rectangle  $AE$ ,  $EB$ , so is the square on  $CF$  to the rectangle  $CF$ ,  $FD$ ,

therefore, alternately,  
 the square on  $AE$  is to the square on  $CF$ , so is the rectangle  $AE$ ,  $EB$  to the rectangle  $CF$ ,  $FD$  [v 16]

But the square on  $AE$  is commensurable with the square on  $CF$ ,

therefore the rectangle  $AE, EB$  is also commensurable with the rectangle  $CF, FD$

If therefore the rectangle  $AE, EB$  is rational,

[x 37]

[x 23, Por]

and each of the straight lines  $AB, CD$  is a second bimedial [x 33]

And for this reason  $CD$  will be the same in order with  $AB$  q e d

### PROPOSITION 68

*A straight line commensurable with a major straight line is itself also major*

Let  $AB$  be major, and let  $CD$  be commensurable with  $AB$ ,

I say that  $CD$  is major

Let  $AB$  be divided at  $E$ ,

therefore  $AE, EB$  are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial [x 39]

Let the same construction be made as before

Then since, as  $AB$  is to  $CD$ , so is  $AE$  to  $CF$ , and  $EB$  to  $FD$ ,

therefore also, as  $AE$  is to  $CF$ , so is  $EB$  to  $FD$



[v 11]

But  $AB$  is commensurable with  $CD$ ,

therefore  $AE, EB$  are also commensurable with  $CF, FD$  respectively [x 11]

And since, as  $AE$  is to  $CF$ , so is  $EB$  to  $FD$ ,

alternately also,

as  $AE$  is to  $EB$ , so is  $CF$  to  $FD$ , [v 16]

therefore also, *componendo*,

as  $AB$  is to  $BE$ , so is  $CD$  to  $DF$ , [v 18]

therefore also as the square on  $AB$  is to the square on  $BE$ , so is the square on  $CD$  to the square on  $DF$  [vi 20]

Similarly we can prove that as the square on  $AB$  is to the square on  $AE$ , so also is the square on  $CD$  to the square on  $CF$

Therefore also, as the square on  $AB$  is to the squares on  $AE, EB$ , so is the square on  $CD$  to the squares on  $CF, FD$ ,

therefore also alternately,

as the square on  $AB$  is to the square on  $CD$  so are the squares on  $AE, EB$  to the squares on  $CF, FD$  [v 16]

But the square on  $AB$  is commensurable with the square on  $CD$ , therefore the squares on  $AE, EB$  are also commensurable with the squares on  $CF, FD$

And the squares on  $AE, EB$  together are rational,

therefore the squares on  $CF, FD$  together are rational

Similarly also twice the rectangle  $AE, EB$  is commensurable with twice the rectangle  $CF, FD$

And twice the rectangle  $AE, EB$  is medial,

therefore twice the rectangle  $CF, FD$  is also medial [x 23, Por]

Therefore  $CF, FD$  are straight lines incommensurable in square which make, at the same time, the sum of the squares on them rational, but the rectangle contained by them medial, therefore the whole  $CD$  is the irrational straight line called major [x 39]

Therefore a straight line commensurable with the major straight line is major

Q E D

## PROPOSITION 69

*A straight line commensurable with the side of a rational plus a medial area is itself also the side of a rational plus a medial area*

Let  $AB$  be the side of a rational plus a medial area, and let  $CD$  be commensurable with  $AB$ ,

it is to be proved that  $CD$  is also the side of a rational plus a medial area



Let  $AB$  be divided into its straight lines at  $E$ , therefore  $AE$ ,  $EB$  are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational [x 40]

Let the same construction be made as before

We can then prove similarly that

$CF$ ,  $FD$  are incommensurable in square, and the sum of the squares on  $AE$ ,  $EB$  is commensurable with the sum of the squares on  $CF$ ,  $FD$ ,

and the rectangle  $AE$ ,  $EB$  with the rectangle  $CF$ ,  $FD$ , so that the sum of the squares on  $CF$ ,  $FD$  is also medial, and the rectangle  $CF$ ,  $FD$  rational

Therefore  $CD$  is the side of a rational plus a medial area Q E D

## PROPOSITION 70

*A straight line commensurable with the side of the sum of two medial areas is the side of the sum of two medial areas*

Let  $AB$  be the side of the sum of two medial areas, and  $CD$  commensurable with  $AB$ ,

it is to be proved that  $CD$  is also the side of the sum of two medial areas



For, since  $AB$  is the side of the sum of two medial areas, let it be divided into its straight lines at  $E$ , therefore  $AE$ ,  $EB$  are straight lines incommensurable in square which make the sum of the squares on them medial the rectangle contained by them medial, and furthermore the sum of the squares on  $AE$ ,  $EB$  incommensurable with the rectangle  $AE$ ,  $EB$  [x 41]

Let the same construction be made as before

We can then prove similarly that

$CF$ ,  $FD$  are also incommensurable in square, the sum of the squares on  $AE$ ,  $EB$  is commensurable with the sum of the squares on  $CF$ ,  $FD$ ,

and the rectangle  $AE$ ,  $EB$  with the rectangle  $CF$ ,  $FD$ , so that the sum of the squares on  $CF$ ,  $FD$  is also medial,

the rectangle  $CF$ ,  $FD$  is medial, and moreover the sum of the squares on  $CF$ ,  $FD$  is incommensurable with the rectangle  $CF$ ,  $FD$

Therefore  $CD$  is the side of the sum of two medial areas Q E D



## PROPOSITION 71

If a rational and a medial area be added together, four irrational straight lines arise, namely a binomial or a first bimedial or a major or a side of a rational plus a medial area

Let  $AB$  be rational, and  $CD$  medial,  
I say that the "side" of the area  $AD$  is a binomial or a first bimedial or a major or a side of a rational plus a medial area  
For  $AB$  is either greater or less than  $CD$   
First, let it be greater,

let a rational straight line  $EF$  be set out, breadth,  
and let  $HI$ , equal to  $DC$ , be applied to  $EF$ , producing  $EH$  as breadth

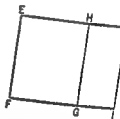
Then, since  $AB$  is rational and is equal to  $EG$ ,

therefore  $EG$  is also rational

And it has been applied to  $EF$ , producing  $EH$  as breadth,

therefore  $EH$  is rational and commensurable in length with  $EF$  [x 20]

Again since  $CD$  is medial and is equal to  $HI$ ,



therefore  $HI$  is also medial

And it is applied to the rational straight line  $EF$ , producing  $HK$  as breadth, therefore  $HK$  is rational and incommensurable in length with  $EF$  [x 22]

And since  $CD$  is medial,

while  $AB$  is rational,

therefore  $AB$  is incommensurable with  $CD$ , so that  $EG$  is also incommensurable with  $HI$

But as  $EG$  is to  $HI$  so is  $EH$  to  $HK$ , therefore  $EH$  is also incommensurable in length with  $HK$  [x 11]

[x 11]

[x 11]

[x 22]

And both are rational,

therefore  $EH$ ,  $HK$  are rational straight lines commensurable in square only, therefore  $EK$  is a binomial straight line divided at  $H$  [x 22]

And, since  $AB$  is greater than  $CD$

while  $AB$  is equal to  $EG$  and  $CD$  to  $HI$ , therefore  $EG$  is also greater than  $HI$ ,

therefore  $EH$  is also greater than  $HK$

square on a straight line commensurable in length with  $EH$  or by the square on a straight line incommensurable with it

First, let the square on it be greater by the square on a straight line commensurable with itself

Now the greater straight line  $HE$  is commensurable in length with the rational straight line  $EF$  set out,

therefore  $EK$  is a first binomial

[x Def II 1]

But  $EF$  is rational,  
and if an area be contained by a rational straight line and the first binomial,  
the side of the square equal to the area is binomial [x 54]

Therefore the "side" of  $EI$  is binomial,  
so that the "side" of  $AD$  is also binomial

Next let the square on  $EH$  be greater than the square on  $HK$  by the square  
with the ra-

therefore  $EK$  is a fourth binomial [x Def II 4]

But  $EF$  is rational,  
and if an area be contained by a rational straight line and the fourth binomial,  
the "side" of the area is the irrational straight line called major [x 57]

Therefore the "side" of the area  $EI$  is major,  
so that the "side" of the area  $AD$  is also major

Next let  $AB$  be less than  $CD$ ,  
therefore  $EG$  is also less than  $HI$ ,  
so that  $EH$  is also less than  $HK$

Now the square on  $HK$  is greater than the square on  $EH$  either by the  
square on a straight line commensurable with  $HK$  or by the square on a  
straight line incommensurable with it

First let the square on it be greater by the square on a straight line com-  
mensurable in length with the ra-

therefore  $EK$  is a second binomial [x Def II 2]

But  $EF$  is rational,  
and if an area be contained by a rational straight line and the second binomial,  
the side of the square equal to it is a first binomial, [x 55]

therefore the "side" of the area  $EI$  is a first binomial  
so that the "side" of  $AD$  is also a first binomial

line  $EF$  set out,

therefore  $EK$  is a fifth binomial [x Def II 5]

But  $EF$  is rational,  
and, if an area be contained by a rational straight line and the fifth binomial,  
the side of the square equal to the area is a side of a rational plus a medial area [x 58]

Therefore the "side" of the area  $EI$  is a side of a rational plus a medial area,  
so that the "side" of the area  $AD$  is also a side of a rational plus a medial area

Therefore etc

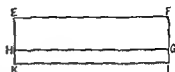
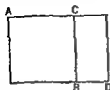
Q E D

## PROPOSITION 72

If two medial areas incommensurable with one another be added together, the re-  
maining two irrational straight lines arise, namely either a second binomial or a  
side of the sum of two medial areas

For let two medial areas  $AB$ ,  $CD$  incommensurable with one another be added together,

I say that the "side" of the area  $AD$  is either a second binomial or a side of the sum of two medial areas



For  $AB$  is either greater or less than  $CD$

First, if it so chance, let  $AB$  be greater than  $CD$

Let the rational straight line  $EF$  be set out,

and to  $EF$  let there be applied the rectangle  $EG$  equal to  $AB$  and producing  $EH$  as breadth, and the rectangle  $HI$  equal to  $CD$  and producing  $HK$  as breadth.

Now, since each of the areas  $AB$ ,  $CD$  is medial,

therefore each of the areas  $EG$ ,  $HI$  is also medial

And they are applied to the rational straight line  $FE$ , producing  $EH$ ,  $HK$  as breadth,

therefore each of the straight lines  $EH$ ,  $HK$  is rational and incommensurable in length with  $EF$  [x 22]

And, since  $AB$  is incommensurable with  $CD$ ,

and  $AB$  is equal to  $EG$ , and  $CD$  to  $HI$ ,

therefore  $EG$  is also incommensurable with  $HI$

But as  $EG$  is to  $HI$ , so is  $EH$  to  $HK$ , [vi 11]

therefore  $EH$  is incommensurable in length with  $HK$  [x 11]

Therefore  $EH$ ,  $HK$  are rational straight lines commensurable in square only,

therefore  $EK$  is binomial [x 36]

But the square on  $EH$  is greater than the square on  $HK$  either by the square on a straight line commensurable with  $EH$  or by the square on a straight line incommensurable with it

First, let the square on it be greater by the square on a straight line commensurable in length with itself

Now neither of the straight lines  $EH$ ,  $HK$  is commensurable in length with the rational straight line  $EF$  set out,

therefore  $EK$  is a third binomial [x Def 11 3]

But  $EF$  is rational,

and if an area be contained by a rational straight line and the third binomial the "side" of the area is a second binomial, [x 36]

therefore the "side" of  $EI$ , that is of  $AD$ , is a second binomial

on  $HK$  by the square

measurable in length

with  $EF$ ,

therefore  $EK$  is a sixth binomial [x Def 11 6]

But, if an area be contained by a rational straight line and the sixth bi

nomial the "side" of the area is the side of the sum of two medial areas, [x 59] so that the "side" of the area  $AD$  is also the side of the sum of two medial areas

Therefore etc

C E D

141        2 - 1 2    2411 neg after 4 neg no then

which it is applied

[**x 44**]

But the square on the binomial, if applied to a rational straight line, produces as breadth the first binomial

The square on the first binomial if applied to a rational straight line produces as breadth the second binomial. [x 61]

The square on the second binomial if applied to a rational straight line produces as breadth the third binomial

The square on the major, if applied to a rational straight line, produces as breadth the fourth binomial. [x 63]

the first because it is rational and from one another because they are not the same in order,  
so that the irrational straight lines themselves also differ from one another

### PROPOSITION 73

If from a rational straight line there be subtracted a rational straight line commensurable with the whole in square only, the remainder is irrational, and let it be called an apotome

For from the rational straight line  $AB$  let the rational straight line  $BC$ , commensurable with the whole in square only, be subtracted,

I say that the remainder  $AC$  is the irrational straight line called *apotome*.

For, since  $AB$  is incommensurable in length with  $BC$ ,  
and as  $AB$  is to  $BC$ , so  $\blacksquare$  the square on  $AB$  to the rectangle  $AB, BC$ ,  
therefore the square on  $AB$   $\blacksquare$  incommensurable with the rectangle  $AB, BC$  [x. 11]

But the squares on  $AB$ ,  $BC$  are commensurable with the square on  $AB$ ,  
and twice the rectangle  $AB$ ,  $BC$  is commensurable with the rectangle  $AB$ ,  $BC$

And inasmuch as the squares on  $AB$ ,  $BC$  are equal to twice the rectangle  $AB$ ,  $BC$  together with the square on  $CA$  [II 7] therefore the squares on  $AB$ ,  $BC$  are also incommensurable with the remainder, the square on  $AC$  [X 13 16]

But the squares on  $AB$ ,  $BC$  are rational,  
therefore  $AC$  is irrational [x Def 4]

And let it be called an *apologe*

## PROPOSITION 74

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a rational rectangle, the remainder is irrational. And let it be called a first apotome of a medial straight line.

For from the medial straight line  $AB$  let there be subtracted the medial straight line  $BC$  which is commensurable with  $AB$  in square only and with  $AB$  makes the rectangle  $AB, BC$  rational,

I say that the remainder  $AC$  is irrational, and let it be called a first apotome of a medial straight line.



For, since  $AB, BC$  are medial,

the squares on  $AB, BC$  are also medial

But twice the rectangle  $AB, BC$  is rational, therefore the squares on  $AB, BC$  are incommensurable with twice the rectangle  $AB, BC$ ,

therefore twice the rectangle  $AB, BC$  is also incommensurable with the remainder the square on  $AC$ , [cf II 7]

since if the whole is incommensurable with one of the magnitudes, the original magnitudes will also be incommensurable [x 16]

But twice the rectangle  $AB, BC$  is rational,

therefore the square on  $AC$  is irrational,

therefore  $AC$  is irrational

[x Def 4]

And let it be called a first apotome of a medial straight line

Q E D

## PROPOSITION 75

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a rational rectangle, the remainder is irrational. And let it be called a second apotome of a medial straight line.

For from the medial straight line  $AB$  let there be subtracted the medial straight line  $CB$  which is commensurable with the whole  $AB$  in square only and such that the rectangle  $AB, BC$  which it contains with the whole  $AB$  is medial, [x 25]

I say that the remainder  $AC$  is irrational and let it be called a second apotome of a medial straight line.



For let a rational straight line  $DI$  be set out, let  $DE$ , equal to the squares on  $AB, BC$ , be applied to  $DI$ , producing  $DG$  as breadth,

and let  $DH$  equal to twice the rectangle  $AB, BC$  be applied to  $DI$ , producing  $DF$  as breadth,

therefore the remainder  $FE$  is equal to the square on  $AC$  [II 7]

Now, since the squares on  $AB, BC$  are medial and commensurable,

therefore  $DE$  is also medial [X 15 and 23, Por.]

And it is equal to  $DH$ ,

therefore  $DH$  is also medial

And it has been applied to the rational straight line  $DI$ , producing  $DF$  as breadth,

therefore  $DF$  is rational and incommensurable in length with  $DI$  [X 22]

And, since  $AB, BC$  are commensurable in square only,

therefore  $AB$  is incommensurable in length with  $BC$ ,

therefore the square on  $AB$  is also incommensurable with the rectangle  $AB, BC$  [X 11]

But the squares on  $AB, BC$  are commensurable with the square on  $AB$ ,

[X 15]

and twice the rectangle  $AB, BC$  is commensurable with the rectangle  $AB, BC$ ,

[X 6]

therefore twice the rectangle  $AB, BC$  is incommensurable with the squares on  $AB, BC$  [X 13]

But  $DE$  is equal to the squares on  $AB, BC$ ,

and  $DH$  to twice the rectangle  $AB, BC$ ,

therefore  $DE$  is incommensurable with  $DH$

But, as  $DE$  is to  $DH$ , so is  $GD$  to  $DF$ ,

[VI 1]

therefore  $GD$  is incommensurable with  $DF$  [X 11]

And both are rational,

therefore  $GD, DF$  are rational straight lines commensurable in square only,

therefore  $FG$  is an apotome [X 73]

But  $DI$  is rational

and the rectangle contained by a rational and an irrational straight line is irrational, [deduction from X 20]

and its "side" is irrational

And  $AC$  is the "side" of  $FE$

therefore  $AC$  is irrational

And let it be called a *second apotome* of a medial straight line Q.E.D.

#### PROPOSITION 76

If a rational straight line be cut by an irrational straight line commensurable in square only, the rectangle contained by them is irrational, and the "side" of it is irrational.

κ

I say that the remainder  $AC$  is the irrational straight line called *minor*

For, since the sum of the squares on  $AB$ ,  $BC$  is rational, while twice the rectangle  $AB$ ,  $BC$  is medial,  
therefore the squares on  $AB$ ,  $BC$  are incommensurable with twice the rectangle  $AB$ ,  $BC$ ,  
and, *conteriendo*, the squares on  $AB$ ,  $BC$  are incommensurable with the remainder, the square on  $AC$  [II 7, x 16]

But the squares on  $AB$ ,  $BC$  are rational,  
therefore the square on  $AC$  is irrational,  
therefore  $AC$  is irrational

And let it be called *minor*

Q E D

### PROPOSITION 77

*If from a straight line there be subtracted a straight line which is incommensurable in square with the whole, and which with the whole makes the sum of the squares on them medial, but twice the rectangle contained by them rational, the remainder is irrational, and let it be called that which produces with a rational area a medial whole*

For from the straight line  $AB$  let there be subtracted the straight line  $BC$  which is incommensurable in square with  $AB$  and fulfils the given conditions, [x 34]

I say that the remainder  $AC$  is the irrational straight line aforesaid

For, since the sum of the squares on  $AB$ ,  $BC$  is medial,  
while twice the rectangle  $AB$ ,  $BC$  is rational,  
therefore the squares on  $AB$ ,  $BC$  are incommensurable with twice the rectangle  $AB$ ,  $BC$ ,  
therefore the remainder also, the square on  $AC$ , is incommensurable with twice the rectangle  $AB$ ,  $BC$  [II 7, x 16]

And twice the rectangle  $AB$ ,  $BC$  is rational,  
therefore the square on  $AC$  is irrational,  
therefore  $AC$  is irrational

And let it be called *that which produces with a rational area a medial whole*

Q E D

### PROPOSITION 78

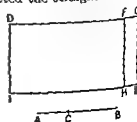
*If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on*

whole

For from the straight line  $AB$  let there be subtracted the straight line  $BC$  incommensurable in square with  $AB$  and fulfilling the given conditions, [x 35]

I say that the remainder  $AC$  is the irrational straight line called *that which produces with a medial area a medial whole*

■ set out,  
the squares



and let  $DH$  equal to twice the rectangle  $AB, BC$  be subtracted  
Therefore the remainder  $FE$  is equal to the square on  $AC$ , [II 7]

so that  $AC$  is the "side" of  $FE$

Now, since the sum of the squares on  $AB, BC$  is medial and is equal to  $DE$ ,  
therefore  $DE$  is medial

And it is applied to the rational straight line  $DI$ , producing  $DG$  as breadth,  
therefore  $DG$  is rational and incommensurable in length with  $DI$  [X 22]

Again since twice the rectangle  $AB, BC$  is medial and is equal to  $DH$ ,  
therefore  $DH$  is medial

And it is applied to the rational straight line  $DI$ , producing  $DF$  as breadth,  
therefore  $DF$  is also rational and incommensurable in length with  $DI$  [X 22]

And, since the squares on  $AB, BC$  are incommensurable with twice the rectangle  $AB, BC$ ,

therefore  $DE$  is also incommensurable with  $DH$

But as  $DE$  is to  $DH$ , so also is  $DG$  to  $DF$ , [VI 1]

therefore  $DG$  is incommensurable with  $DF$  [X 11]

And both are rational,  
therefore  $GD, DF$  are rational straight lines commensurable in square only

Therefore  $FG$  is an apotome [X 73]

And  $FH$  is rational,  
but the rectangle contained by a rational straight line and an apotome is irrational,  
[deduction from X 20]

and its "side" is irrational

And  $AC$  is the "side" of  $FE$ ,

therefore  $AC$  is irrational

And let it be called that which produces with a medial area a medial whole

Q E D

PROPOSITION 79

To an apotome only one rational straight line can be annexed which is commensurable with the whole in square only

Let  $AB$  be an apotome and  $BC$  an annex to it,

$A \quad B \quad C \quad D$  therefore  $AC, CB$  are rational straight lines  
commensurable in square only [X 73]

I say that no other rational straight line can be annexed to  $AB$  which is commensurable with the whole in square only

For, if possible, let  $BD$  be so annexed,  
therefore  $AD, DB$  are also rational straight lines commensurable in square only [X 73]

Now, since the excess of the squares on  $AD, DB$  over twice the rectangle  $AD, DB$  is also the excess of the squares on  $AC, CB$  over twice the rectangle  $AC, CB$ ,

for both exceed by the same, the square on  $AB$ , [II 7]

therefore, alternately, the excess of the squares on  $AD, DB$  over the squares on  $AC, CB$  is the excess of twice the rectangle  $AD, DB$  over twice the rectangle  $AC, CB$

But the squares on  $AD, DB$  exceed the squares on  $AC, CB$  by a rational area,  
for both are rational,  
therefore twice the rectangle  $AD, DB$  also exceeds twice the rectangle  $AC, CB$   
by a rational area



which is impossible,

for both are medial [x. 21], and a medial area does not exceed a medial by a rational area [x. 26]

Therefore no other rational straight line can be annexed to  $AB$  which is commensurable with the whole in square only

Therefore only one rational straight line can be annexed to an apotome which is commensurable with the whole in square only Q. E. D.

### PROPOSITION 80

*To a first apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a rational rectangle*

For let  $AB$  be a first apotome of a medial straight line, and let  $BC$  be an annex to  $AB$ ; therefore  $AC$ ,  $CB$  are medial straight lines commensurable in square only and such that the rectangle  $AC$ ,  $CB$  which they contain is rational, [x. 74]

I say that no other medial straight line can be annexed to  $AB$  which is commensurable with the whole in square only and which contains with the whole a rational area.

For, if possible, let  $DB$  also be so annexed, therefore  $AD$ ,  $DB$  are medial straight lines commensurable in square only and such that the rectangle  $AD$ ,  $DB$  which they contain is rational [x. 74]

Now, since the excess of the squares on  $AD$ ,  $DB$  over twice the rectangle  $AD$ ,  $DB$  is also the excess of the squares on  $AC$ ,  $CB$  over twice the rectangle  $AC$ ,  $CB$ ,

for they exceed by the same the square on  $AB$ , [ii. 7] therefore, alternately, the excess of the squares on  $AD$ ,  $DB$  over the squares on  $AC$ ,  $CB$  is also the excess of twice the rectangle  $AD$ ,  $DB$  over twice the rectangle  $AC$ ,  $CB$

But twice the rectangle  $AD$ ,  $DB$  exceeds twice the rectangle  $AC$ ,  $CB$  by a rational area

for both are rational Therefore the squares on  $AD$ ,  $DB$  also exceed the squares on  $AC$ ,  $CB$  by a rational area

which is impossible for both are medial [x. 15 and 23, Por.], and a medial area does not exceed a medial by a rational area [x. 26]

Therefore etc Q. E. D.

### PROPOSITION 81

*To a second apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a medial rectangle*

Let  $AB$  be a second apotome of a medial straight line and  $BC$  an annex to  $AB$ ; therefore  $AC$ ,  $CB$  are medial straight lines commensurable in square only and such that the rectangle  $AC$ ,  $CB$  which they contain is medial [x. 75]

I say that no other medial straight line can be annexed to  $AB$  which is com-

measurable with the whole in square only and which contains with the whole a medial rectangle

For, if possible, let  $BD$  also be so annexed,

therefore  $AD$ ,  $DB$  are also medial straight lines commensurable in square only and such that the rectangle  $AD$ ,  $DB$  which they contain is medial [x 75]

Let a rational straight line  $EF$  be set out, let  $EG$  equal to the squares on  $AC$ ,  $CB$  be applied to  $EF$ , producing  $EM$  as breadth, and let  $HG$  equal to twice the rectangle  $AC$ ,  $CB$  be subtracted, producing  $HM$  as breadth,

therefore the remainder  $EL$  is equal to the square on  $AB$ , [II 7]  
so that  $AB$  is the "side" of  $EL$

Again, let  $EI$  equal to the squares on  $AD$ ,  $DB$  be applied to  $EF$ , producing  $EN$  as breadth

But  $EL$  is also equal to the square on  $AB$ , therefore the remainder  $HI$  is equal to twice the rectangle  $AD$ ,  $DB$  [II 7]

Now, since  $AC$ ,  $CB$  are medial straight lines, therefore the squares on  $AC$ ,  $CB$  are also medial

And they are equal to  $EG$ ,

therefore  $EG$  is also medial [x 15 and 23, Por.]

there  
Ag

twice the rectangle  $AC$ ,  $CB$  is also medial [x 23, Por.]

And it is equal to  $HG$ ,

therefore  $HG$  is also medial

And

therefore  $AC$  is incommensurable in length with  $CB$

But, as  $AC$  is to  $CB$ , so is the square on  $AC$  to the rectangle  $AC$ ,  $CB$ ; therefore the square on  $AC$  is incommensurable with the rectangle  $AC$ ,  $CB$  [x 11]

But the squares on  $AC$ ,  $CB$  are commensurable with the square on  $AC$ , while twice the rectangle  $AC$ ,  $CB$  is commensurable with the rectangle  $AC$ ,  $CB$ , [x 6]

therefore the squares on  $AC$ ,  $CB$  are incommensurable with twice the rectangle  $AC$ ,  $CB$  [x 13]

And  $EG$  is equal to the squares on  $AC$ ,  $CB$ ,

while  $GH$  is equal to twice the rectangle  $AC$ ,  $CB$ ,

therefore  $EG$  is incommensurable with  $HG$

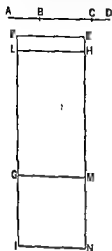
But, as  $EG$  is to  $HG$ , so is  $EM$  to  $HM$ , [VI 1]

therefore  $EM$  is incommensurable in length with  $MH$  [x 11]

And both are rational,

therefore  $EM$ ,  $MH$  are rational straight lines commensurable in square only;

therefore  $EH$  is an apotome, and  $HM$  an annex to it [x 73]



Similarly we can prove that  $HN$  is also an annex to it; therefore to an apotome different straight lines are annexed which are commensurable with the wholes in square only which is impossible [x 79]

Therefore etc

Q E D

### PROPOSITION 82

To a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes, with the whole, the sum of the squares on them rational but twice the rectangle contained by them medial

Let  $AB$  be the minor straight line, and let  $BC$  be an annex to  $AB$ , therefore  $AC$ ,  $CB$  are straight lines incommensurable in square which make the sum of the squares on them rational, but twice the rectangle contained by them medial [x 76]

I say that no other straight line can be annexed to  $AB$  fulfilling the same conditions

For, if possible, let  $BD$  be so annexed, therefore  $AD$ ,  $DB$  are also straight lines incommensurable in square which fulfil the aforesaid conditions [x 76]

Now, since the excess of the squares on  $AD$ ,  $DB$  over the squares on  $AC$ ,  $CB$  is also the excess of twice the rectangle  $AD$ ,  $DB$  over twice the rectangle  $AC$ ,  $CB$ ,

while the squares on  $AD$ ,  $DB$  exceed the squares on  $AC$ ,  $CB$  by a rational area, for both are rational,

therefore twice the rectangle  $AD$ ,  $DB$  also exceeds twice the rectangle  $AC$ ,  $CB$  by a rational area

which is impossible, for both are medial [x 26]

Therefore to a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes the squares on them added together rational, but twice the rectangle contained by them medial Q E D

### PROPOSITION 83

To a straight line which produces with a rational area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial, but twice the rectangle contained by them rational

Let  $AB$  be the straight line which produces with a rational area a medial whole,

and let  $BC$  be an annex to  $AB$ ; therefore  $AC$ ,  $CB$  are straight lines



[x 77]  
the same

conditions

For, if possible let  $BD$  be so annexed, therefore  $AD$ ,  $DB$  are also straight lines incommensurable in square which fulfil the given conditions [x 77]

Since then, as in the preceding cases, the excess of the squares on  $AD$ ,  $DB$  over the squares on  $AC$ ,  $CB$  is also the

excess of twice the rectangle  $AD, DB$  over twice the rectangle  $AC, CB$ , while twice the rectangle  $AD, DB$  exceeds twice the rectangle  $AC, CB$  by a rational area

for both are rational,  
therefore the squares on  $AD, DB$  also exceed the squares on  $AC, CB$  by a rational area

which is impossible, for both are medial [X 26]

Therefore no other straight line can be annexed to  $AB$  which is incommensurable in square with the whole and which with the whole fulfils the aforesaid conditions,

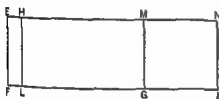
therefore only one straight line can be so annexed Q E D

### PROPOSITION 84

*To a straight line which produces with a medial area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial and twice the rectangle contained by them both medial and also incommensurable with the sum of the squares on them*

Let  $AB$  be the straight line which produces with a medial area a medial whole

and  $BC$  an annex to it,  
therefore  $AC, CB$  are straight lines incommensurable in square which fulfil the aforesaid conditions [X 78]



I say that no other straight line can be annexed to  $AB$  which fulfils the aforesaid conditions

For if possible, let  $BD$  be so annexed

so that  $AD, DB$  are also straight lines incommensurable in square which make the squares on  $AD, DB$  added together medial, twice the rectangle  $AD, DB$  medial, and also the squares [X 78]

on  $AD, DB$  incommensurable with twice the rectangle  $AD, DB$

Let a rational straight line  $EF$  be set out,  
let  $EG$  equal to the squares on  $AC, CB$  be applied to  $EF$ , producing  $EM$  as breadth,  
and let  $HG$  equal to twice the rectangle  $AC, CB$  be applied to  $EF$ , producing  $HM$  as breadth,

therefore the remainder, the square on  $AB$  [XI 7], is equal to  $EL$ ,

therefore  $AB$  is the "side" of  $EL$

Again let  $EI$  equal to the squares on  $AD, DB$  be applied to  $EF$ , producing  $EN$  as breadth

But the square on  $AB$  is also equal to  $EL$ ,  
therefore the remainder twice the rectangle  $AD, DB$  [XI 7] is equal to  $HI$

Now, since the sum of the squares on  $AC, CB$  is medial and is equal to  $EG$ ,  
therefore  $EG$  is also medial

And it is applied to the rational straight line  $EF$ , producing  $EM$  as breadth,  
therefore  $EM$  is rational and incommensurable in length with  $EF$  [X 22]

Again, since twice the rectangle  $AC, CB$  is medial and is equal to  $HG$ ,  
therefore  $HG$  is also medial

And it is applied to the rational straight line  $EF$ , producing  $HM$  as breadth,  
therefore  $HM$  is rational and incommensurable in length with  $FF$  [x 22]

And, since the squares on  $AC, CB$  are incommensurable with twice the rectangle  $AC, CB$ ,

$EG$  is also incommensurable with  $HG$ ,

therefore  $EM$  is also incommensurable in length with  $MH$  [vi 1, x 11]

And both are rational,

therefore  $EM, MH$  are rational straight lines commensurable in square only  
therefore  $EH$  is an apotome, and  $HM$  an annex to it [x 73]

Similarly we can prove that  $EH$  is again an apotome and  $HN$  an annex to it.

Therefore to an apotome different rational straight lines are annexed which  
are commensurable with the wholes in square only

which was proved impossible [x 70]

Therefore no other straight line can be so annexed to  $AB$

Therefore to  $AB$  only one straight line can be annexed which is incommensurable in square with the whole and which with the whole makes the squares on them added together medial twice the rectangle contained by them medial, and also the squares on them incommensurable with twice the rectangle contained by them

Q E D

### DEFINITIONS III

1 Given a rational straight line and an apotome, if the square on the whole be greater than the square on the annex by the square on a straight line commensurable in length with the whole and the whole be commensurable in length with the rational straight line set out, let the apotome be called a *first apotome*

2 But if the annex be commensurable in length with the rational straight line set out and the square on the whole be greater than that on the annex by the square on a straight line commensurable with the whole, let the apotome be called a *second apotome*

3 But if neither be commensurable in length with the rational straight line set out and the square on the whole be greater than the square on the annex by the square on a straight line commensurable with the whole, let the apotome be called a *third apotome*

4 Again if the square on the whole be greater than the square on the annex by the square on a straight line incommensurable with the whole then, if the whole be commensurable in length with the rational straight line set out let the apotome be called a *fourth apotome*

5 if the annex be so commensurable a *fifth*,

6 and if neither a *sixth*

### PROPOSITION 85

To find the first apotome

Let a rational straight line  $A$  be set out

and let  $BG$  be commensurable in length with  $A$ ,

therefore  $BG$  is also rational

Let two square numbers  $DE$ ,  $EF$  be set out, and let their difference  $FD$  not be square,

$A$  —————  
 $H$  —————

$B$   $C$  —————  $G$   
 $E$   $F$  —————  $D$

therefore neither has  $ED$  to  $DF$  the ratio which a square number has to a square number

as  $ED$  is to  $DF$ , so is the square on  $BG$  to the square on  $GC$ , [v 6, Por]  
 therefore the square on  $BG$  is commensurable with the square on  $GC$  [x 6]

But the square on  $BG$  is rational,

therefore the square on  $GC$  is also rational,

therefore  $GC$  is also rational

And, since  $ED$  has not to  $DF$  the ratio which a square number has to a square number,

therefore neither has the square on  $BG$  to the square on  $GC$  the ratio which a square number has to a square number,

therefore  $BG$  is incommensurable in length with  $GC$  [v 9]

And both are rational,

therefore  $BG$ ,  $GC$  are rational straight lines commensurable in square only,

therefore  $BC$  is an apotome [x 73]

I say next that it is also a first apotome

For let the square on  $H$  be that by which the square on  $BG$  is greater than the square on  $GC$

Now since, as  $ED$  is to  $FD$ , so is the square on  $BG$  to the square on  $GC$ ,  
 therefore also, *convertendo*, [v 19, Por]

as  $DE$  is to  $EF$ , so is the square on  $GB$  to the square on  $H$

But  $DE$  has to  $EF$  the ratio which a square number has to a square number,  
 for each is square,

therefore the square on  $GB$  also has to the square on  $H$  the ratio which a square number has to a square number,

therefore  $BG$  is commensurable in length with  $H$  [x 9]

And the square on  $BG$  is greater than the square on  $GC$  by the square on  $H$ ,  
 therefore the square on  $BG$  is greater than the square on  $GC$  by the square on a straight line commensurable in length with  $BG$

And the whole  $BG$  is commensurable in length with the rational straight line  $A$  set out

Therefore  $BC$  is a first apotome [x Def III 1]

Therefore the first apotome  $BC$  has been found

(Being) that which it was required to find Q E D

### PROPOSITION 36

To find the second apotome

Let a rational straight line  $A$  be set out and  $GC$  commensurable with  $A$ ,

therefore  $GC$  is rational

Let two square numbers  $DE$ ,  $EF$  be set out and let their difference  $FD$  be square

Now let it be contrived that as  $FD$  is to  $DE$ , so is the square on  $GB$  to the square on  $GC$

Therefore the square on  $CG$  is commensurable with the square on  $GB$

But the square on  $CG$  is rational,  
 therefore the square on  $GB$  is also rational,  
 therefore  $BG$  is rational

And, since the square on  $GC$  has not to the square on  $GB$  the ratio which a square number has to a square number,  
 $CG$  is incommensurable in length with  $GB$ .

[x 9]

And both are rational,  
 therefore  $CG$ ,  $GB$  are rational straight lines commensurable in square only,  
 therefore  $BC$  is an apotome

[x 73]

I say next that it is also a second apotome

For let the square on  $H$  be that by which the square on  $BG$  is greater than the square on  $GC$

Since then as the square on  $BG$  is to the square on  $GC$ , so is the number  $ED$  to the number  $DF$ ,

therefore, *convertendo*,

as the square on  $BG$  is to the square on  $H$ , so is  $DE$  to  $EF$  [v 19 Por]

And each of the numbers  $DE$ ,  $EF$  is square,  
 therefore the square on  $BG$  has to the square on  $H$  the ratio which a square number has to a square number,

therefore  $BG$  is commensurable in length with  $H$  [x 9]

And the square on  $BG$  is greater than the square on  $GC$  by the square on  $H$ ,  
 therefore the square on  $BG$  is greater than the square on  $GC$  by the square on a straight line commensurable in length with  $BG$

And  $CG$ , the annex, is commensurable with the rational straight line  $A$  set out

Therefore  $BC$  is a second apotome

[x Def in 2]

Therefore the second apotome  $BC$  has been found

Q E D

### PROPOSITION 87

To find the third apotome

Let a rational straight line  $A$  be set out  
 let three numbers  $E$ ,  $BC$ ,  $CD$  be set out which have not to one another the ratio which a square number has to a square number but let  $CB$  have to  $BD$  the ratio which a square number has to a square number

Let it be contrived that as  $E$  is to  $BC$ , so is the square on  $A$  to the square on  $FG$ ,  
 and, as  $BC$  is to  $CD$ , so is the square on  $FG$  to the square on  $GH$

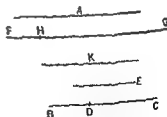
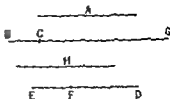
[x 6 Por]

Since then as  $E$  is to  $BC$  so is the square on  $A$  to the square on  $FG$ ,  
 therefore the square on  $A$  is commensurable with the square on  $FG$  [x 6]

But the square on  $A$  is rational

therefore the square on  $FG$  is also rational,

therefore  $FG$  is rational



And, since  $E$  has not to  $BC$  the ratio which a square number has to a square number,  
therefore neither has the square on  $A$  to the square on  $FG$  the ratio which a square number has to a square number,

therefore  $A$  is incommensurable in length with  $FG$  [x 9]

Again, since, as  $BC$  is to  $CD$ , so is the square on  $FG$  to the square on  $GH$ ,  
therefore the square on  $FG$  is commensurable with the square on  $GH$  [x 6]

But the square on  $FG$  is rational,

therefore the square on  $GH$  is also rational,

therefore  $GH$  is rational

And, since  $BC$  has not to  $CD$  the ratio which a square number has to a square number,

therefore neither has the square on  $FG$  to the square on  $GH$  the ratio which a square number has to a square number,

therefore  $FG$  is incommensurable in length with  $GH$  [x 9]

And both are rational,

therefore  $FG$ ,  $GH$  are rational straight lines commensurable in square only,

therefore  $FH$  is an apotome [x 73]

I say next that it is also a third apotome

For since, as  $E$  is to  $BC$ , so is the square on  $A$  to the square on  $FG$ ,

and, as  $BC$  is to  $CD$ , so is the square on  $FG$  to the square on  $GH$ ,

therefore, *ex aequali*, as  $E$  is to  $CD$ , so is the square on  $A$  to the square on  $HG$  [v 22]

But  $E$  has not to  $CD$  the ratio which a square number has to a square number,  
therefore neither has the square on  $A$  to the square on  $GH$  the ratio which a square number has to a square number,

therefore  $A$  is incommensurable in length with  $GH$  [x 11]

Therefore neither of the straight lines  $FG$ ,  $GH$  is commensurable in length with the rational straight line  $A$  set out

Now let the square on  $K$  be that by which the square on  $FG$  is greater than the square on  $GH$

Since then, as  $BC$  is to  $CD$ , so is the square on  $FG$  to the square on  $GH$ ,  
therefore, *convertendo*, as  $BC$  is to  $BD$ , so is the square on  $FG$  to the square on  $K$  [v 19 Por]

But  $BC$  has to  $BD$  the ratio which a square number has to a square number,  
therefore the square on  $FG$  also has to the square on  $K$  the ratio which a square number has to a square number

Therefore  $FG$  is commensurable in length with  $K$ , [x 9]

and the square on  $FG$  is greater than the square on  $GH$  by the square on a straight line commensurable with  $FG$

with

III 3]

Therefore the third apotome  $FH$  has been found

Q E D

# PROPOSITION 88

To find the fourth apotome

Let a rational straight line  $A$  be set out, and  $BG$  commensurable in length with it,



therefore  $BG$  is also rational

Let two numbers  $DF$ ,  $FE$  be set out such that the whole  $DE$  has not to either of the numbers  $DF$ ,  $FE$  the ratio which a square number has to a square number



Let it be contrived that, as  $DE$  is to  $EF$ , so is the square on  $BG$  to the square on  $GC$ , [x 6, Por]

therefore the square on  $BG$  is commensurable with the square on  $GC$  [x 6]

But the square on  $BG$  is rational,

therefore the square on  $GC$  is also rational,

therefore  $GC$  is rational

Now, since  $DE$  has not to  $EF$  the ratio which a square number has to a square number,

therefore neither has the square on  $BG$  to the square on  $GC$  the ratio which a square number has to a square number,

therefore  $BG$  is incommensurable in length with  $GC$  [x 9]

And both are rational,

therefore  $BC$  is rational

Now

the square on  $GC$

Since then as  $DE$  is to  $EF$ , so is the square on  $BG$  to the square on  $GC$ , therefore also *convertendo* as  $ED$  is to  $DF$ , so is the square on  $GB$  to the square on  $H$  [v 10 Por]

But  $ED$  has not to  $DF$  the ratio which a square number has to a square number

therefore neither has the square on  $GB$  to the square on  $H$  the ratio which a square number has to a square number,

therefore  $GB$  is incommensurable in length with  $H$  [x 9]

$A$  set out

Therefore  $BC$  is a fourth apotome

[x Def m 4]

Therefore the fourth apotome has been found

$Q.E.D.$

# PROPOSITION 89

To find the fifth apotome

Let a rational straight line  $A$  be set out and let  $CG$  be commensurable in length with  $A$ ,

therefore  $CG$  is rational

Let two numbers  $DF$ ,  $FE$  be set out such that  $DE$  again has not to either of the numbers  $DF$ ,  $FE$  the ratio which a square number has to a square number,

and let it be contrived that as  $FE$  is to  $ED$ , so is the square on  $CG$  to the square on  $GB$

Therefore the square on  $GB$  is also rational,

[x 6]

therefore  $BG$  is also rational

Now since as  $DE$  is to  $EF$ , so is the square on  $BG$  to the square on  $GC$ ,

while  $DE$  has not to  $EF$  the ratio which a square number has to a square number

therefore neither has the square on  $BG$  to the square on  $GC$  the ratio which a square number has to a square number,

therefore  $BG$  is incommensurable in length with  $GC$  [x 9]

And both are rational,

therefore  $BG$   $GC$  are rational straight lines commensurable in square only,

[x 73]

than

the square on  $GC$

Since then as the square on  $BG$  is to the square on  $GC$ , so is  $DE$  to  $EF$ , therefore, *convertendo*, as  $ED$  is to  $DF$ , so is the square on  $BG$  to the square on  $H$  [v 10 Por]

But  $ED$  has not to  $DF$  the ratio which a square number has to a square number,

therefore neither has the square on  $BG$  to the square on  $H$  the ratio which a square number has to a square number,

a straight line incommensurable in length with  $GB$

And the annex  $CG$  is commensurable in length with the rational straight line  $A$  set out,

therefore  $BC$  is a fifth apotome

[x Def III 5]

Therefore the fifth apotome  $BC$  has been found

Q E D

### PROPOSITION 90

To find the sixth apotome

Let a rational straight line  $A$  be set out and three numbers  $E$   $BC$ ,  $CD$  not

$A$  \_\_\_\_\_

$F$  \_\_\_\_\_  $G$

$H$  \_\_\_\_\_

$E$  \_\_\_\_\_

$B$  \_\_\_\_\_  $C$

square number has to a square number

Let it be contrived that as  $E$  is to  $BC$ , so is

$CG$  to

Por]

Now since as  $E$  is to  $BC$  so is the square on  $A$  to the square on  $FG$

therefore the square on  $A$  is commensurable with the square on  $FG$  [x 6]

But the square on  $A$  is rational,

therefore the square on  $FG$  is also rational,  
therefore  $FG$  is also rational

And, since  $E$  has not to  $BC$  the ratio which a square number has to a square number,

therefore neither has the square on  $A$  to the square on  $FG$  the ratio which a square number has to a square number,

therefore  $A$  is incommensurable in length with  $FG$  [x 9]

Again since, as  $BC$  is to  $CD$ , so is the square on  $FG$  to the square on  $GH$ ,  
therefore the square on  $FG$  is commensurable with the square on  $GH$  [x 6]

But the square on  $FG$  is rational,

therefore the square on  $GH$  is also rational,

therefore  $GH$  is also rational

And, since  $BC$  has not to  $CD$  the ratio which a square number has to a square number,

therefore neither has the square on  $FG$  to the square on  $GH$  the ratio which a square number has to a square number,

therefore  $FG$  is incommensurable in length with  $GH$  [x 9]

And both are rational,

therefore  $FG$ ,  $GH$  are rational straight lines commensurable in square only,

therefore  $FH$  is an apotome [x 73]

I say next that it is also a sixth apotome

For since, as  $E$  is to  $BC$ , so is the square on  $A$  to the square on  $FG$

and, as  $BC$  is to  $CD$ , so is the square on  $FG$  to the square on  $GH$ ,

therefore, *ex æquali* as  $E$  is to  $CD$ , so is the square on  $A$  to the square on  $GH$  [v 22]

But  $E$  has not to  $CD$  the ratio which a square number has to a square number,

therefore neither has the square on  $A$  to the square on  $GH$  the ratio which a square number has to a square number,

therefore  $A$  is incommensurable in length with  $GH$ , [x 9]

therefore neither of the straight lines  $FG$   $GH$  is commensurable in length with the rational straight line  $A$

Now let the square on  $K$  be that by which the square on  $FG$  is greater than the square on  $GH$

Since then as  $BC$  is to  $CD$ , so is the square on  $FG$  to the square on  $GH$   
therefore, *conterclendo* as  $CB$  is to  $BD$ , so is the square on  $FG$  to the square on  $K$  [v 19, For]

But  $CB$  has not to  $BD$  the ratio which a square number has to a square number,

therefore neither has the square on  $FG$  to the square on  $K$  the ratio which a square number has to a square number,

therefore  $K$  is incommensurable in length with  $FG$ , [x 9]

therefore  $K$ , [x 9]

is incommensurable in length with  $A$ , [x 9]

therefore  $K$ , [x 9]

is incommensurable in length with  $A$ , [x 9]

And neither of the straight lines  $FG$   $GH$  is commensurable with the rational straight line  $A$  set out

Therefore  $FH$  is a sixth apotome

Therefore the sixth apotome  $FH$  has been found

[x Def III 6]

Q E D

## PROPOSITION 91

If an area be contained by a rational straight line and a first apotome the "side" of the area is an apotome

For let the area  $AB$  be contained by the rational straight line  $AC$  and the first apotome  $AD$ ,

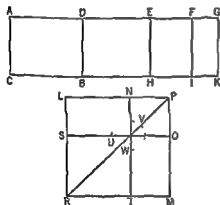
I say that the "side" of the area  $AB$  is an apotome

For since  $AD$  is a first apotome let  $DG$  be its annex,  
therefore  $AG$ ,  $GD$  are rational straight lines commensurable in square only [x 73]

And the whole  $AG$  is commensurable with the rational straight line  $AC$  set out  
and the square on  $AG$  is greater than the square on  $GD$  by the square on a

surable parts

[x 17]



Let  $DG$  be bisected at  $E$ ,  
let there be applied to  $AG$  a parallelogram equal to the square on  $EG$  and deficient by a square figure  
and let it be the rectangle  $AF$   $FG$ ,  
therefore  $AF$  is commensurable with  $FG$

And through the points  $E$   $F$ ,  $G$  let  $EH$   $FI$   $GK$  be drawn parallel to  $AC$

Now since  $AF$  is commensurable in length with  $FG$

therefore  $AG$  is also commensurable in length with each of the straight lines  $AF$   $FG$  [x 15]

But  $AG$  is commensurable with  $AC$ ,  
therefore each of the straight lines  $AF$ ,  $FG$  is commensurable in length with  $AC$  [x 12]

And  $AC$  is rational

therefore each of the straight lines  $AF$   $FG$  is also rational

[x 10]

th

$DL$   $EG$

straight lines

[x 15]

But  $DG$  is rational and incommensurable in length with  $AC$ ,  
therefore each of the straight lines  $DE$   $EG$  is also rational and incommensurable in length with  $AC$  [x 13]

therefore each of the rectangles  $DH$ ,  $EA$  is medial [x 21]

Now let the square  $LM$  be made equal to  $AI$  and let there be subtracted the square  $NO$  having a common angle with it the angle  $LPN$ , and equal to  $FA$ ,

therefore the squares  $LM$   $NO$  are about the same diameter [vi 26]

Let  $PR$  be their diameter and let the figure be drawn

Since then the rectangle contained by  $AF$ ,  $FG$  is equal to the square on  $EG$ ,  
therefore, as  $AF$  is to  $EG$ , so is  $EG$  to  $FG$  [vi 17]

But, as  $AF$  is to  $LG$ , so is  $AI$  to  $EK$ ,  
and, as  $EG$  is to  $FG$ , so is  $EK$  to  $KF$ , [vi 1]

therefore  $EK$  is a mean proportional between  $AI$ ,  $KF$  [v 11]

But  $MN$  is also a mean proportional between  $LM$ ,  $NO$ , as was before proved,  
[Lemmas after x 53]

and  $AI$  is equal to the square  $LM$ , and  $KF$  to  $NO$ ,

therefore  $MN$  is also equal to  $EK$ .

But  $EK$  is equal to  $DH$ , and  $MN$  to  $LO$ ,

therefore  $DK$  is equal to the gnomon  $UVW$  and  $NO$ .

But  $AK$  is also equal to the squares  $LM$ ,  $NO$ ,

therefore the remainder  $AB$  is equal to  $ST$ .

But  $ST$  is the square on  $LN$ ,

therefore the square on  $LN$  is equal to  $AB$ ;

therefore  $LN$  is the "side" of  $AB$ .

I say next that  $LN$  is an apotome

For, since each of the rectangles  $AI$ ,  $FK$  is rational,

and they are equal to  $LM$ ,  $NO$ ,

therefore each of the squares  $LM$ ,  $NO$ , that is, the squares on  $LP$ ,  $PN$  respectively, is also rational,

therefore each of the straight lines  $LP$ ,  $PN$  is also rational

Again, since  $DH$  is medial and is equal to  $LO$ ,

therefore  $LO$  is also medial

Since, then,  $LO$  is medial,

while  $NO$  is rational,

therefore  $LO$  is incommensurable with  $NO$

But as  $LO$  is to  $NO$ , so is  $LP$  to  $PN$ , [vi 1]

therefore  $LP$  is incommensurable in length with  $PN$  [x 11]

And both are rational,

therefore  $LP$ ,  $PN$  are rational straight lines commensurable in square only,

therefore  $LN$  is an apotome [x 73]

And it is the "side" of the area  $AB$ ,

therefore the "side" of the area  $AB$  is an apotome

Therefore etc

Q E D

# PROPOSITION 92

If an area be contained by a rational straight line and a second apotome, the "side" of the area is a first apotome of a medial straight line

For let the area  $AB$  be contained by the rational straight line  $AC$  and the second apotome  $AD$ ,

I say that the "side" of the area  $AB$  is a first apotome of a medial straight line

For let  $DG$  be the annex to  $AD$ ,

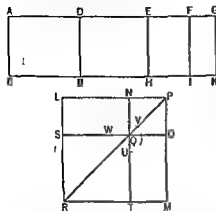
therefore  $AG$ ,  $GD$  are rational straight lines commensurable in square only, [x 73]

and the annex  $DG$  is commensurable with the rational straight line  $AC$  set out,

while the square on the whole  $AG$  is greater than the square on the annex  $GD$

by the square on a straight line commensurable in length with  $AG$  [x Defn iii 2]

Since, then the square on  $AG$  is greater than the square on  $GD$  by the square on a straight line commensurable with  $AG$ ,



therefore, if there be applied to  $AG$  a parallelogram equal to the fourth part of the square on  $GD$  and deficient by a square figure, it divides it into commensurable parts [x 17]

Let then  $DG$  be bisected at  $E$ , let there be applied to  $AG$  a parallelogram equal to the square on  $EG$  and deficient by a square figure,

and let it be the rectangle  $AF, FG$ , therefore  $AF$  is commensurable in length with  $FG$

Therefore  $AG$  is also commensurable in length with each of the straight lines  $AF, FG$  [x 15]

But  $AG$  is rational and incommensurable in length with  $AC$ , therefore each of the straight lines  $AF, FG$  is also rational and incommensurable in length with  $AC$ , [x 13]

therefore each of the rectangles  $AI, FK$  is medial [x 21]

Again since  $DE$  is commensurable with  $EG$ , therefore  $DG$  is also commensurable with each of the straight lines  $DE, EG$  [x 15]

But  $DG$  is commensurable in length with  $AC$

Therefore each of the rectangles  $DH, EK$  is rational [x 10]

Let then the square  $LM$  be constructed equal to  $AI$ , and let there be subtracted  $NO$  equal to  $FK$  and being about the same angle with  $LM$ , namely the angle  $LPM$ ,

therefore the squares  $LM, NO$  are about the same diameter [vi 26]

Let  $PR$  be their diameter, and let the figure be drawn

Since then  $AI, FK$  are medial and are equal to the squares on  $LP, PN$ ,

the squares on  $LP, PN$  are also medial,

therefore  $LP, PN$  are also medial straight lines commensurable in square only

And since the rectangle  $AF, FG$  is equal to the square on  $EG$ ,

therefore, as  $AF$  is to  $EG$ , so is  $EG$  to  $FG$ , [vi 17]

while, as  $AF$  is to  $EG$ , so is  $AI$  to  $EK$ ,

and, as  $EG$  is to  $FG$ , so is  $EK$  to  $FK$ , [vi 11]

therefore  $EK$  is a mean proportional between  $AI, FK$  [v 11]

But  $MN$  is also a mean proportional between the squares  $LM, NO$ , and  $AI$  is equal to  $LM$ , and  $FK$  to  $NO$ ,

therefore  $MN$  is also equal to  $EK$

But  $DH$  is equal to  $EK$ , and  $LO$  equal to  $MN$ ,

therefore the whole  $DK$  is equal to the gnomon  $UVW$  and  $NO$

Since, then, the whole  $AK$  is equal to  $LM, NO$

and, in these  $DK$  is equal to the gnomon  $UVW$  and  $NO$ ,

therefore the remainder  $AB$  is equal to  $TS$

But  $TS$  is the square on  $LN$ ,

therefore the square on  $LN$  is equal to the area  $AB$ ,



$AC$ ,  
 [x 13]  
 [x 21]  
 the straight lines  
 $DE, EG$  [x 15]  
 with  $AC$ ,  
 rational and incommensu-  
 [x 13]  
 therefore each of the rectangles  $AE, EG$  is medial [x 21]

$G$ ,  
 therefore  $AF$  is incommensurable in length with  $EG$  [x 13]  
 But, as  $AF$  is to  $EG$ , so is  $AI$  to  $EK$ , [vi 1]  
 therefore  $AI$  is incommensurable with  $EK$  [x 11]  
 Now let the square  $LM$  be constructed equal to  $AI$ ,  
 and let there be subtracted  $NO$  equal to  $EK$  and being about the same angle  
 with  $LM$ ,  
 therefore  $LM, NO$  are about the same diameter [vi 26]  
 Let  $PR$  be their diameter, and let the figure be drawn  
 Now, since the rectangle  $AF, FG$  is equal to the square on  $EG$ ,  
 therefore, as  $AF$  is to  $EG$ , so is  $EG$  to  $FG$  [vi 17]  
 But, as  $AF$  is to  $EG$ , so is  $AI$  to  $EK$ ,  
 and, as  $EG$  is to  $FG$ , so is  $EK$  to  $FK$ , [vi 1]  
 therefore also, as  $AI$  is to  $EK$ , so is  $EK$  to  $FK$ , [v. 11]  
 therefore  $EK$  is a mean proportional between  $AI, FK$   
 But  $MN$  is also a mean proportional between the squares  $LM, NO$ ,  
 and  $AI$  is equal to  $LM$  and  $EK$  to  $NO$ ,  
 therefore  $EK$  is also equal to  $MN$   
 But  $MA$  is equal to  $LO$  and  $EA$  equal to  $DH$ ,  
 therefore the whole  $DH$  is also equal to the gnomon  $UVW$  and  $NO$   
 But  $AK$  is also equal to  $LM, NO$ ,  
 therefore the remainder  $AB$  is equal to  $ST$ , that is, to the square on  $LN$ ,  
 therefore  $LN$  is the "side" of the area  $AB$   
 I say that  $LN$  is a second apotome of a medial straight line  
 For, since  $AI, FK$  were proved medial, and are equal to the squares on  $LP$ ,  
 $PN$ ,

therefore each of the squares on  $LP, PN$  is also medial,  
 therefore each of the straight lines  $LP, PN$  is medial

And since  $AI$  is commensurable with  $FK$  [vi 1, x 11]  
 therefore the square on  $LP$  is also commensurable with the square on  $PN$   
 Again, since  $AI$  was proved incommensurable with  $EK$ ,

[vi 1, x 11]  
 therefore  $LP, PN$  are medial straight lines commensurable in square only  
 I say next that they also contain a medial rectangle  
 For, since  $EK$  was proved medial and is equal to the rectangle  $LP, PN$ ,



therefore the rectangle  $LP, PN$  is also medial  
so that  $LP, PN$  are medial straight lines commensurable in square only which  
contain a medial rectangle

Therefore  $LN$  is a second apotome of a medial straight line,  
and it is the "side" of the area  $AB$  [x 75]

Therefore the "side" of the area  $AB$  is a second apotome of a medial straight line  $QEP$

### PROPOSITION 94

*If an area be contained by a rational straight line and a fourth apotome, the "side" of the area is minor*

For let the area  $AB$  be contained by the rational straight line  $AC$  and the fourth apotome  $AD$ ,

I say that the "side" of the area  $AB$  is minor

For let  $DG$  be the annex to  $AD$ ,  
therefore  $AG$   $GD$  are rational straight lines commensurable in square only,  
 $AG$  is commensurable in length with the rational straight line  $AC$  set out  
and the square on the whole  $AG$  is greater than the square on the annex  $DG$  by  
the square on a straight line incommensurable in length with  $AG$ ,

{x Def III 4}

Since then the square on  $AG$  is greater than the square on  $GD$  by the square on a straight line incommensurable in length with  $AG$

therefore if there be applied to  $AG$  a parallelogram equal to the fourth part of the square on  $DG$  and deficient by a square figure it will divide it into incommensurable parts. [x 18]

Let then  $DG$  be bisected at  $E$ ,  
let there be applied to  $AG$  a parallelogram equal to the square on  $EG$  and deficient by a square figure

and let it be the rectangle  $AF$   $FG$ ,  
therefore  $AF$  is incommensurable in  
length with  $FG$

Let  $EH$   $FI$   $GA$  be drawn through  $E$   $F$   $G$  parallel to  $AC$ ,  $BD$

Since then  $4G$  is rational and commensurable in length with  $AC$ ,

therefore the whole  $AK$  is rational. [x 19]

Again, since  $DG$  is incommensurable in length with  $AC$ , and both are ra-

therefore  $DK$  is medial

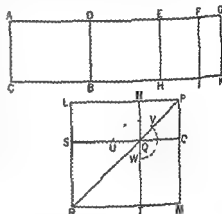
Again since  $AF$  is incommensurable in length with  $FG$

therefore  $AI$  is also incommensurable with  $FK$  [VI 1, x 11]

Now let the square  $LM$  be constructed equal to  $AI$ , and let there be subtracted  $VO$  equal to  $FA$  and about the same angle, the angle  $LPM$ .

Therefore the squares  $LM$  &  $NO$  are about the same diameter

Let  $PR$  be their diameter and let the figure be drawn



Since, then, the rectangle  $AF, FG$  is equal to the square on  $EG$ ,  
therefore, proportionally, as  $AF$  is to  $EG$ , so  $\square EG$  to  $FG$  [vi. 17]

But, as  $AF$  is to  $EG$ , so is  $AI$  to  $EK$ ,  
and, as  $EG$  is to  $FG$ , so is  $EK$  to  $FK$ , [VI. 1]

therefore  $EK$  is a mean proportional between  $AI$ ,  $EK$  [v 11]

But  $MN$  is also a mean proportional between the squares  $LM$ ,  $NO$ ,  
and  $AI$  is equal to  $LM$ , and  $FA$  to  $NO$ .

therefore  $EK$  is also equal to  $MN$

But  $DH$  is equal to  $EK$ , and  $LO$  is equal to  $MN$ ,  
therefore the whole  $DK$  is equal to the gnomon  $UVW$  and  $NO$

Since, then, the whole  $AK$  is equal to the squares  $LM$ ,  $NO$ , and, in these,  $DK$  is equal to the gnomon  $UVW$  and the square  $NO$ , therefore the remainder  $AB$  is equal to  $ST$ , that is, to the square on  $LN$ ; therefore  $LN$  is the "side" of the area  $AB$ .

I say that  $LN$  is the irrational straight line called minor

For, since  $AK$  is rational and is equal to the squares on  $LP$ ,  $PN$ , therefore the sum of the squares on  $LP$ ,  $PN$  is rational

Again, since  $DK$  is medial,  
and  $DK$  is equal to twice the rectangle  $LP, PN$ ,  
therefore twice the rectangle  $LP, PN$  is medial

And, since  $AI$  was proved incommensurable with  $FA$ , therefore the square on  $LP$  is also incommensurable with the square on  $PN$ .

Therefore  $LP$ ,  $PN$  are straight lines incommensurable in square which make the sum of the squares on them rational, but twice the rectangle contained by them medial.

Therefore  $LN$  is the irrational straight line called minor,  
and it is the "side" of the area  $AB$  [x 76]

Therefore the "side" of the area  $AB$  is minor Q.E.D.

### PROPOSITION 95

*If an area be contained by a rational straight line and a fifth apotome, the "side" of the area is a straight line which produces with a rational area a medial whole*

For let the area  $AB$  be contained by the rational straight line  $AC$  and the fifth apotome  $AD$ .

I say that the "side" of the area  $AB$  is a straight line which produces with a rational area a medial whole

For let  $DG$  be the annex to  $AD$ ,  
therefore  $AG$ ,  $GD$  are rational straight lines commensurable in square only,  
the annex  $GD$  is commensurable in length with the rational straight line  $AC$   
set out,

and it is less than the one on the inner DCB

commensurable parts [x 18]

Let then  $DG$  be bisected at the point  $E$ ,

Now, since  $AG$  is incommensurable in length with  $CA$ , and both are rational  
therefore  $AK$  is medial [x 21]

Again, since  $DG$  is rational and commensurable in length with  $AC$ ,

$DK$  is rational [x 19]

Now let the square  $LM$  be constructed equal to  $AI$ , and let the square  $NO$  equal to  $FK$  and about the same angle, the angle  $LPM$ , be subtracted,

therefore the squares  $LM$ ,  $NO$  are about the same diameter [vi 26]

Let  $PR$  be their diameter, and let the figure be drawn

Similarly then we can prove that  $LN$  is the "side" of the area  $AB$

I say that  $LN$  is the straight line which produces with a rational area a medial whole

For, since  $AK$  was proved medial and is equal to the squares on  $LP$ ,  $PN$ ,  
therefore the sum of the squares on  $LP$ ,  $PN$  is medial

Again, since  $DK$  is rational and is equal to twice the rectangle  $LP$ ,  $PN$ ,  
the latter is itself also rational

And, since  $AI$  is incommensurable with  $FK$ ,  
therefore the square on  $LP$  is also incommensurable with the square on  $PN$ ,  
therefore  $LP$ ,  $PN$  are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational

Therefore the remainder  $LN$  is the irrational straight line called that which produces with a rational area a medial whole, [x 77]

and it is the "side" of the area  $AB$

Therefore the "side" of the area  $AB$  is a straight line which produces with a rational area a medial whole

Q E D

### PROPOSITION 96

If an area be contained by a rational straight line and a sixth apotome, the "side" of the area is a straight line which produces with a medial area a medial whole

For let the area  $AB$  be contained by the rational straight line  $AC$  and the sixth apotome  $AD$ ,

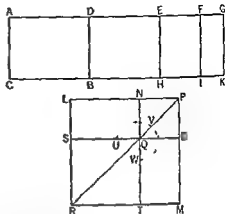
I say that the "side" of the area  $AB$  is a straight line which produces with a medial area a medial whole

For let  $DG$  be the annex to  $AD$ ,  
therefore  $AG$ ,  $GD$  are rational straight lines commensurable in square only,  
neither of them is commensurable in length with the rational straight line  $AC$   
set out,

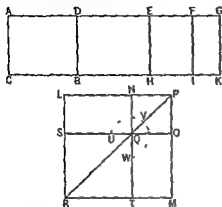
and the square on the whole  $AG$  is greater than the square on the annex  $DG$  by the square on a straight line incommensurable in length with  $AG$ .

[x Def iii 6]

Since, then, the square on  $AG$  is greater than the square on  $GD$  by the square on a straight line incommensurable in length with  $AG$ ,



therefore, if there be applied to  $AG$  a parallelogram equal to the fourth part of the square on  $DG$  and deficient by a square figure, it will divide it into incommensurable parts [x 18]



Let then  $DG$  be bisected at  $E$ , let there be applied to  $AG$  a parallelogram equal to the square on  $EG$  and deficient by a square figure, and let it be the rectangle  $AF, FG$ , therefore  $AF$  is incommensurable in length with  $FG$

But, as  $AF$  is to  $FG$ , so is  $AI$  to  $FK$ , [vi 1]  
therefore  $AI$  is incommensurable with  $FK$  [x 11]

And, since  $AG, AC$  are rational straight lines commensurable in square only,

$AK$  is medial [x 21]

Again, since  $AC, DG$  are rational straight lines and incommensurable in length,

$DK$  is also medial [x 21]

Now, since  $AG, GD$  are commensurable in square only,  
therefore  $AG$  is incommensurable in length with  $GD$

But, as  $AG$  is to  $GD$ , so is  $AK$  to  $KD$ , [vi 1]  
therefore  $AK$  is incommensurable with  $KD$  [x 11]

Now let the square  $LM$  be constructed equal to  $AI$ ,  
and let  $NO$  equal to  $FK$ , and about the same angle be subtracted,  
therefore the squares  $LM, NO$  are about the same diameter [vi 26]

Let  $PR$  be their diameter, and let the figure be drawn

Then in manner similar to the above we can prove that  $LN$  is the "side" of the area  $AB$

I say that  $LV$  is a straight line which produces with a medial area a medial whole

For, since  $AK$  was proved medial and is equal to the squares on  $LP, PN$ ,  
therefore the sum of the squares on  $LP, PN$  is medial

Again, since  $DK$  was proved medial and is equal to twice the rectangle  $LP, PN$ ,

twice the rectangle  $LP, PN$  is also medial

And since  $AK$  was proved incommensurable with  $DK$ ,  
the squares on  $LP, PN$  are also incommensurable with twice the rectangle  $LP, PN$

And since  $AI$  is incommensurable with  $FK$ ,  
therefore the square on  $LP$  is also incommensurable with the square on  $PN$ ,

tangle contained by them

Therefore  $LV$  is the irrational straight line called that which produces with a medial area a medial whole, [x 76]

and it is the "side" of the area  $AB$

Therefore the "side" of the area is a straight line which produces with a medial area a medial whole Q E D

### PROPOSITION 97

*The square on an apotome applied to a rational straight line produces as breadth a first apotome*

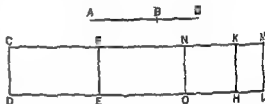
Let  $AB$  be an apotome, and  $CD$  rational, and to  $CD$  let there be applied  $CE$  equal to the square on  $AB$  and producing  $CF$  as breadth,

I say that  $CF$  is a first apotome

For let  $BG$  be the annex to  $AB$ ;

therefore  $AG$ ,  $GB$  are rational straight lines commensurable in square only [x 73]

To  $CD$  let there be applied  $CH$  equal to the square on  $AG$ , and  $KL$  equal to the square on  $BG$



Therefore the whole  $CL$  is equal to the squares on  $AG$ ,  $GB$ , and, in these,  $CE$  is equal to the square on  $AB$ ,

therefore the remainder  $FL$  is equal to twice the rectangle  $AG$ ,  $GB$  [ii 7]

Let  $FM$  be bisected at the point  $N$ ,

and let  $NO$  be drawn through  $N$  parallel to  $CD$ ,

therefore each of the rectangles  $FO$ ,  $LN$  is equal to the rectangle  $AG$ ,  $GB$

Now, since the squares on  $AG$ ,  $GB$  are rational,

and  $DM$  is equal to the squares on  $AG$ ,  $GB$ ,

therefore  $DM$  is rational

And it has been applied to the rational straight line  $CD$ , producing  $CM$  as breadth,

therefore  $CM$  is rational and commensurable in length with  $CD$  [x 20]

Again, since twice the rectangle  $AG$ ,  $GB$  is medial, and  $FL$  is equal to twice the rectangle  $AG$ ,  $GB$ ,

therefore  $FL$  is medial

And it is applied to the rational straight line  $CD$ , producing  $FM$  as breadth,

therefore  $FM$  is rational and incommensurable in length with  $CD$  [x 22]

And, since the squares on  $AG$ ,  $GB$  are rational,

while twice the rectangle  $AG$ ,  $GB$  is medial,

therefore the squares on  $AG$ ,  $GB$  are incommensurable with twice the rectangle  $AG$ ,  $GB$

And  $CL$  is equal to the squares on  $AG$ ,  $GB$ ,

and  $FL$  to twice the rectangle  $AG$ ,  $GB$ ,

therefore  $DM$  is incommensurable with  $FL$

But, as  $DM$  is to  $FL$ , so is  $CM$  to  $FM$ ,

therefore  $CM$  is incommensurable in length with  $FM$

And both are rational,

therefore  $CM$ ,  $MF$  are rational straight lines commensurable in square only,

therefore  $CF$  is an apotome

[vi 1]

[x 11]

[x 73]

I say next that it is also a first apotome

For, since the rectangle  $AG, GB$  is a mean proportional between the squares on  $AG, GB$ ,

and  $CH$  is equal to the square on  $AG$ , 1

$KL$  equal to the square on  $BG$ ,

and  $NL$  equal to the rectangle  $AG, GB$ .

therefore  $NL$  is also a mean proportional between  $CH$ ,  $KL$ .

therefore, as  $CH$  is to  $NL$ , so is  $NL$  to  $KL$

But, as  $CH$  is to  $NL$ , so is  $CK$  to  $NM$ .

and, as  $NL$  is to  $KL$  so is  $NM$  to  $KM$ , [vi 1]

therefore the rectangle  $CK, KM$  is equal to the square on  $NM$  [vi 17] that is, to the fourth part of the square on  $FM$

And, since the square on  $AG$  is commensurable with the square on  $GB$ ,

$CH$  is also commensurable with  $KL$

But, as  $CH$  is to  $KL$  so is  $CK$  to  $KM$ . [VI. 11]

therefore  $CK$  is commensurable with  $KM$  [x 11]

"qual to the fourth

while  $CK$  is commensurable with  $KM$

out,

therefore  $CF$  is a first apotome [x Def in 1]

Therefore etc

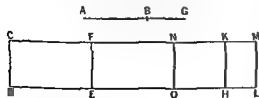
### PROPOSITION 98

*The square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome*

Let  $AB$  be a first apotome of a medial straight line and  $CD$  a rational straight line, and to  $CD$  let there be applied  $CE$  equal to the square on  $AB$ , producing  $CF$  as breadth

I say that  $CF$  is a second apotome

For let  $BG$  be the annex to  $AB$ ,  
therefore  $AG, GB$  are medial straight lines commensurable in square only  
which contain a rational rectangle [x 74]



To *CD* let there be applied *CH* equal to the square on *AG*, producing *CK* as breadth, and *KL* equal to the square on *GB*, producing *KM* as breadth, therefore the whole *CL* is equal to the squares on *AG*, *GB*.

therefore  $CL$  is also medial [x 15 and 24 Por]

And it is applied to the rational straight line  $CD$ , producing  $CM$  as breadth; therefore  $CM$  is rational and incommensurable in length with  $CD$  [x 22]

Now, since  $CL$  is equal to the squares on  $AG$ ,  $GB$ ,  
 and, in these, the square on  $AB$  is equal to  $CE$ ,  
 therefore the remainder, twice the rectangle  $AG$ ,  $GB$ , is equal to  $FL$  [ii 7]  
 But twice the rectangle  $AG$ ,  $GB$  is rational  
 therefore

And it is as breadth, [x 20]  
 that  $FL$  is rational while  
 to  $CL$  is medial

therefore  $CL$  is incommensurable with  $FL$

But, as  $CL$  is to  $FL$ , so is  $CM$  to  $FM$ , [vi 1]  
 therefore  $CM$  is incommensurable in length with  $FM$  [x 11]

And both are rational,  
 therefore  $CM$ ,  $MF$  are rational straight lines commensurable in square only,  
 therefore  $CF$  is an apotome [x 73]

I say next that it is also a second apotome

For let  $FM$  be bisected at  $N$ ,

and let  $NO$  be drawn through  $N$  parallel to  $CD$ ,

therefore each of the rectangles  $FO$ ,  $NL$  is equal to the rectangle  $AG$ ,  $GB$

Now, since the rectangle  $AG$ ,  $GB$  is a mean proportional between the squares  
 on  $AG$ ,  $GB$ ,

and the square on  $AO$  is equal to  $CH$ ,

the rectangle  $AG$ ,  $GB$  to  $NL$ ,

and the square on  $BO$  to  $KL$ ,

therefore  $NL$  is also a mean proportional between  $CH$ ,  $KL$ ,

therefore as  $CH$  is to  $NL$  so is  $NL$  to  $KL$

But as  $CH$  is to  $NL$  so is  $CK$  to  $NM$

and as  $NL$  is to  $KL$  so is  $NM$  to  $MA$ , [vi 1]

therefore, as  $CH$  is to  $NM$ , so is  $NM$  to  $MA$ , [vi 11]

therefore the rectangle  $CK$ ,  $MA$  is equal to the square on  $NM$  [vi 17], that is  
 to the fourth part of the square on  $FM$

Since then,  $CM$ ,  $MF$  are two unequal straight lines and the rectangle  $CK$ ,  
 $MA$  equal to the fourth part of the square on  $MF$  and deficient by a square  
 figure has been applied to the greater  $CM$ , and divides it into commensurable  
 parts

therefore the square on  $CM$  is greater than the square on  $MF$  by the square on  
 a straight line commensurable in length with  $CM$  [x 17]

And the annex  $FM$  is commensurable in length with the rational straight  
 line  $CD$  set out,

therefore  $CF$  is a second apotome [x Def III 2]

Therefore etc

Q E D

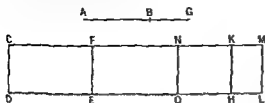
# PROPOSITION 99

The square on a second apotome of a medial straight line applied to a rational  
 straight line produces as breadth a third apotome

Let  $AB$  be a second apotome of a medial straight line and  $CD$  rational  
 and to  $CD$  let there be applied  $CE$  equal to the square on  $AB$ , producing  $CF$  as  
 breadth,

I say that  $CF$  is a third apotome

For let  $BG$  be the annex to  $AB$ ,  
 therefore  $AG, GB$  are medial straight lines commensurable in square only  
 which contain a medial rectangle [x 75]



therefore  $CL$  is also medial [x 13 and 23, Por.]

Let  $CH$  equal to the square  
 on  $AG$  be applied to  $CD$ , pro-  
 ducing  $CH$  as breadth,  
 and let  $KL$  equal to the square  
 on  $GB$  be applied to  $AH$ , pro-  
 ducing  $KL$  as breadth,  
 therefore the whole  $CL$  is equal  
 to the squares on  $AG, GB$ ,

th

$CE$  is equal to the square on  $AB$

therefore the remainder  $LF$  is equal to twice the rectangle  $AG, GB$  [ii 7]

Let then  $FM$  be bisected at the point  $N$ ,

and let  $NO$  be drawn parallel to  $CD$ ,

therefore each of the rectangles  $FO, NL$  is equal to the rectangle  $AG, GB$

But the rectangle  $AG, GB$  is medial,

therefore  $FL$  is also medial

And it is applied to the rational straight line  $EF$ , producing  $FM$  as breadth,  
 therefore  $FM$  is also rational and incommensurable in length with  $CD$  [x 22]

And, since  $AG, GB$  are commensurable in square only,

therefore  $AG$  is incommensurable in length with  $GB$ ,

therefore the square on  $AG$  is also incommensurable with the rectangle  $AG, GB$   
 $GB$  [vi 1, x 11]

But the squares on  $AG, GB$  are commensurable with the square on  $AG$ ,

and twice the rectangle  $AG, GB$  with the rectangle  $AG, GB$ ,

therefore the squares on  $AG, GB$  are incommensurable with twice the rectangle  
 $AG, GB$  [x 13]

But  $CL$  is equal to the squares on  $AG, GB$ ,

and  $FL$  is equal to twice the rectangle  $AG, GB$ ,

therefore  $CL$  is also incommensurable with  $FL$

But as  $CL$  is to  $FL$  so is  $CM$  to  $FM$ ,

therefore  $CM$  is incommensurable in length with  $FM$  [vi 1]

And both are rational,

therefore  $CM, FM$  are rational straight lines commensurable in square only

therefore  $CF$  is an apotome [x 73]

I say next that it is also a third apotome

For

And since the rectangle  $AG, GB$  is a mean proportional between the squares  
 on  $AG, GB$ ,

and  $CH$  is equal to the square on  $AG$ ,

$KL$  equal to the square on  $GB$ ,

and  $NL$  equal to the rectangle  $AG, GB$ ,



therefore  $NL$  is also a mean proportional between  $CH$ ,  $KL$ ,

therefore, as  $CH$  is to  $NL$ , so is  $NL$  to  $KL$

But, as  $CH$  is to  $NL$ , so is  $CK$  to  $NM$ ,

and, as  $NL$  is to  $KL$ , so is  $NM$  to  $KM$ , [vi 1]

therefore, as  $CK$  is to  $MN$ , so is  $MN$  to  $KM$ , [v 11]

therefore the rectangle  $CH$ ,  $AM$  is equal to [the square on  $MN$ , that is, to] the fourth part of the square on  $FM$

Since, then  $CM$ ,  $MF$  are two unequal straight lines and a parallelogram equal to the fourth part of the square on  $FM$  and deficient by a square figure has been applied to  $CM$ , and divides it into commensurable parts

therefore the square on  $CM$  is greater than the square on  $MF$  by the square on a straight line commensurable with  $CM$  [x 17]

And neither of the straight lines  $CM$ ,  $MF$  is commensurable in length with the rational straight line  $CD$  set out,

therefore  $CF$  is a third apotome [x Def III 3]

Therefore etc

Q E D

### PROPOSITION 100

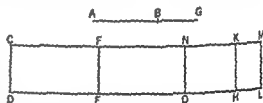
The square on a minor straight line applied to a rational straight line produces as breadth a fourth apotome

Let  $AB$  be a minor and  $CD$  a rational straight line and to the rational straight line  $CD$  let  $CE$  be applied equal to the square on  $AB$  and producing  $CF$  as breadth,

I say that  $CF$  is a fourth apotome

For let  $BG$  be the annex to  $AB$ ,

therefore  $AG$ ,  $GB$  are straight lines incommensurable in square which make the sum of the squares on  $AG$ ,  $GB$  rational but twice the rectangle  $AG$ ,  $GB$  medial [x 70]



To  $CD$  let there be applied  $CH$  equal to the square on  $AG$  and producing  $CL$  as breadth

and  $AL$  equal to the square on  $BG$  producing  $KM$  as breadth,

therefore the whole  $CL$  is equal to the squares on  $AG$ ,  $GB$

And the sum of the squares on  $AG$ ,  $GB$  is rational,

therefore  $CL$  is also rational

And it is applied to the rational straight line  $CD$  producing  $CM$  as breadth, therefore  $CM$  is also rational and commensurable in length with  $CD$  [x 20]

And since the whole  $CL$  is equal to the squares on  $AG$ ,  $GB$ , and in these  $CE$  is equal to the square on  $AB$ ,

therefore the remainder  $FL$  is equal to twice the rectangle  $AG$ ,  $GB$  [ii 7]

Let then  $FM$  be bisected at the point  $N$

then  $FN$  is equal to either of the straight lines  $CD$ ,  $ML$ ,

$VL$  is equal to the rectangle  $AG$ ,  $GB$

$GB$  is medial and is equal to  $FL$ ,

therefore  $FL$  is also medial

And it is applied to the rational straight line  $FE$  producing  $FM$  as breadth

therefore  $FM$  is rational and incommensurable in length with  $CD$  [x 22]

And, since the sum of the squares on  $AG, GB$  is rational,  
while twice the rectangle  $AG, GB$  is medial,

the squares on  $AG, GB$  are incommensurable with twice the rectangle  $AG, GB$ .

But  $GL$  is equal to the squares on  $AG, GB$ ,

and  $FL$  equal to twice the rectangle  $AG, GB$ ,

therefore  $CL$  is incommensurable with  $FL$

But, as  $CL$  is to  $FL$ , so is  $CM$  to  $MF$ ,

[vi 1]

therefore  $CM$  is incommensurable in length with  $MF$

[x 11]

And both are rational,

therefore  $CM, MF$  are rational straight lines commensurable in square only,

therefore  $CF$  is an apotome

[x 73]

I say that it is also a fourth apotome

For, since  $AG, GB$  are incommensurable in square,

therefore the square on  $AG$  is also incommensurable with the square on  $GB$

And  $CH$  is equal to the square on  $AG$ ,

and  $KL$  equal to the square on  $GB$ ;

therefore  $CH$  is incommensurable with  $KL$

But, as  $CH$  is to  $KL$ , so is  $CK$  to  $KM$ ,

[vi 1]

therefore  $CK$  is incommensurable in length with  $KM$

[x 11]

And, since the rectangle  $AG, GB$  is a mean proportional between the squares on  $AG, GB$ ,

and the square on  $AG$  is equal to  $CH$ ,

the square on  $GB$  to  $KL$ ,

and the rectangle  $AG, GB$  to  $NL$ ,

therefore  $NL$  is a mean proportional between  $CH, KL$ ,

therefore, as  $CH$  is to  $NL$ , so is  $NL$  to  $KL$

But, as  $CH$  is to  $NL$ , so is  $CK$  to  $NM$ ,

BY THE ALGEBRA, CK : NM :: CH : NL

[x 11]

Since then  $CM, MF$  are two unequal straight lines, and the rectangle  $CK, KM$  equal to the fourth part of the square on  $MF$  and deficient by a square figure has been applied to  $CM$  and divides it into incommensurable parts, therefore the square on  $CM$  is greater than the square on  $MF$  by the square on a straight line incommensurable with  $CM$  [x 18]

And the whole  $CM$  is commensurable in length with the rational straight line  $CD$  set out,

therefore  $CF$  is a fourth apotome

[x Def III 4]

Therefore etc

Q E D

# PROPOSITION 101

The square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome

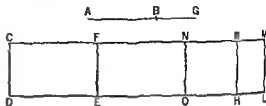
W  
th

For let  $BG$  be the annex to  $AB$ ,

therefore  $AG, GB$  are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational [x 77]

To  $CD$  let there be applied  $CH$  equal to the square on  $AG$  and  $KL$  equal to the square on  $GB$ , therefore the whole  $CL$  is equal to the squares on  $AG, GB$

But the sum of the squares on  $AG, GB$  together is medial, therefore  $CL$  is medial



and, in these,  $CE$  is equal to the square on  $AB$ , therefore the remainder  $FL$  is equal to twice the rectangle  $AG, GB$  [ii 7]

Let then  $FM$  be bisected at  $N$ , and through  $N$  let  $NO$  be drawn parallel to either of the straight lines  $CD, ML$ , therefore each of the rectangles  $FO, NL$  is equal to the rectangle  $AG, GB$

And, since twice the rectangle  $AG, GB$  is rational and equal to  $FL$ , therefore  $FL$  is rational

And if applied to  $FM$  as breadth [x 20]

therefore  $CL$  is incommensurable with  $FL$  But, as  $CL$  is to  $FL$ , so is  $CM$  to  $MF$ , [vi 1]

therefore  $CM$  is incommensurable in length with  $MF$  [x 11]

And both are rational, therefore  $CM, MF$  are rational straight lines commensurable in square only, therefore  $CF$  is an apotome [x 73]

I say next that it is also a fifth apotome For we can prove similarly that the rectangle  $CK, KM$  is equal to the square on  $NM$ , that is to the fourth part of the square on  $FM$

And since the square on  $AG$  is incommensurable with the square on  $GB$ , while the square on  $AG$  is equal to  $CH$ , and the square on  $GB$  to  $KL$ , therefore  $CH$  is incommensurable with  $KL$

But, as  $CH$  is to  $KL$  so is  $CK$  to  $KM$ , [vi 1] therefore  $CK$  is incommensurable in length with  $KM$  [x 11]

Since then  $CM, MF$  are two unequal straight lines and a parallelogram equal to the fourth part of the square on  $FM$  and deficient by a square figure has been applied to  $CM$ , and divides it into incommensurable parts,

re on  
[x 13]  
out,  
[ii 5]

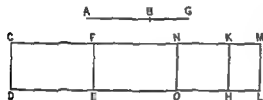
## PROPOSITION 102

*The square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome*

Let  $AB$  be the straight line which produces with a medial area a medial whole, and  $CD$  a rational straight line, and to  $CD$  let  $CE$  be applied equal to the square on  $AB$  and producing  $CF$  as breadth,

I say that  $CF$  is a sixth apotome

For let  $BG$  be the annex to  $AB$ ,



therefore  $AG, GB$  are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle  $AG, GB$  medial and the squares on  $AG, GB$  incommensurable with twice the rectangle  $AG, GB$

[x 78]

Now to  $CD$  let there be applied  $CH$  equal to the square on  $AG$  and producing  $CK$  as breadth,

and  $KL$  equal to the square on  $BG$ ,

therefore the whole  $CL$  is equal to the squares on  $AG, GB$ ,

therefore  $CL$  is also medial

And since  $CE$  is equal to the square on  $AB$ , and  $CH$  equal to the square on  $AG$ , and  $KL$  equal to the square on  $BG$ ,  
[x 79]

and, in these,  $CE$  is equal to the square on  $AB$ ,

therefore the remainder  $FL$  is equal to twice the rectangle  $AG, GB$  [x 7]

And twice the rectangle  $AG, GB$  is medial,

therefore  $FL$  is also medial

And since  $CE$  is equal to the square on  $AB$ , and  $CH$  equal to the square on  $AG$ , and  $KL$  equal to the square on  $BG$ ,  
[x 80]

tangle  $AG, GB$ ,

and  $CL$  is equal to the squares on  $AG, GB$ ,

and  $FL$  equal to twice the rectangle  $AG, GB$ ,

therefore  $CL$  is incommensurable with  $FL$

But, as  $CL$  is to  $FL$ , so is  $CM$  to  $MF$ , [vi 1]

therefore  $CM$  is incommensurable in length with  $MF$  [x 11]

And both are rational

Therefore  $CM, MF$  are rational straight lines commensurable in square only,

therefore  $CF$  is an apotome [x 73]

I say next that it is also a sixth apotome

For

to  $CD$ ,

therefore each of the rectangles  $CE, CH$  is equal to the rectangle  $AG, GB$

And, since  $AG, GB$  are incommensurable in square,

therefore the square on  $AG$  is incommensurable with the square on  $GB$   
 But  $CH$  is equal to the square on  $AG$ ,

and  $KL$  is equal to the square on  $GB$ ,

therefore  $CH$  is incommensurable with  $KL$

But as  $CH$  is to  $KL$ , so is  $CK$  to  $KM$ , [vi 1]

therefore  $CK$  is incommensurable with  $KM$  [v 11]

And since the rectangle  $AG, GB$  is a mean proportional between the squares on  $AG, GB$ ,

and  $CH$  is equal to the square on  $AG$ ,

$KL$  equal to the square on  $GB$ ,

and  $NL$  equal to the rectangle  $AG, GB$ ,

therefore  $NL$  is also a mean proportional between  $CH, KL$ ,

therefore as  $CH$  is to  $NL$  so is  $NL$  to  $KL$

And for the same reason as before the square on  $CM$  is greater than the square on  $MF$  by the square on a straight line incommensurable with  $CM$  [x 18]

And neither of them is commensurable with the rational straight line  $CD$  set out,

therefore  $CF$  is a sixth apotome [v Def III 6]  
 Q E D

### PROPOSITION 103

*A straight line commensurable in length with an apotome is an apotome and the same in order*

Let  $AB$  be an apotome

and let  $CD$  be commensurable in length with  $AB$ ,

I say that  $CD$  is also an apotome and the same in order with  $AB$

For since  $AB$  is an apotome let  $BE$  be the annex to it, therefore  $AE, EB$  are rational straight lines commensurable in square only [x 73]

Let it be contrived that the ratio of  $BE$  to  $DF$  is the same as the ratio of  $AE$  to  $CD$  [vi 12]

therefore also as one is to one so are all to all, [v 12]

therefore also as the whole  $AE$  is to the whole  $CF$ , so is  $AB$  to  $CD$

But  $AB$  is commensurable in length with  $CD$

Therefore  $AE$  is also commensurable with  $CF$  and  $BE$  with  $DF$  [x 11]

And  $AE, EB$  are rational straight lines commensurable in square only, therefore  $CF, FD$  are also rational straight lines commensurable in square only [x 11]

Now since as  $AE$  is to  $CF$  so is  $BE$  to  $DF$

alternately therefore as  $EF$  is to  $EB$  so is  $CF$  to  $FD$  [v 18]

And the square on  $AE$  is greater than the square on  $EB$  either by the square on a straight line commensurable with  $AE$  or by the square on a straight line incommensurable with it

If then the square on  $EF$  is greater than the square on  $EB$  by the square on a straight line commensurable with  $AE$  the square on  $CF$  will also be greater than the square on  $FD$  by the square on a straight line commensurable with  $CF$  [x 11]

And, if  $AE$  is commensurable in length with the rational straight line set out,

$CF$  is so also, [x 12]

if  $BE$ , then  $DF$  also, [id]

and, if neither of the straight lines  $AE$ ,  $EB$ , then neither of the straight lines  $CF$ ,  $FD$  [x 13]

But, if the square on  $AE$  is greater than the square on  $EB$  by the square on a straight line incommensurable with  $AE$ ,

the square on  $CF$  will also be greater than the square on  $FD$  by the square on a straight line incommensurable with  $CF$  [x 14]

And, if  $AE$  is commensurable in length with the rational straight line set out,

$CF$  is so also,

if  $BE$ , then  $DF$  also, [x 12]

and if neither of the straight lines  $AE$ ,  $EB$ , then neither of the straight lines  $CF$ ,  $FD$  [x 13]

Therefore  $CD$  is an apotome and the same in order with  $AB$  Q E D

#### PROPOSITION 104

*A straight line commensurable with an apotome of a medial straight line is an apotome of a medial straight line and the same in order*

C ——— D ——— F For, since  $AB$  is an apotome of a medial straight line, let  $EB$  be the annex to it

Therefore  $AE$ ,  $EB$  are medial straight lines commensurable in square only [x 74, 75]

Therefore  $CF$ ,  $FD$  are also medial straight lines [x 23] commensurable in square only, [x 13]

therefore  $CD$  is an apotome of a medial straight line [x 71, 75]

I say next that it is also the same in order with  $AB$

Since, as  $AE$  is to  $EB$ , so is  $CF$  to  $FD$ ,

therefore also, as the square on  $AE$  is to the rectangle  $AE$ ,  $EB$ , so is the square on  $CF$  to the rectangle  $CF$ ,  $FD$

But the square on  $AE$  is commensurable with the square on  $CF$ , therefore the rectangle  $AE$ ,  $EB$  is also commensurable with the rectangle  $CF$ ,  $FD$  [v 16, x 11]

Therefore, if the rectangle  $AE$ ,  $EB$  is rational, the rectangle  $CF$ ,  $FD$  will also be rational, [x Def 4]

and if the rectangle  $AE$ ,  $EB$  is medial, the rectangle  $CF$ ,  $FD$  is also medial [x 23, Por]

Therefore  $CD$  is an apotome of a medial straight line and the same in order with  $AB$  [x 74, 75]

Q E D

## PROPOSITION 105

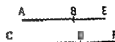
*A straight line commensurable with a minor straight line is minor.*

Let  $AB$  be a minor straight line, and  $CD$  commensurable with  $AB$ ,

I say that  $CD$  is also minor

Let the same construction be made as before;

then, since  $AE$ ,  $EB$  are incommensurable in square,



[x 76]

therefore  $CF$ ,  $FD$  are also incommensurable in square [x 13]

Now since, as  $AE$  is to  $EB$ , so  $\square CF$  to  $FD$ , [v 12, v 16]

therefore also, as the square on  $AE$  is to the square on  $EB$ , so  $\square$  the square on  $CF$  to the square on  $FD$  [v 12]

Therefore, *componendo*, as the squares on  $AE$ ,  $EB$  are to the square on  $EB$ , so are the squares on  $CF$ ,  $FD$  to the square on  $FD$  [v 18]

But the square on  $BE$  is commensurable with the square on  $DF$ , therefore the sum of the squares on  $AE$ ,  $EB$   $\square$  also commensurable with the sum of the squares on  $CF$ ,  $FD$  [v 16, x 11]

But the sum of the squares on  $AE$ ,  $EB$  is rational, [x 76]

therefore the sum of the squares on  $CF$ ,  $FD$  is also rational [x Def 4]

Again, since, as the square on  $AE$  is to the rectangle  $AE$ ,  $EB$ , so  $\square$  the square on  $CF$  to the rectangle  $CF$ ,  $FD$ ,

while the square on  $AE$  is commensurable with the square on  $CF$ , therefore the rectangle  $AE$ ,  $EB$  is also commensurable with the rectangle  $CF$ ,  $FD$

But the rectangle  $AE$ ,  $EB$  is medial, [x 76]

therefore the rectangle  $CF$ ,  $FD$  is also medial, [x 33, Por]

therefore  $CF$ ,  $FD$  are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial

Therefore  $CD$  is minor [x 76]

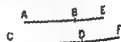
Q E D

## PROPOSITION 106

*A straight line commensurable with that which produces with a rational area a medial whole is a straight line which produces with a rational area a medial whole*

Let  $AB$  be a straight line which produces with a rational area a medial whole, and  $CD$  commensurable with  $AB$ ,

I say that  $CD$  is also a straight line which produces with a rational area a medial whole



For let  $BE$  be the annex to  $AB$ ,

therefore  $AE$ ,  $EB$  are straight lines incommensurable in square which make the sum of the squares on  $AE$ ,  $EB$  medial but the rectangle contained by them rational [x 77]

Let the same construction be made

Then we can prove, in manner similar to the foregoing, that  $CF$ ,  $FD$  are in the same ratio as  $AE$ ,  $EB$ ,

the sum of the squares on  $AE$ ,  $EB$  is commensurable with the sum of the squares on  $CF$ ,  $FD$

and the rectangle  $AE$ ,  $EB$  with the rectangle  $CF$ ,  $FD$ ,

so that  $CF$ ,  $FD$  are also straight lines incommensurable in square which make the sum of the squares on  $CF$ ,  $FD$  medial, but the rectangle contained by them rational

Therefore  $CD$  is a straight line which produces with a rational area a medial whole

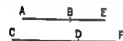
[x 77]

Q E D

## PROPOSITION 107

*A straight line commensurable with that which produces with a medial area a medial whole is itself also a straight line which produces with a medial area a medial whole*

Let  $AB$  be a straight line which produces with a medial area a medial whole, and let  $CD$  be commensurable with  $AB$ ,  
I say that  $CD$  is also a straight line which produces with a medial area a medial whole  
For let  $BE$  be the annex to  $AB$ ,  
and let the same construction be made,



th -  $AE$ ,  $EB$  are commensurable with  $CF$ ,  $FD$ ,  
th  
di  
rectangle contained by them

[x 78]

Now, as was proved,  $AE$ ,  $EB$  are commensurable with  $CF$ ,  $FD$ ,  
the sum of the squares on  $AE$ ,  $EB$  with the sum of the squares on  $CF$ ,  $FD$ ,  
and the rectangle  $AE$ ,  $EB$  with the rectangle  $CF$ ,  $FD$ ,  
therefore  $CF$ ,  $FD$  are also straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and further, the sum of the squares on them incommensurable with the rectangle contained by them

Therefore  $CD$  is a straight line which produces with a medial area a medial whole

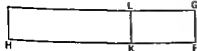
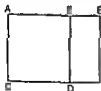
[x 78]

Q E D

## PROPOSITION 108

*If from a rational area a medial area be subtracted, the "side" of the remaining area becomes one of two irrational straight lines, either an apotome or a minor straight line*

For from the rational area  $BC$  let the medial area  $BD$  be subtracted,  
I say that the "side" of the remainder  $EC$  becomes one of two irrational straight lines either an apotome or a minor straight line



For let a rational straight line  $FG$  be set out,  
to  $FG$  let there be applied the rectangular parallelogram  $GH$  equal to  $BC$ ,  
and let  $GK$  equal to  $DB$  be subtracted,  
therefore the remainder  $EC$  is equal to  $LH$   
Since, then,  $BC$  is rational, and  $BD$  medial,



while  $BC$  is equal to  $GH$ , and  $BD$  to  $GK$ ,  
therefore  $GH$  is rational, and  $GK$  medial.

And they are applied to the rational straight line  $FG$ ;

Now the square on  $HF$  is greater than the square on  $FK$  by the square on a straight line either commensurable with  $HF$  or not commensurable

First, let the square on it be greater by the square on a straight line commensurable with it

Now the whole  $HF$  is commensurable in length with the rational straight line  $FG$  set out,

therefore  $KH$  is a first apotome [x Def III 1]

But the "side" of the rectangle contained by a rational straight line and a first apotome is an apotome [x 91]

$FG$  set out,

$KH$  is a fourth apotome. [x Def III 4]

But the "side" of the rectangle contained by a rational straight line and a fourth apotome is minor [x. 94]

Q E D

### PROPOSITION 109

If from a medial area a rational area be subtracted there arise two other irrational straight lines either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole

For from the medial area  $BC$  let the rational area  $BD$  be subtracted

I say that the "side" of the remainder  $EC$  becomes one of two irrational straight lines either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole

For let a rational straight line  $FG$  be set out,

and let the areas be similarly applied

It follows then that  $FH$  is rational and incommensurable in length with  $FG$ , while  $KF$  is rational and commensurable in length with  $FG$ , therefore  $FH$   $FK$  are rational straight lines commensurable in square only, [x 13]



therefore  $KH$  is an apotome and  $FK$  the annex to it [x 73]

Now the square on  $HF$  is greater than the square on  $FK$  either by the square on a straight line commensurable with  $HF$  or by the square on a straight line incommensurable with it

If then the square on  $HF$  is greater than the square on  $FK$  by the square on a straight line commensurable with  $HF$ , while the annex  $FK$  is commensurable in length with the rational straight line  $FG$  set out,

$KH$  is a second apotome [x Deff III 2]

But  $FG$  is rational, so that the "side" of  $LH$ , that is, of  $EC$ , is a first apotome of a medial straight line [x 92]

But, if the square on  $HF$  is greater than the square on  $FK$  by the square on a straight line incommensurable with  $HF$ , while the annex  $FK$  is commensurable in length with the rational straight line  $FG$  set out,

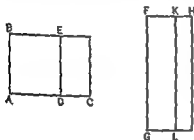
$KH$  is a fifth apotome, so that the "side" of  $EC$  is a straight line which produces with a rational area a medial whole [x Deff III 5] [x 95]

Q E D

### PROPOSITION 110

If from a medial area there be subtracted a medial area incommensurable with the whole, the two remaining irrational straight lines arise, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole

For, as in the foregoing figures, let there be subtracted from the medial area  $BC$  the medial area  $BD$  incommensurable with the whole,



I say that the "side" of  $EC$  is one of two irrational straight lines, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole

For, since each of the rectangles  $BC$ ,  $BD$  is medial, and  $BC$  is incommensurable with  $BD$ , it follows that each of the straight lines  $FH$ ,  $FK$  will be rational and incommensurable in length with  $FG$  [x 22]

And, since  $BC$  is incommensurable with  $BD$ ,

that is,  $GH$  with  $GK$ ,

$HF$  is also incommensurable with  $FK$ , [vi 1, x 11]

therefore  $FH$ ,  $FK$  are rational straight lines commensurable in square only,

therefore  $AH$  is an apotome [x 73]

If then the square on  $FH$  is greater than the square on  $FK$  by the square on a straight line commensurable with  $FH$ , while neither of the straight lines  $FH$ ,  $FK$  is commensurable in length with the rational straight line  $FG$  set out,

$KH$  is a third apotome [x Deff III 3]

But  $AL$  is rational and the rectangle contained by a rational straight line and a third apotome is irrational,

and the "side" of it is irrational, and is called a second apotome of a medial straight line, [x 93]  
 so that the "side" of  $LH$ , that is, of  $EC$ , is a second apotome of a medial straight line

... are on  $FK$  by the square on

ensurable in length with  $FG$ ,

[x Deff in 6]

But the "side" of the rectangle contained by a rational straight line and a sixth apotome is a straight line which produces with a medial area a medial whole [x 96]

Therefore the "side" of  $LH$  that is, of  $EC$ , is a straight line which produces with a medial area a medial whole Q E D

PROPOSITION 111

The apotome is not the same with the binomial straight line

Let  $AB$  be an apotome,

I say that  $AB$  is not the same with the binomial straight line

For if possible, let it be so,

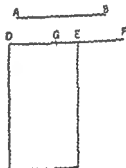
let a rational straight line  $DC$  be set out, and to  $CD$  let there be applied the rectangle  $CE$  equal to the square on  $AB$  and producing  $DE$  as breadth

Then, since  $AB$  is an apotome,

$DE$  is a first apotome [x 97]

Let  $EF$  be the annex to it,

therefore  $DF$ ,  $FE$  are rational straight lines commensurable in square only,



$DC$  set out  
 [x Deff III 1]

Again, since  $AB$  is binomial,

therefore  $DE$  is a first binomial straight line [x 60]

Let it be divided into its terms at  $G$

and let  $DG$  be the greater term,

therefore  $DG$   $GE$  are rational straight lines commensurable in square only, the square on  $DG$  is greater than the square on  $GE$  by the square on a straight line commensurable with  $DG$ , and the greater term  $DG$  is commensurable in [x Deff II 1]

with  $DG$ , [x 12]  
 [x 15]

But  $DF$

Therefore  $GF$   $FF$  are rational straight lines commensurable in square only, [x 13]  
 therefore  $EG$  is an apotome [x 73]

But it is also rational

which is impossible

Therefore the apotome is not the same with the binomial straight line

Q E D

*The apotome and the irrational straight lines following it are neither the same with the medial straight line nor with one another*

For the square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied, [x 22]

while the square on an apotome, if applied to a rational straight line, produces as breadth a first apotome, [x 97]

the square on a first apotome of a medial straight line, if applied to a rational straight line, produces as breadth a second apotome, [x 98]

the square on a second apotome of a medial straight line, if applied to a rational straight line, produces as breadth a third apotome, [x 99]

the square on a minor straight line, if applied to a rational straight line, produces as breadth a fourth apotome, [x 100]

the square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome, [x 101]

and the square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome [x 102]

Since then the said breadths differ from the first and from one another, from the first because it is rational, and from one another since they are not the same in order,

it is clear that the irrational straight lines themselves also differ from one another

And, since the apotome has been proved not to be the same as the binomial straight line, [x 111]

but, if applied to a rational straight line, the straight lines following the apotome produce, as breadths, each according to its own order, apotomes, and those following the binomial straight line themselves also, according to their order, produce the binomials as breadths,

following the thirteen irra-

Medial,

Binomial,

First bimedial,

Second bimedial,

Major,

"Side" of a rational plus a medial area,

"Side" of the sum of two medial areas,

Apotome,

First apotome of a medial straight line,

Second apotome of a medial straight line,

Minor,

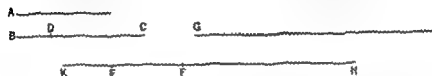
Producing with a rational area a medial whole,

Producing with a medial area a medial whole

PROPOSITION 11<sup>o</sup>

The square on a rational  
as breadth an  
binomial and  
have the same

Let  $A$  be a <sup>square</sup> ~~square~~ line  
let  $BC$  be a binomial and let  $DC$  be its greater term  
let the rectangle  $BC$   $EF$  be equal to the square on  $A$ ,



I say that  $EF$  is an apotome the terms of which are commensurable with  $CD$   
will have the same order as  $BC$   
ual to the square on  $A$   
ual to the rectangle  $BD$   $G$

therefore as  $CB$  is to  $BD$  so is  $G$  to  $EF$  [vi 16]

But  $CB$  is greater than  $BD$

therefore  $G$  is also greater than  $EF$  [v 10 v 14]

Let  $EH$  be equal to  $G$

therefore as  $CB$  is to  $BD$  so is  $HE$  to  $EF$

therefore *separando* as  $CD$  is to  $BD$  so is  $HF$  to  $FE$  [v 17]

Let it be contrived that as  $HF$  is to  $FE$  so is  $FA$  to  $AE$

therefore also the whole  $HA$  is to the whole  $AF$  as  $FA$  is to  $AE$

for as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents [v 12]

But as  $FA$  is to  $AE$  so is  $CD$  to  $DB$  [v 11]

therefore also as  $HA$  is to  $AF$  so is  $CD$  to  $DB$  [id]

But the square on  $CD$  is commensurable with the square on  $DB$  [x 36]

therefore the square on  $HA$  is also commensurable with the square on  $AF$  [vi 22 v 11]

to  $AE$  since the  
[v Def 9]

so that  $HA$  is also commensurable in length with  $EA$  [x 13]

Now since the square on  $HA$  is equal to the rectangle  $EH$   $BD$

hale the square on  $A$  is rational

therefore the rectangle  $EH$   $BD$  is also rational

And

Since then as  $CD$  is to  $DB$  so is  $FA$  to  $AE$

while  $CD$   $DB$  are straight lines commensurable in square only

therefore  $FA$   $AE$  are also commensurable in square only [x 11]

But  $KE$  is rational;

therefore  $FK$  is also rational

incommensurable with it

[x 14]

And, if  $CD$  is commensurable in length with the rational straight line set out,

so also is  $FK$ ,

[x 11 12]

if  $BD$  is so commensurable,

so also is  $KE$ ,

[x 12]

the square on  $FK$  is also greater than the square on  $KE$  by the square on a straight line incommensurable with  $FK$

[x 14]

And, if  $CD$  is commensurable with the rational straight line set out,

so also is  $FK$ ,

if  $BD$  is so commensurable,

so also is  $KE$ ,

so that the terms  $CD$ ,  $DB$  of the binomial straight line and in the same ratio and it has the same order as  $BC$

Q.E.D.

### PROPOSITION 113

The square on a rational straight line, if applied to an apotome produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio, and further, the binomial so arising has the same order as the apotome

Let  $A$  be a rational straight line and  $BD$  an apotome, and let the rectangle  $BD \cdot KH$  be equal to the square on  $A$ , so that the square on the rational straight line  $A$  when applied to the apotome  $BD$  produces  $KH$  as breadth,

I say that  $AH$  is a binomial straight line the terms of which are commensurable with the terms of  $BD$  and in the same ratio, and further,  $KH$  has the same order as  $BD$

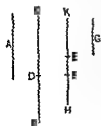
For let  $DC$  be the annex to  $BD$ ,

therefore  $BC$ ,  $CD$  are rational straight lines commensurable in square only

[x 73]

Let the rectangle  $BC$ ,  $G$  be also equal to the square on  $A$

But the square on  $A$  is rational,



therefore the rectangle  $BC$ ,  $G$  is also rational

with  $BC$  [v 20]

$KH$ ,

to  $G$  [vi 16]

therefore  $KH$  is also greater than  $v$

[v 16, v 14]

with  $BC$

therefore, *convertendo*, as  $BC$  is to  $CD$ , so is  $AH$  to  $HE$  [v 18, Por]

Let it be contrived that, as  $KH$  is to  $HE$ , so is  $HF$  to  $FE$ ,  
therefore also the remainder  $KF$  is to  $FH$  as  $KH$  is to  $HE$ , that is, as  $BC$  is to  
 $CD$  [v 19]

[x. 11]

so that also as the first is to the third, so is the square on the first to the  
square on the second, [v Def 9]

therefore also, as  $KF$  is to  $FE$ , so is the square on  $KF$  to the square on  $FE$

But the square on  $KF$  is commensurable with the square on  $FH$ ,

for  $KF$ ,  $FH$  are commensurable in square,

therefore  $KF$  is also commensurable in length with  $FE$ , [x 11]

so that  $KF$  is also commensurable in length with  $KE$  [x 13]

But  $KE$  is rational and commensurable in length with  $BC$ ,

therefore  $KF$  is also rational and commensurable in length with  $BC$  [x 12]

And, since, as  $BC$  is to  $CD$ , so is  $KF$  to  $FH$ ,

alternately, as  $BC$  is to  $KF$ , so is  $DC$  to  $FH$  [v 16]

But  $BC$  is commensurable with  $KF$ ,

therefore  $FH$  is also commensurable in length with  $CD$  [x 11]

But  $BC$ ,  $CD$  are rational straight lines commensurable in square only,

therefore  $KF$ ,  $FH$  are also rational straight lines [x Def 3] commensurable in  
square only,

therefore  $AH$  is binomial

[x 36]

If now the square on  $BC$  is greater than the square on  $CD$  by the square on a

are on a

[x 14]

set out,

so also is  $AE$ ,

if  $CD$  is commensurable in length with the rational straight line set out,

so also is  $FH$ ,

on 5

on 5

straight line incommensurable with  $KF$

[x 14]

And, if  $BC$  is commensurable with the rational straight line set out,  
so also is  $KF$ ,

if  $CD$  is so commensurable,

so also is  $FH$ ,

but if neither of the straight lines  $BC$ ,  $CD$ ,  
then neither of the straight lines  $KF$ ,  $FH$ .

Therefore  $KH$  is a binomial straight line, the terms of which  $KF$ ,  $FH$  are  
commensurable with the terms  $BC$ ,  $CD$  of the apotome and in the same ratio,  
and further,  $KH$  has the same order as  $BD$  Q E D

## PROPOSITION 114

If an area be contained by an apotome and the binomial straight line the terms of  
which are commensurable with the terms of the apotome and in the same ratio, the  
"side" of the area is rational

For let an area, the rectangle  $AB$ ,  $CD$ , be contained by the apotome  $AB$  and  
the binomial straight line  $CD$ ,

and let  $CE$  be the greater term of the latter,  
let the terms  $CE$ ,  $ED$  of the binomial straight  
line be commensurable with the terms  $AF$ ,  $FB$  of  
the apotome and in the same ratio,  
and let the "side" of the rectangle  $AB$ ,  $CD$  be  $G$ ,

I say that  $G$  is rational

For let a rational straight line  $H$  be set out,  
and to  $CD$  let there be applied a rectangle equal  
to the square on  $H$  and producing  $KL$  as breadth,

Therefore  $KL$  is an apotome

Let its terms be  $KM$ ,  $ML$  commensurable with the terms  $CE$ ,  $ED$  of the bi-  
nomial straight line and in the same ratio [x 112]

But  $CE$ ,  $ED$  are also commensurable with  $AF$ ,  $FB$  and in the same ratio,  
therefore, as  $AF$  is to  $FB$ , so is  $KM$  to  $ML$

Therefore, alternately, as  $AF$  is to  $KM$ , so is  $BF$  to  $LM$ ,  
therefore also the remainder  $AB$  is to the remainder  $AL$  as  $AF$  is to  $KM$  [v 10]

But  $AF$  is commensurable with  $KM$ , [x 12]

therefore  $AB$  is also commensurable with  $AL$  [x 11]

And as  $AB$  is to  $KL$ , so is the rectangle  $CD$ ,  $AB$  to the rectangle  $CD$ ,  $KL$ ,  
[v 11]

therefore the rectangle  $CD$ ,  $AB$  is also commensurable with the rectangle  $CD$ ,  
 $KL$  [x 11]

But the square on  $H$  is rational

therefore the square on  $G$  is also rational,

therefore  $G$  is rational

And it is the "side" of the rectangle  $CD$ ,  $AB$

Therefore etc



Porism. And it is made manifest to us by this also that it is possible for a rational area to be contained by irrational straight lines Q E D

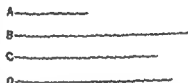
## PROPOSITION 115

*From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding*

Let  $A$  be a medial straight line,

I say that from  $A$  there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding

Let a rational straight line  $B$  be set out, and let the square on  $C$  be equal to the rectangle  $B, A$ ,



therefore  $C$  is irrational, [x Def 4]

for that which is contained by an irrational and a rational straight line is irrational [deduction from x 20]

And it is not the same with any of the preceding, for the square on none of the preceding if applied to a rational straight line produces as breadth a medial straight line

Again, let the square on  $D$  be equal to the rectangle  $B, C$ , therefore the square on  $D$  is irrational [deduction from x 20]

Therefore  $D$  is irrational, [x Def 4]  
and it is not the same with any of the preceding

Similar

the rest

and non

preceding

Q E D

## BOOK ELEVEN

### DEFINITIONS

- 1 A *solid* is that which has length, breadth, and depth
- 2 An extremity of a solid is a surface
- 3 A *straight line* is at right angles to a plane, when it makes right angles with all the straight lines which meet it and are in the plane
- 4 A plane is at right angles to a plane when the straight lines drawn in one of the planes, at right angles to the common section of the planes are at right angles to the remaining plane
- 5 The *inclination of a straight line to a plane* is, assuming a perpendicular drawn from the extremity of the straight line which is elevated above the plane to the plane, and a straight line joined from the point thus arising to the extremity of the straight line which is in the plane, the angle contained by the straight line so drawn and the straight line standing up
- 6 The *inclination of a plane to a plane* is the acute angle contained by the straight lines drawn at right angles to the common section at the same point, one in each of the planes
- 7 A plane is said to be *similarly inclined* to a plane as another is to another when the said angles of the inclinations are equal to one another
- 8 *Parallel planes* are those which do not meet
- 9 *Similar solid figures* are those contained by similar planes equal in multitude
- 10 *Equal and similar solid figures* are those contained by similar planes equal in multitude and in magnitude
- 11 A *solid angle* is the inclination constituted by more than two lines which meet one another and are not in the same surface towards all the lines  
Otherwise A *solid angle* is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point
- 12 A *pyramid* is a solid figure, contained by planes, which is constructed from one plane to one point
- 13 A *prism* is a solid figure contained by planes two of which namely those which are opposite are equal, similar and parallel, while the rest are parallelograms
- 14 When the diameter of a semicircle remaining fixed the semicircle is carried round and restored again to the same position from which it began to be moved the figure so comprehended is a *sphere*
- 15 The *axis of the sphere* is the straight line which remains fixed and about which the semicircle is turned
- 16 The *centre of the sphere* is the same as that of the semicircle
- 17 A *diameter of the sphere* is any straight line drawn through the centre and

terminated in both directions by the surface of the sphere

18 When one side of those about the right angle in a right angled triangle remaining fixed the triangle is carried round and restored again to the same

less *obtuse-angled* and if greater *acute-angled*

19 The *axis of the cone* is the straight line which remains fixed and about which the triangle is turned

20 And the *base* is the circle described by the straight line which is carried round

21 When one side of those about the right angle in a rectangular parallelogram remaining fixed the parallelogram is carried round and restored again to the same position from which it began to be moved the figure so comprehended is a *cylinder*

22 The *axis of the cylinder* is the straight line which remains fixed and about which the parallelogram is turned

23 And the *bases* are the circles described by the two sides opposite to one another which are carried round

24 *Similar cones and cylinders* are those in which the axes and the diameters of the bases are proportional

25 A *cube* is a solid figure contained by six equal squares

26 An *octahedron* is a solid figure contained by eight equal and equilateral triangles

27 An *icosahedron* is a solid figure contained by twenty equal and equilateral triangles

28 A *dodecahedron* is a solid figure contained by twelve equal equilateral and equiangular pentagons

## BOOK VI PROPOSITIONS

### PROPOSITION I

A part of a straight line cannot be in the plane of reference and a part in a plane more elevated

I or if possible let a part  $AB$  of the straight line  $ABC$  be in the plane of reference and a part  $BC$  in a plane more elevated

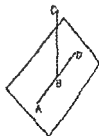
There will then be in the plane of reference some straight line continuous with  $AB$  in a straight line

Let it be  $BD$

therefore  $AB$  is a common segment of the two straight lines  $ABC$  and  $BD$

which is impossible inasmuch as if we describe a circle with centre  $B$  and distance  $AB$  the diameters will cut off unequal circumferences of the circle

Therefore a part of a straight line cannot be in the plane of reference and a part in a plane more elevated



## PROPOSITION 2

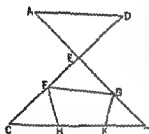
*If two straight lines cut one another, they are in one plane, and every triangle is in one plane*

For let the two straight lines  $AB$ ,  $CD$  cut one another at the point  $E$ ,  
I say that  $AB$ ,  $CD$  are in one plane and every triangle is in one plane

For let points  $F$ ,  $G$  be taken at random on  $EC$ ,  $EB$ ,

let  $CB$ ,  $FG$  be joined,

and let  $FH$ ,  $GK$  be drawn across,



a part also of one of the straight lines  $EC$ ,  $EB$  will be in the plane of reference, and a part in another

But, if the part  $FCBG$  of the triangle  $ECB$  be in the plane of reference, and the rest in another,

a part also of both the straight lines  $EC$ ,  $EB$  will be in the plane of reference and a part in another

which was proved absurd [xi 1]

Therefore the triangle  $ECB$  is in one plane

and, in whatever  
 $AB$ ,  $CD$  also

[xi 1]

Therefore the straight lines  $AB$ ,  $CD$  are in one plane,  
and every triangle is in one plane

Q E D

## PROPOSITION 3

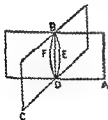
*If two planes cut one another their common section is a straight line*

For let the two planes  $AB$ ,  $BC$  cut one another,

and let the line  $DB$  be their common section,

I say

be



which is absurd

Therefore  $DEB$ ,  $DFB$  are not straight lines

Similarly we can prove that neither will there be any other straight line joined from  $D$  to  $B$  except  $DB$  the common section of the planes  $AB$ ,  $BC$

Therefore etc

Q E D

## PROPOSITION 4

*If a straight line be set up at right angles to two straight lines which cut one another, at their common point of section, it will also be at right angles to the plane through them*

For let a straight line  $EF$  be set up at right angles to the two straight lines  $AB, CD$  which cut one another at the point  $E$  from  $E$ ,

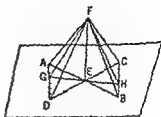
I say that  $EF$  is also at right angles to the plane through  $AB, CD$

For let  $AE, EB, CE, ED$  be cut off equal to one another

and let any straight line  $GEH$  be drawn across through  $E$  at random

let  $AD, CB$  be joined,

and further let  $FA, FG, FD, FC, FH, FB$  be joined from the point  $F$  taken at random <on  $EF$ >



Now since the two straight lines  $AE, ED$  are equal to the two straight lines  $CE, EB$ , and contain equal angles [1 16]

therefore the base  $AD$  is equal to the base  $CB$ ,

and the triangle  $AED$  will be equal to the triangle  $CEB$ , [1 4]

so that the angle  $DAE$  is also equal to the angle  $EBC$

But the angle  $AGE$  is also equal to the angle  $BEH$  [1 15]

therefore  $AGE, BEH$  are two triangles which have two angles equal to two angles respectively and one side equal to one side namely that adjacent to the equal angles that is to say  $AE$  to  $EB$

therefore they will also have the remaining sides equal to the remaining sides [1 17]

Therefore  $GE$  is equal to  $EH$  and  $AG$  to  $BH$

And since  $AE$  is equal to  $EB$

while  $FE$  is common and at right angles

therefore the base  $FA$  is equal to the base  $FB$  [1 4]

For the same reason

$FC$  is also equal to  $FD$

And since  $AD$  is equal to  $CB$

and  $FA$  is also equal to  $FB$

the two sides  $FA, AD$  are equal to the two sides  $FB, BC$  respectively [1 8]

therefore

And since  $a$

and further  $FA$  also equal to  $FB$

the two sides  $FA, AG$  are equal to the two sides  $FB, BH$

And the angle  $FAG$  was proved equal to the angle  $FBH$

therefore the base  $FG$  is equal to the base  $FH$  [1 4]

Now since again  $GE$  was proved equal to  $EH$

and  $EF$  is common

the two sides  $GE, EF$  are equal to the two sides  $HE, EF$ ,

and the base  $FG$  is equal to the base  $FH$

therefore the angle  $GFE$  is equal to the angle  $HEF$  [1 8]

Therefore each of the angles  $CEP, HFF$  is right

Therefore  $FE$  is at right angles to  $GH$  drawn at random through  $E$

Similarly we can prove that  $FE$  will also make right angles with all the straight lines which meet it and are in the plane of reference

But a straight line is at right angles to a plane when it makes right angles

with all the straight lines which meet it and are in that same plane, [xi Def 3]

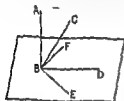
Therefore etc

Q E D

### PROPOSITION 5

*If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section, the three straight lines are in one plane*

For let a straight line  $AB$  be set up at right angles to the three straight lines  $BC, BD, BE$ , at their point of meeting at  $B$ ,



elevated

let the plane through  $AB, BC$  be produced, it will thus make, as common section in the plane of reference, a straight line [xi 3]

Let it make  $BF$

Therefore the three straight lines  $AB, BC, BF$  are in one plane, namely that drawn through  $AB, BC$

Not theref But I 4]

Thus  $AB$  will also make right angles with all the straight lines which meet it and are in the plane of reference [xi Def 3]

But  $BF$  which is in the plane of reference meets it, therefore the angle  $ABF$  is right

But, by hypothesis the angle  $ABC$  is also right, therefore the angle  $ABF$  is equal to the angle  $ABC$

And they are in one plane

which is impossible

Therefore the straight line  $BC$  is not in a more elevated plane,

therefore the three straight lines  $BC, BD, BE$  are in one plane

Therefore, if a straight line be set up at right angles to three straight lines, at their point of meeting, the three straight lines are in one plane Q E D

### PROPOSITION 6

*If two straight lines be at right angles to the same plane the straight lines will be parallel*

For let the two straight lines  $AB, CD$  be at right angles to the plane of reference,

I say that  $AB$  is parallel to  $CD$

For let them meet the plane of reference at the points  $B, D$ ,

let the straight line  $BD$  be joined

let  $DE$  be drawn, in the plane of reference, at right angles to  $BD$ ,

let  $DE$  be made equal to  $AB$ ,

and let  $BE$ ,  $AE$ ,  $AD$  be joined

Now, since  $AB$  is at right angles to the plane of reference it will also make right angles with all the straight lines which meet it and are in the plane of reference [XI Def 3]

But each of the straight lines  $BD$ ,  $BE$  is in the plane of reference and meets  $AB$ ,

therefore each of the angles  $ABD$ ,  $ABE$  is right

For the same reason

each of the angles  $CDB$ ,  $CDE$  is also right

And, since  $AB$  is equal to  $DE$ ,

and  $BD$  is common

the two sides  $AB$ ,  $BD$  are equal to the two sides  $ED$ ,  $DB$ ,

and they include right angles,

therefore the base  $AD$  is equal to the base  $BE$  [I 4]

And, since  $AB$  is equal to  $DE$ ,

while  $AD$  is also equal to  $BE$ ,

the two sides  $AB$ ,  $BE$  are equal to the two sides  $ED$ ,  $DA$ ,

and  $AE$  is their common base,

therefore the angle  $ABE$  is equal to the angle  $EDA$  [I 8]

But the angle  $ABE$  is right,

therefore the angle  $EDA$  is also right,

therefore  $ED$  is at right angles to  $DA$

But it is also at right angles to each of the straight lines  $BD$ ,  $DC$ , therefore  $ED$  is set up at right angles to the three straight lines  $BD$ ,  $DA$ ,  $DC$  at their point of meeting,

therefore the three straight lines  $BD$ ,  $DA$ ,  $DC$  are in one plane [XI 5]

But, in whatever plane  $DB$ ,  $DA$  are in that plane is  $AB$  also

for every triangle is in one plane, [XI 2]

therefore  $AB$  is also in that plane, and  $AB$  is in one plane

And  $ED$

[I 28]

Therefore etc

Q E D

### PROPOSITION 7

If two straight lines be parallel and points be taken at random on each of them the straight line joining the points is in the same plane with the parallel straight lines

Let  $AB$ ,  $CD$  be two parallel straight lines

and let points  $E$ ,  $F$  be taken at random on them

respectively,

then that the straight line  $EF$  is in the same plane with  $AB$ ,  $CD$

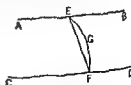
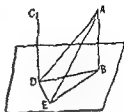
a more elevated plane as  $EGF$

and let a plane be drawn through  $EGF$ ,

it will then make, as section in the plane of reference, a straight line [XI 3]

Let it make it as  $EF$ ,

therefore the two straight lines  $EGF$ ,  $EF$  will enclose an area which is impossible



Therefore the straight line joined from  $E$  to  $F$  is not in a plane more elevated, therefore the straight line joined from  $E$  to  $F$  is in the plane through the parallel straight lines  $AB, CD$

Therefore etc

Q E D

### PROPOSITION 8

*If two straight lines be parallel, and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane*

Let  $AB, CD$  be two parallel straight lines,

and let one of them,  $AB$ , be at right angles to the plane of reference,

I say that the remaining one,  $CD$ , will also be at right angles to the same plane

For let  $AB, CD$  meet the plane of reference at the points  $B, D$ ,

and let  $BD$  be joined,

therefore  $AB, CD, BD$  are in one plane [XI 7]

Let  $DE$  be drawn, in the plane of reference, at right angles to  $BD$ ,

let  $DE$  be made equal to  $AB$ ,

and let  $BE, AE, AD$  be joined

Now, since  $AB$  is at right angles to the plane of reference, therefore  $AB$  is also at right angles to all the straight lines which meet it and are in the plane of reference, [XI Def 3]

therefore each of the angles  $ABD, ABE$  is right

And since the straight line  $BD$  has fallen on the parallels  $AB, CD$ ,

therefore the angles  $ABD, CDB$  are equal to two right angles [I 29]

But the angle  $ABD$  is right,

therefore the angle  $CDB$  is also right,

therefore  $CD$  is at right angles to  $BD$

And since  $AB$  is equal to  $DE$ ,

and  $BD$  is common,

the two sides  $AB, BD$  are equal to the two sides  $ED, DB$ ,

and the angle  $ABD$  is equal to the angle  $EDB$ ,

for each is right,

therefore the base  $AD$  is equal to the base  $BE$

And, since  $AB$  is equal to  $DE$ ,

and  $BE$  to  $AD$ ,

the two sides  $AB, BE$  are equal to the two sides  $ED, DA$  respectively,

and  $AE$  is their common base,

therefore the angle  $ABE$  is equal to the angle  $EDA$

But the angle  $ABE$  is right,

therefore the angle  $EDA$  is also right,

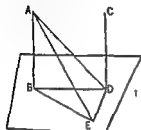
therefore  $ED$  is at right angles to  $AD$

But it is also at right angles to  $DB$ ,

therefore  $ED$  is also at right angles to the plane through  $BD, DA$  [XI 4]

Therefore  $CD$  will also make right angles with all the straight lines which meet it and are in the plane through  $BD, DA$

But  $DC$  is in the plane through  $BD, DA$ , inasmuch as  $AB, BD$  are in the





plane through  $BD, DA$ ,

[xi. 2]

and  $DC$  is also in the plane in which  $AB, BD$  are

Therefore  $ED$  is at right angles to  $DC$ ,

so that  $CD$  is also at right angles to  $DE$

But  $CD$  is also at right angles to  $BD$ .

Therefore  $CD$  is set up at right angles to the two straight lines  $DE, DB$  which cut one another, from the point of section at  $D$ ;

[xi. 4]

Therefore etc

Q E D

### PROPOSITION 9

*Straight lines which are parallel to the same straight line and are not in the same plane with it are also parallel to one another*

For let each of the straight lines  $AB, CD$  be parallel to  $EF$ , not being in the same plane with it,

I say that  $AB$  is parallel to  $CD$

For let a point  $G$  be taken at random on  $EF$ ,

and from it let there be drawn  $GH$ , in the plane through  $EF, AB$ , at right angles to  $EF$ , and  $GK$  in the plane through  $FE, CD$  again at right angles to  $EF$

Now, since  $EF$  is at right angles to each of the straight lines  $GH, GK$ ,

therefore  $EF$  is also at right angles to the plane through  $GH, GK$  [xi. 4]

And  $EF$  is parallel to  $AB$ ;

therefore  $AB$  is also at right angles to the plane through  $HG, GK$  [xi. 6]

For the same reason

$CD$  is also at right angles to the plane through  $HG, GK$ ;

therefore each of the straight lines  $AB, CD$  is at right angles to the plane through  $HG, GK$

But, if two straight lines be at right angles to the same plane, the straight lines are parallel, [xi. 6]

therefore  $AB$  is parallel to  $CD$

Q E D

### PROPOSITION 10

*If two straight lines meeting one another be parallel to two straight lines meeting one another not in the same plane, they will contain equal angles*

For let the two straight lines  $AB, BC$  meeting one another be parallel to the two straight lines  $DE, EF$  meeting one another, not in the same plane,

I say that the angle  $ABC$  is equal to the angle  $DEF$ .

For let  $BA, BC, ED, EF$  be cut off equal to one another, and let  $AD, CF, BE, AC, DF$  be joined

Now, since  $BA$  is equal and parallel to  $ED$ ,

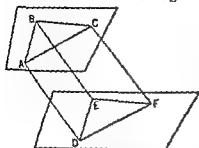
therefore  $AD$  is also equal and parallel to  $BE$

[i. 33]

For the same reason

$CF$  is also equal and parallel to  $BE$ .

Therefore each of the straight lines  $AD$ ,  $CF$  is equal and parallel to  $BE$



But straight lines which are parallel to the same straight line and are not in the same plane with it are parallel to one another, [xi 9]

therefore  $AD$  is parallel and equal to  $CF$

And  $AC$ ,  $DF$  join them, therefore  $AC$  is also equal and parallel to  $DF$  [i 33]

Now, since the two sides  $AB$ ,  $BC$  are equal to the two sides  $DE$ ,  $EF$ ,

and the base  $AC$  is equal to the base  $DF$ , therefore the angle  $ABC$  is equal to the angle  $DEF$  [i 8]

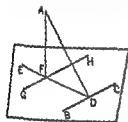
Therefore etc

Q E D

### PROPOSITION 11

From a given elevated point to draw a straight line perpendicular to a given plane

Let  $A$  be the given elevated point and the plane of reference the given plane, thus it is required to draw from the point  $A$  a straight line perpendicular to the plane of reference



Let any straight line  $BC$  be drawn at random in the plane of reference

and let  $AD$  be drawn from the point  $A$  perpendicular to  $BC$  [i 12]

If then  $AD$  is also perpendicular to the plane of reference that which was enjoined will have been done

But if not let  $DE$  be drawn from the point  $D$  at right angles to  $BC$  and in the plane of reference [i 11]

let  $AF$  be drawn from  $A$  perpendicular to  $DE$  [i 12]

and let  $GH$  be drawn through the point  $F$  parallel to  $BC$  [i 31]

the  $AD$  is perpendicular to  $BC$  and  $DE$  [xi 4]

but if two straight lines be parallel and one of them be at right angles to any plane the remaining one will also be at right angles to the same plane, [xi 8] therefore  $GH$  is also at right angles to the plane through  $ED$   $DA$

Therefore  $GH$  is also at right angles to all the straight lines which meet it [xi Def 3]

so that  $FA$  is also at right angles to  $GH$

the  $FA$  is perpendicular to  $GH$  and  $FA$  is perpendicular to  $DE$  and  $DA$  through them [xi 4]

therefore  $FA$  is at right angles to the plane through  $ED$ ,  $GH$

But the plane through  $ED$   $GH$  is the plane of reference,

therefore  $AF$  is at right angles to the plane of reference

Therefore from the given elevated point  $A$  the straight line  $AF$  has been drawn perpendicular to the plane of reference Q E F

### PROPOSITION 12

To set up a straight line at right angles to a given plane from a given point in it

Let the plane of reference be the given plane,

and  $A$  the point in it,

thus it is required to set up from the point  $A$  a straight line at right angles to the plane of reference

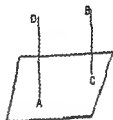
Let any elevated point  $B$  be conceived

from  $B$  let  $BC$  be drawn perpendicular to

the plane of reference, [xi 11]

and through the point  $A$  let  $AD$  be drawn

parallel to  $BC$  [i 31]



Then since  $AD$   $CB$  are two parallel straight lines while one of them  $BC$  is at right angles to the plane of reference therefore the remaining one  $AD$  is also at right angles to the plane of reference [xi 8]

Therefore  $AD$  has been set up at right angles to the given plane from the point  $A$  in it Q E F

### PROPOSITION 13

From the same point two straight lines cannot be set up at right angles to the same plane on the same side

For if possible from the same point  $A$  let the two straight lines  $AB$ ,  $AC$  be set up at right angles to the plane of reference and on the same side

and let a plane be drawn through  $BA$   $AC$

it will then make as section through  $A$  in the plane of reference a straight line [xi 3]

Let it make  $DAE$ ,

therefore the straight lines  $AB$   $AC$ ,

$DAE$  are in one plane

And since  $CA$  is at right angles to the plane of reference it will also make right angles with all the straight lines which meet it and are in the plane of reference [xi Def 3]

But  $DAE$  meets it and is in the plane of reference

therefore the angle  $CAE$  is right

For the same reason

the angle  $BAE$  is also right

therefore the angle  $CAE$  is equal to the angle  $BAE$

And they are in one plane

which is impossible

Therefore etc

Q E D

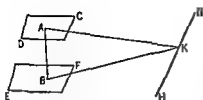
### PROPOSITION 14

Planes to which the same straight line is at right angles will be parallel

For let any straight line  $AB$  be at right angles to each of the planes  $CD$   $EF$ ,

I say that the planes are parallel

For, if not, they will meet when produced



Let them meet,  
they will then make, as common section,  
a straight line [XI 3]

Let them make  $GH$ ,  
let a point  $K$  be taken at random on  $GH$ ,  
and let  $AK$ ,  $BK$  be joined

Now, since  $AB$  is at right angles to  
the plane  $EF$ ,

therefore  $AB$  is also at right angles to  $BK$  which is a straight line in the plane  
 $EF$  produced, [XI Def 3]

therefore the angle  $ABK$  is right

For the same reason

the angle  $BAK$  is also right

Thus in the triangle  $ABK$ , the two angles  $ABK$ ,  $BAK$  are equal to two  
right angles

which is impossible [I 17]

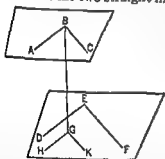
Therefore the planes  $CD$ ,  $EF$  will not meet when produced,  
therefore the planes  $CD$ ,  $EF$  are parallel [XI Def 8]

Therefore planes to which the same straight line is at right angles are par-  
allel Q E D

### PROPOSITION 15

If two straight lines meeting one another be parallel to two straight lines meeting  
one another not being in the same plane, the planes through them are parallel

For let the two straight lines  $AB$ ,  $BC$  meeting one another be parallel to the  
two straight lines  $DE$ ,  $EF$  meeting one another,  
not being in the same plane, I say that the  
planes produced through  $AB$ ,  $BC$  and  $DE$ ,  $EF$   
will not meet one another



For let  $BG$  be drawn from the point  $B$  per-  
pendicular to the plane through  $DE$ ,  $EF$  [XI 11],  
and let it meet the plane at the point  $G$ ,

through  $G$  let  $GH$  be drawn parallel to  $ED$ , and  
 $GK$  parallel to  $EF$  [I 31]

Now since  $BG$  is at right angles to the plane  
through  $DE$ ,  $EF$

therefore it will also make right angles with all the straight lines which meet it  
and are in the plane through  $DE$ ,  $EF$  [XI Def 3]

But each of the straight lines  $GH$ ,  $GK$  meets it and is in the plane through  
 $DE$ ,  $EF$ ,

therefore each of the angles  $BGH$ ,  $BGK$  is right

And, since  $BA$  is parallel to  $GH$ , [XI 9]

therefore the angles  $GBA$ ,  $BGH$  are equal to two right angles [I 29]

But the angle  $BGH$  is right,

therefore the angle  $GBA$  is also right,

therefore  $GB$  is at right angles to  $BA$

For the same reason

plane through  $BA, BC$ .

[XI 4]

But planes to which the same straight line is at right angles are parallel,

[XI 14]

therefore the plane through  $AB, BC$  is parallel to the plane through  $DE, EF$

Therefore, if two straight lines meeting one another be parallel to two straight lines meeting one another, not in the same plane, the planes through them are parallel

Q.E.D.

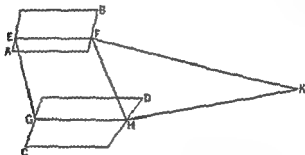
## PROPOSITION 16

If two parallel planes be cut by any plane, their common sections are parallel

For let the two parallel planes  $AB, CD$  be cut by the plane  $EFGH$ ,

and let  $EF, GH$  be their common sections,

I say that  $EF$  is parallel to  $GH$



For if not  $EF, GH$  will, when produced, meet either in the direction of  $F, H$  or of  $E, G$

Let them be produced, as in the direction of  $F, H$ , and let them, first, meet at  $K$

Now, since  $EFH$  is in the plane  $AB$ ,

therefore all the points on  $FFK$  are also in the plane  $AB$  [XI 1]

But  $A$  is one of the points on the straight line  $EFK$ ,

therefore  $K$  is in the plane  $AB$

For the same reason

$K$  is also in the plane  $CD$ ,

therefore the planes  $AB, CD$  will meet when produced

But they do not meet because they are by hypothesis, parallel, therefore the straight lines  $EF, GH$  will not meet when produced in the direction of  $F, H$

Similarly we can prove that neither will the straight lines  $EF, GH$  meet when produced in the direction of  $E, G$

But straight lines which do not meet in either direction are parallel [I Def 23]

Therefore  $EF$  is parallel to  $GH$

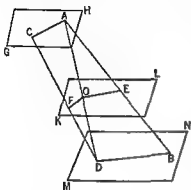
Therefore etc

Q.E.D.

## PROPOSITION 17

*If two straight lines be cut by parallel planes, they will be cut in the same ratios*

For let the two straight lines  $AB$ ,  $CD$  be cut by the parallel planes  $GH$ ,  $KL$ ,  $MN$  at the points  $A$ ,  $E$ ,  $B$  and  $C$ ,  $F$ ,  $D$ ,



I say that, as the straight line  $AE$  is to  $EB$  so is  $CF$  to  $FD$

For let  $AC$ ,  $BD$ ,  $AD$  be joined,  
let  $AD$  meet the plane  $KL$  at the point  $O$ ,  
and let  $EO$ ,  $OF$  be joined

Now, since the two parallel planes  $KL$ ,  $MN$  are cut by the plane  $EBDO$ ,  
their common sections   $BD$  are parallel

[XI 16]

For the same reason since the two parallel planes  $GH$   $KL$  are cut by the plane  $AOFC$ ,

their common sections  $AC$ ,  $OF$  are parallel

[id]

And since the straight line  $EO$  has been drawn parallel to  $BD$ , one of the sides of the triangle  $ABD$ ,

therefore proportionally, as  $AE$  is to  $EB$ , so is  $AO$  to  $OD$  [VI 2]

Again since the straight line  $OF$  has been drawn parallel to  $AC$ , one of the sides of the triangle  $ADC$

proportionally, as  $AO$  is to  $OD$ , so is  $CF$  to  $FD$  [id]

But it was also proved that as  $AO$  is to  $OD$ , so is  $AE$  to  $EB$ ,

therefore also as  $AE$  is to  $EB$ , so is  $CF$  to  $FD$

[V 11]

Therefore etc

Q E D

## PROPOSITION 18

*If a straight line be at right angles to any plane, all the planes through it will also be at right angles to the same plane*

For let any straight line  $AB$  be at right angles to the plane of reference,

I say that all the planes through  $AB$  are also at right angles to the plane of reference

For let the plane  $DE$  be drawn through  $AB$ ,  
let  $CE$  be the common section of the plane  $DE$   
and the plane of reference,

let a point  $F$  be taken at random on  $CE$ ,  
and from  $F$  let  $FG$  be drawn in the plane  $DE$   
at right angles to  $CE$  [I 11]

Now, since  $AB$  is at right angles to the plane of reference,  $AB$  is also at right angles to all the straight lines which meet it and are in the plane of reference,

[XI Def 3]

so that it is also at right angles to  $CE$ ,

therefore the angle  $ABF$  is right

But the angle  $GFB$  is also right,

therefore  $AB$  is parallel to  $FG$

[I 28]

But  $AB$  is at right angles to the plane of reference,

therefore  $FG$  is also at right angles to the plane of reference [xi 8]

Now a plane is at right angles to a plane when the straight lines drawn in one of the planes at right angles to the common section of the planes are at right angles to the remaining plane [xi Def 4]

And  $FG$  drawn in one of the planes  $DE$  at right angles to  $CE$ , the common section of the planes was proved to be at right angles to the plane of reference therefore the plane  $DE$  is at right angles to the plane of reference

Similarly also it can be proved that all the planes through  $AB$  are at right angles to the plane of reference

Therefore etc

Q E D

### PROPOSITION 19

If two planes which cut one another be at right angles to any plane their common section will also be at right angles to the same plane

For let the two planes  $AB$   $BC$  be at right angles to the plane of reference and let  $BD$  be their common section,

I say that  $BD$  is at right angles to the plane of reference

For suppose it is not and from the point  $D$  let  $DE$  be drawn in the plane  $AB$  at right angles to the straight line  $AD$  and  $DF$  in the plane  $BC$  at right angles to  $CD$

Now since the plane  $AB$  is at right angles to the plane of reference

and  $DE$  has been drawn in the plane  $AB$  at right angles to  $AD$  their common section,

therefore  $DE$  is at right angles to the plane of reference

Similarly we can prove that

$DF$  is also at right angles to the plane of reference

Therefore from the same point  $D$  two straight lines have been set up at right angles to the plane of reference on the same side

which is impossible

Therefore no straight line except the common section  $DB$  of the planes  $AB$   $BC$  can be set up from the point  $D$  at right angles to the plane of reference

Therefore etc

[xi Def 4]

[xi 13]

Q E D

### PROPOSITION 20

If a solid angle be contained by three plane angles any two taken together in any manner are greater than the remaining one

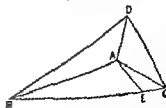
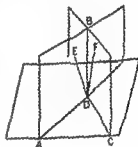
For let the solid angle at  $A$  be contained by the three plane angles  $BAC$   $CAD$   $DAB$

I say that any two of the angles  $BAC$   $CAD$   $DAB$  taken together in any manner are greater than the remaining one

If now the angles  $BAC$   $CAD$   $DAB$  are equal to one another it is manifest that any two are greater than the remaining one

But, if not let  $BAC$  be greater

and on the straight line  $AB$  and at the point  $A$  on it let the angle  $BAE$  be



constructed, in the plane through  $BA, AC$ , equal to the angle  $DAB$ ,

let  $AE$  be made equal to  $AD$

and let  $BEC$ , drawn across through the point  $E$ , cut the straight lines  $AB, AC$  at the points  $B, C$ ,

let  $DB, DC$  be joined

Now, since  $DA$  is equal to  $AE$ ,

and  $AB$  is common

two sides are equal to two sides,

and the angle  $DAB$  is equal to the angle  $BAE$ ,

therefore the base  $DB$  is equal to the base  $BE$

[I 4]

And since the two sides  $BD, DC$  are greater than  $BC$ ,

[I 20]

and of these  $DB$  was proved equal to  $BE$ ,

therefore the remainder  $DC$  is greater than the remainder  $EC$

Now, since  $DA$  is equal to  $AE$ ,

and  $AC$  is common,

and the base  $DC$  is greater than the base  $EC$ ,

therefore the angle  $DAC$  is greater than the angle  $EAC$

[I 25]

But the angle  $DAB$  was made equal to the angle  $BAE$ ,

therefore the angles  $DAB, DAC$  are greater than the angle  $BAC$

Similarly we can prove that the remaining angles also taken together two and two, are greater than the remaining one

Therefore etc

Q E D

### PROPOSITION 21

*Any solid angle is contained by plane angles less than four right angles*

Let the angle at  $A$  be a solid angle contained by the plane angles  $BAC, CAD, DAB$ ,

I say that the angles  $BAC, CAD, DAB$  are less than four right angles

For let points  $B, C, D$  be taken at random on the straight lines  $AB, AC, AD$  respectively,

and let  $BC, CD, DB$  be joined

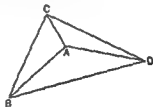
Now, since the solid angle at  $B$  is contained by the three plane angles  $CBA, ABD, CBD$

any two are greater than the remaining one,

[XI 20]

therefore the angles  $CBA, ABD$  are greater than the angle  $CBD$

For the same reason



two right angles

And since the three angles of each of the triangles  $ABC, ACD, ADB$  are equal to two right angles

therefore the nine angles of the three triangles the angles  $CBA, ACD, CAD, CDA, ADB, DBA, BAD$  are equal to six right an



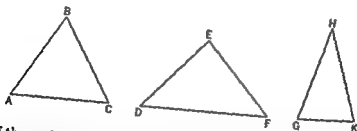
and of them the six angles  $ABC, BCA, ACD, CDA, ADB, DBA$  are greater than two right angles; therefore the remaining three angles  $BAC, CAD, DAB$  containing the solid angle are less than four right angles. Therefore etc.

Q E D

## PROPOSITION 22

... three plane angles  $ABC, DEF, GHK$ , of which two, taken together in any manner, are greater than the remaining one, namely the angles  $ABC, DEF$  greater than the angle  $GHK$ , the angles  $DEF, GHK$  greater than the angle  $ABC$ , and, further, the three straight lines  $AB, BC, CA$  are greater than the three straight lines  $DE, EF, FD$ , let

I say that it is possible to construct a triangle whose sides shall be equal to the three straight lines  $AB, BC, CA$ .  
 Let  $AC, DF, GK$  be equal to one another, and let the angles  $ABC, DEF, GHK$  be equal to one another, and let the straight lines  $AB, BC, CA$  be greater than the straight lines  $DE, EF, FD$ , that is



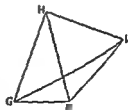
Now, if the angles  $ABC, DEF, GHK$  are equal to one another, it is manifest that,  $AC, DF, GK$  being equal also, it is possible to construct a triangle out of straight lines equal to  $AC, DF, GK$ .

But, if not, let them be unequal, and on the straight line  $HK$ , and at the point  $H$  on it, let the angle  $KHL$  be constructed equal to the angle  $ABC$ , let  $HL$  be made equal to one of the straight lines  $AB, BC, DE, EF, GH, HK$ , and let  $KL, GL$  be joined.

Now, since the two sides  $AB, BC$  are equal to the two sides  $KH, HL$ , and the angle at  $B$  is equal to the angle  $KHL$ ,

therefore the base  $AC$  is equal to the base  $KL$ . And, since the angles  $ABC, GHK$  are greater than the angle  $DEF$ , while the angle  $ABC$  is equal to the angle  $KHL$ , therefore the angle  $GHL$  is greater than the angle  $DEF$ .

And, since the two sides  $GH, HL$  are equal to the two sides  $DE, EF$ ,



{ 14 }

than the angle  $DEF$ ,  
than the base  $DF$ . [1 24]

therefore  $GA, AL$  are much greater than  $DF$

But  $KL$  is equal to  $AC$ ;

therefore  $AC, GK$  are greater than the remaining straight line  $DF$

Similarly we can prove that

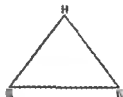
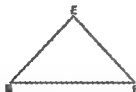
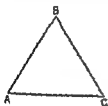
$AC, DF, GK$

Q E D

### PROPOSITION 23

To construct a solid angle out of three plane angles two of which, taken together in any manner, are greater than the remaining one thus the three angles must be less than four right angles

Let the angles  $ABC, DEF, GHK$  be the three given plane angles, and let two of these, taken together in any manner, be greater than the remaining one, while, further, the three are less than four right angles, thus it is required to construct a solid angle out of angles equal to the angles  $ABC, DEF, GHK$



Let  $AB, BC, DE, EF, GH, HA$  be cut off equal to one another,  
and let  $AC, DF, GK$  be joined,  
it is therefore possible to construct a triangle out of straight lines equal to  $AC, DF, GK$  [xi 22]

Let  $LMN$  be so constructed that  $AC$  is equal to  $LM, DF$  to  $MN$ , and further,  $GK$  to  $NL$ ,  
let the circle  $LMN$  be described about the triangle  $LMN$ ,

let its centre be taken and let it be  $O$ ,

let  $LO, MO, NO$  be joined,

I say that  $AB$  is greater than  $LO$

For if not  $AB$  is either equal to  $LO$ , or less

First, let it be equal

Then since  $AB$  is equal to  $LO$ ,

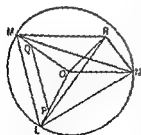
while  $AB$  is equal to  $BC$ , and  $OL$  to  $OM$ ,

the two sides  $AB, BC$  are equal to the two sides  $LO, OM$  respectively;

and by hypothesis, the base  $AC$  is equal to the base  $LM$ ,

therefore the angle  $ABC$  is equal to the angle  $LOM$  [1 8]

For the same reason



the angle  $DEF$  is also equal to the angle  $MON$ ,  
 and further the angle  $GKH$  to the angle  $NOL$ ,  
 therefore the three angles  $ABC, DEF, GKH$  are equal to the three angles  $LOM, MON, NOL$

But the three angles  $LOM, MON, NOL$  are equal to four right angles,  
 therefore the angles  $ABC, DEF, GKH$  are equal to four right angles

But they are also, by hypothesis, less than four right angles  
 which is absurd

Therefore  $AB$  is not equal to  $LO$

I say next that neither is  $AB$  less than  $LO$

For, if possible, let it be so,

and let  $OP$  be made equal to  $AB$ , and  $OQ$  equal to  $BC$ ,  
 and let  $PQ$  be joined

Then, since  $AB$  is equal to  $BC$ ,

$OP$  is also equal to  $OQ$ ,

so that the remainder  $LP$  is equal to  $QM$

Therefore  $LM$  is parallel to  $PQ$ ,

and  $LMO$  is equiangular with  $PQO$ ,

therefore, as  $OL$  is to  $LM$ , so is  $OP$  to  $PQ$ ,

and alternately, as  $LO$  is to  $OP$ , so is  $LM$  to  $PQ$ .

But  $LO$  is greater than  $OP$ ,

therefore  $LM$  is also greater than  $PQ$

But  $LM$  was made equal to  $AC$ ,

therefore  $AC$  is also greater than  $PQ$

Since, then, the two sides  $AB, BC$  are equal to the two sides  $PO, OQ$ ,

and the base  $AC$  is greater than the base  $PQ$ ,

therefore the angle  $ABC$  is greater than the angle  $POQ$  [I 25]

Similarly we can prove that

the angle  $DEF$  is greater than the angle  $MOQ$ ,

Therefore the

angles  $LOM, MON, NOL$  are greater than the three

angles  $LOM, MON, NOL$

But, by hypothesis, the angles  $ABC, DEF, GKH$  are less than four right angles,

therefore the angles  $LOM, MON, NOL$  are much less than four right angles

But they are also equal to four right angles

which is absurd

Therefore  $AB$  is not less than  $LO$

And it was proved that neither is it equal,

therefore  $AB$  is greater than  $LO$

Let then  $OR$  be set up from the point  $O$  at right angles to the plane of the circle  $LMN$ ,

and let the square on  $OR$  be equal to that area by which the square on  $AB$  is greater than the square on  $LO$ , [XI 12]

let  $RL, RM, RN$  be joined

Then since  $RO$  is at right angles to the plane of the circle  $LMN$ ,  
 therefore  $RO$  is also at right angles to each of the straight lines  $LO, MO, NO$

And, since  $LO$  is equal to  $OV$ ,

while  $OR$  is common and at right angles,

therefore the base  $RL$  is equal to the base  $RM$ .

[1 4]

For the same reason

$RN$  is also equal to each of the straight lines  $RL, RM$ ;

therefore the three straight lines  $RL, RM, RN$  are equal to one another

which

th

therefore the square on  $AB$  is equal to the squares on  $LO, OR$

But the square on  $LR$  is equal to the squares on  $LO, OR$ , for the angle  $LOR$

is right,

[1 47]

But

is equal to each of the straight lines  $AB, BC, AC$

And since the two sides  $LR, RM$  are equal to the two sides  $AB, BC$ ,

and the base  $LM$  is by hypothesis equal to the base  $AC$ ,

therefore the angle  $LRM$  is equal to the angle  $ABC$

[1 8]

For the same reason

the angle  $MNR$  is also equal to the angle  $DEF$ ,

and the angle  $LRN$  to the angle  $GHE$

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and into the semicircle  $ABC$  let  $AC$  be fitted equal to the straight line  $LO$ , not being greater than the diameter  $AB$ ,

[11 1]

let  $CB$  be joined

Since

Th  
He

CB

But  $AC$  is equal to  $LO$

Therefore the square on  $AB$  is greater than the square on  $LO$  by the square on  $CB$

If then we cut off  $OR$  equal to  $BC$ , the square on  $AB$  will be greater than the square on  $LO$  by the square on  $OR$

#### PROPOSITION 21

If a solid be contained by parallel planes, the opposite planes are parallelogrammic

For let the solid  $CDHG$  be contained by the parallel planes  $AC$ ,  $GF$ ,  $AH$ ,  $DF$ ,  $BF$ ,  $AE$ ,

I say that the opposite planes in it are equal and parallelogrammic

For, since the two parallel planes  $BG$ ,  $CE$  are cut by the plane  $AC$ ,

their common sections are parallel [xi 16]

Therefore  $AB$  is parallel to  $DC$

Again since the two parallel planes  $BF$ ,  $AE$  are cut by the plane  $AC$ ,

their common sections are parallel [xi 16]

Therefore  $BC$  is parallel to  $AD$

But  $AB$  was also proved parallel to  $DC$ ,

therefore  $AC$  is a parallelogram

Similarly we can prove that each of the planes  $DF$ ,  $FG$ ,  $GB$ ,  $BF$ ,  $AE$  is a parallelogram

Let  $AH$   $DF$  be joined

Then since  $AB$  is parallel to  $DC$  and  $BH$  to  $CF$ , the two straight lines  $AB$ ,  $BH$  which meet one another are parallel to the two straight lines  $DC$   $CF$  which meet one another, not in the same plane,

therefore they will contain equal angles, [xi 10]

therefore the angle  $ABH$  is equal to the angle  $DCF$

And since the two sides  $AB$ ,  $BH$  are equal to the two sides  $DC$ ,  $CF$ , [i 3]

and the angle  $ABH$  is equal to the angle  $DCF$ ,

therefore the base  $AH$  is equal to the base  $DF$ ,

and the triangle  $ABH$  is equal to the triangle  $DCF$  [i 4]

And the parallelogram  $BG$  is double of the triangle  $ABH$ , and the parallelogram  $CE$  double of the triangle  $DCF$ , [i 34]

therefore the parallelogram  $BG$  is equal to the parallelogram  $CE$

Similarly we can prove that

$AC$  is also equal to  $GF$ ,

and  $AE$  to  $BF$

Therefore etc

Q E D

### PROPOSITION 25

If a parallelepipedal solid be cut by a plane which is parallel to the opposite planes, then as the base is to the base so will the solid be to the solid

For let the parallelepipedal solid  $ABCD$  be cut by the plane  $FG$  which is parallel to the opposite planes  $RA$   $DH$ ,

I say that as the base  $APFV$  is to the base  $EHCF$ , so is the solid  $ABFU$  to the solid  $EGCD$

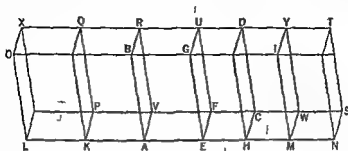
For let  $AH$  be produced in each direction, let any number of straight lines whatever,  $AK$   $KL$ , be made equal to  $AE$ .

and any number whatever  $HM$ ,  $MN$ , equal to  $EH$ ;

and let the parallelograms  $IP$   $AV$   $HW$ ,  $VS$  and the solids  $LQ$ ,  $KE$ ,  $DM$ ,  $MT$  be completed

Then, since the straight lines  $LK$   $KA$ ,  $AE$  are equal to one another, the parallelograms  $IP$   $AV$ ,  $IP$  are also equal to one another,

$KO, KB, AG$  are equal to one another,  
and further,  $LX, KQ, AR$  are equal to one another, for they are opposite  
[XI 24]



For the same reason

the parallelograms  $EC, HW, MS$  are also equal to one another,

—  $HG, HI, IN$  are equal to one another,

and further,  $DH, MY, NT$  are equal to one another

Therefore the solid  $EC, HW, MS$  are equal to three planes

to one another

For the same reason

the three solids  $ED, DM, MT$  are also equal to one another

Therefore, whatever multiple the base  $LF$  is of the base  $AF$ , the same multiple also is the solid  $LU$  of the solid  $AU$

For the same reason,

whatever multiple the base  $NF$  is of the base  $FH$ , the same multiple also is the solid  $NU$  of the solid  $HU$

And if the base  $LF$  is equal to the base  $NF$ , the solid  $LU$  is also equal to the solid  $NU$ ,

if the base  $LF$  exceeds the base  $NF$  the solid  $LU$  also exceeds the solid  $NU$ ,  
and, if one falls short the other falls short

Therefore there being four magnitudes, the two bases  $AF, FH$ , and the two solids  $AU, HU$ ,

equimultiples have been taken of the base  $AF$  and the solid  $AU$ , namely the base  $LF$  and the solid  $LU$ ,

and equimultiples of the base  $FH$  and the solid  $HU$ , namely the base  $NF$  and the solid  $NU$

and it has been proved that if the base  $LF$  exceeds the base  $NF$ , the solid  $LU$  also exceeds the solid  $NU$ ,

if the bases are equal, the solids are equal,

and if the base falls short, the solid falls short,

Therefore, as the base  $AF$  is to the base  $FH$ , so is the solid  $AU$  to the solid  $HU$

[v Def 5]

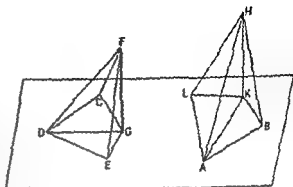
Q E D

### PROPOSITION 26

On a given straight line, and at a given point on it, to construct a solid angle equal to a given solid angle

For let a point  $F$  be taken at random on  $DF$ , let  $FG$  be drawn from  $F$  perpendicular to the plane through  $ED, DC$ , and let it meet the plane at  $G$ , [xi 11]

let  $DG$  be joined, let there be constructed on the straight line  $AB$  and at the point  $A$  on it the angle  $BAL$  equal to the angle  $EDC$ , and the angle  $BAK$  equal to the angle  $EDG$ ,



[i 23]

let  $AK$  be made equal to  $DG$ ,

let  $KH$  be set up from the point  $K$  at right angles to the plane through  $BA, AL$ , [xi 12]

let  $AH$  be made equal to  $GF$ ,

and let  $HA$  be joined,

I say that the solid angle at  $A$ , contained by the angles  $BAL, BAH, HAL$  is equal to the solid angle at  $D$  contained by the angles  $EDC, EDF, FDC$

For let  $AB, DE$  be cut off equal to one another

and let  $HB, KB, FE, GE$  be joined

Then since  $FG$  is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of reference, [xi Def 3]

therefore each of the angles  $FGD, FGE$  is right

For the same reason

each of the angles  $HKA, HKB$  is also right

And since the two sides  $KA, AB$  are equal to the two sides  $GD, DE$  respectively,

and they contain equal angles

therefore the base  $KB$  is equal to the base  $GE$  [i 4]

But  $KH$  is also equal to  $GF$ ,

and they contain right angles,

therefore  $HB$  is also equal to  $FE$  [i 4]

Again since the two sides  $AK, KH$  are equal to the two sides  $DG, GF$ ,

and they contain right angles,

therefore the base  $AH$  is equal to the base  $FD$  [i 4]

But  $AB$  is also equal to  $DE$ ,

therefore the two sides  $HA, AB$  are equal to the two sides  $DF, DE$

And the base  $HB$  is equal to the base  $FE$ ,

therefore the angle  $BAH$  is equal to the angle  $EDF$  [i 8]

For the same reason

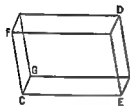
the angle  $HAL$  is also equal to the angle  $FDC$

And the angle  $BAL$  is also equal to the angle  $EDC$

Therefore on the straight line  $AB$ , and at the point  $A$  on it, a solid angle has been constructed equal to the given solid angle at  $D$  Q E F

## PROPOSITION 27

On a given straight line to describe a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid



For on the straight line  $AB$  and at the point  $A$  on it let the solid angle, contained by the angles  $BAH$ ,  $HAK$ ,  $KAB$ , be constructed equal to the solid angle at  $C$ , so that the angle  $BAH$  is equal to the angle  $ECF$ , the angle  $BAK$  equal to the

angle  $ECG$ , and the angle  $KAH$  to the angle  $GCF$ ,

and let it be contrived that,

as  $EC$  is to  $CG$ , so is  $BA$  to  $AK$ ,

and, as  $GC$  is to  $CF$ , so is  $KA$  to  $AH$  [vi 12]

Therefore also, *ex aequali*,

as  $EC$  is to  $CF$ , so is  $BA$  to  $AH$  [v 22]

Let the parallelogram  $HB$  and the solid  $AL$  be completed

Now since, as  $EC$  is to  $CG$ , so is  $BA$  to  $AK$ ,

and the sides about the equal angles  $ECG$ ,  $BAK$  are thus proportional,

therefore the parallelogram  $GE$  is similar to the parallelogram  $KB$

For the same reason

the parallelogram  $KH$  is also similar to the parallelogram  $GF$ , and further,  $FE$  is to  $HB$ ,

therefore three parallelograms of the solid  $CD$  are similar to three parallelograms of the solid  $AL$

But the former three are both equal and similar to the three opposite parallelograms,

and the latter three are both equal and similar to the three opposite parallelograms,

therefore the whole solid  $CD$  is similar to the whole solid  $AL$  [xi Def 9]

Therefore on the given straight line  $AB$  there has been described  $AL$  similar and similarly situated to the given parallelepipedal solid  $CD$  Q E F

## PROPOSITION 28

If a parallelepipedal solid be cut by a plane through the diagonals of the opposite planes, the solid will be bisected by the plane

For let the parallelepipedal solid  $AB$  be cut by the plane  $CDEF$  through the diagonals  $CF$ ,  $DE$  of opposite planes,

I say that the solid  $AB$  will be bisected by the plane  $CDEF$

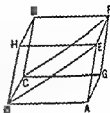
For, since the triangle  $CGF$  is equal to the triangle  $CFB$ ,

[i 34]



and  $ADE$  to  $DEH$ ,  
 while the parallelogram  $CA$  is also equal to the parallelogram  $EB$ , for they are opposite, and  $GE$  to  $CH$ , therefore the prism contained by the two triangles  $CGF$ ,  $ADE$  and the three parallelograms  $GE$ ,  $AC$ ,  $CE$  is also equal to the prism contained by the two triangles  $CFB$ ,  $DEH$  and the three parallelograms  $GE$ ,  $AC$ ,  $CE$  for they are contained under the same altitude and in magnitude [xi Def 10]

Hence the whole solid  $AB$  is bisected by the plane  $CDEF$ .



Q E D

## PROPOSITION 29

*Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are on the same straight lines, are equal to one another*

Let  $CM$ ,  $CN$  be parallelepipedal solids on the same base  $AB$  and of the same height,

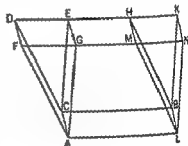
and let the extremities of their sides which stand up, namely  $AG$ ,  $AF$ ,  $LM$ ,  $LN$ ,  $CD$ ,  $CE$ ,  $BH$ ,  $BK$ , be on the same straight lines  $FN$ ,  $DK$ ,

I say that the solid  $CM$  is equal to the solid  $CN$

For, since each of the figures  $CH$ ,  $CK$  is a parallelogram,  $CB$  is equal to each of the straight lines  $DH$ ,  $EK$ , [i 34]

hence  $DH$  is also equal to  $EK$

Let  $EH$  be subtracted from each,



PK.

[i 8, 4]  
[i 30]

For the same reason

the triangle  $AFG$  is also equal to the triangle  $MLN$

But the parallelogram  $CF$  is equal to the parallelogram  $BM$ , and  $CG$  to  $BN$ , for they are opposite,

therefore the prism contained by the two triangles  $AFG$ ,  $DCE$  and the three parallelograms  $AD$ ,  $DG$ ,  $CG$  is equal to the prism contained by the two triangles  $MLN$ ,  $HBA$  and the three parallelograms  $BM$ ,  $HN$ ,  $BN$

Let there be added to each the solid of which the parallelogram  $AB$  is the base and  $GEHM$  its opposite, therefore the whole parallelepipedal solid  $CM$  is equal to the whole parallelepipedal solid  $CN$

Therefore etc

Q E D

## PROPOSITION 30

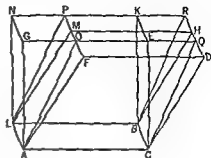
*Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are not on the same straight lines, are equal to one another*

Let  $CM$ ,  $CN$  be parallelepipedal solids on the same base  $AB$  and of the same height,  
and let the extremities of their sides which stand up, namely  $AF$ ,  $AG$ ,  $LM$ ,  $LN$ ,  $CD$ ,  $CE$ ,  $BH$ ,  $BK$ , not be on the same straight lines,

I say that the solid  $CM$  is equal to the solid  $CN$

For let  $NK$ ,  $DH$  be produced and meet one another at  $R$ ,  
and further, let  $FM$ ,  $GE$  be produced to  $P$ ,  $Q$ ,

let  $AO$ ,  $LP$ ,  $CQ$ ,  $BR$  be joined



ties of their sides which stand up, namely  $AF$ ,  $AO$ ,  $LM$ ,  $LP$ ,  $CD$ ,  $CQ$ ,  $BH$ ,  $BR$ , are on the same straight lines  $FP$ ,  $DR$  [xi 29]

But the solid  $CP$ , of which the parallelogram  $ACBL$  is the base, and  $OQRP$  its opposite, is equal to the solid  $CN$ , of which the parallelogram  $ACBL$  is the base and  $GEKN$  its opposite,

Therefore etc

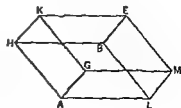
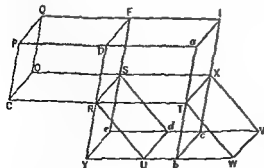
Q E D

### PROPOSITION 31

*Parallelepipedal solids which are on equal bases and of the same height are equal to one another*

Let the parallelepipedal solids  $AE$ ,  $CF$ , of the same height, be on equal bases  $AB$ ,  $CD$

I say that the solid  $AE$  is equal to the solid  $CF$



First, let the sides which stand up,  $HK$ ,  $BE$ ,  $AG$ ,  $LM$ ,  $PQ$ ,  $DF$ ,  $CO$ ,  $RS$ , be at right angles to the bases  $AB$ ,  $CD$ ,

let the straight line  $RT$  be produced in a straight line with  $CR$ ,  
on the straight line  $RT$ , and at the point  $R$  on it, let the angle  $TRU$  be con-  
structed equal to the angle  $ALB$ , [r 23]

let  $RT$  be made equal to  $AL$ , and  $RU$  equal to  $LB$ ,  
and let the base  $RW$  and the solid  $XU$  be completed

Now, since the two sides  $TR$ ,  $RU$  are equal to the two sides  $AL$ ,  $LB$ ,  
and they contain equal angles,  
therefore the parallelogram  $RW$  is equal and similar to the parallelogram  $HL$   
Since again  $AL$  is equal to  $RT$ , and  $LM$  to  $RS$ ,  
and they contain right angles,  
therefore the parallelogram  $RX$  is equal and similar to the parallelogram  $AM$   
For the same reason

$LE$  is also equal and similar to  $SU$ ,  
therefore three parallelograms of the solid  $AE$  are equal and similar to three  
parallelograms of the solid  $XU$

But the former three are equal and similar to the three opposite, and the  
latter three to the three opposite, [r 24]  
therefore the whole parallelepipedal solid  $AE$  is equal to the whole parallele-  
pipedal solid  $XU$ . [x1 Def 10]

Let  $DR$ ,  $WU$  be drawn through and meet one another at  $Y$ ,  
let to  $DY$ ,

Then the solid  $XY$ , of which the parallelogram  $RX$  is the base and  $Yc$  its op-  
posite parallelogram  $RA$  is the base and

the height and the extremities  
 $Tb$ ,  $TH$ ,  $Se$ ,  $Sd$ ,  $Xc$ ,  $AV$ , are [r 29]

But the solid  $XU$  is equal to  $AE$ ,

therefore the solid  $XY$  is also equal to the solid  $AE$

And, since the parallelogram  $RUWT$  is equal to the parallelogram  $YT$ ,  
for they are on the same base  $RT$  and in the same parallels  $RT$ ,  $YW$ , [r 35]

while  $RUWT$  is equal to  $CD$ , since it is also equal to  $AB$ ,

therefore the parallelogram  $YT$  is also equal to  $CD$

But  $DT$  is another parallelogram,

therefore as the base  $CD$  is to  $DT$ , so is  $YT$  to  $DT$  [v 7]

And since the parallelepipedal solid  $CI$  has been cut by the plane  $RF$  which  
is parallel to opposite planes

as the base  $CD$  is to the base  $DT$ , so is the solid  $CF$  to the solid  $RI$  [x1 25]

For the same reason,

since the parallelepipedal solid  $YI$  has been cut by the plane  $RX$  which is par-  
allel to opposite planes

as the base  $YT$  is to the base  $DT$ , [r 26]

But as the base  $CD$

therefore also as the solid  $CF$  to  $RI$  [v 11]

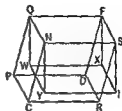
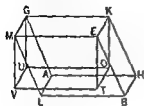
Therefore each of the solids  $CF$ ,  $YX$  has to  $RI$  the same ratio,

therefore the solid  $CF$  is equal to the solid  $YX$ . [v 9]

But  $YX$  was not as good as  $XY$ .

Next, let the : - - " "  $PQ, DF, RS$ , not be  
at right angles t .

I say again that the solid  $AE$  is equal to the solid  $CF$



For from the points  $K, E, G, M, Q, F, N, S$  let  $KO, ET, GU, MV, QW, FX, NY, SI$  be drawn perpendicular to the plane of reference, and let them meet the plane at the points  $O, T, U, V, W, X, Y, I$ .

and let  $OT, OU, UV, TV, WX, WY, YI, IX$  be joined

Then the solid  $KV$  is equal to the solid  $QI$ ,  
for they are on the same base  $VI$  and have equal heights,  
which stand

same height, and their sides  
(First part of this Prop)

But the sol

and *QI* to *CF*.

for they are on the same base and of the same height, while the extremities of their sides which stand up are not on the same straight lines [xi 30]

Therefore the solid  $AE$  is also equal to the solid  $CF$

Therefore etc

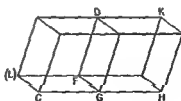
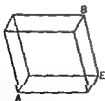
O E D

### PROPOSITION 32

**Parallelepipedal solids which are of the same height are to one another as their bases**

Let  $AB$ ,  $CD$  be parallelepipedal solids of the same height,

I say that the parallelepipedal solids  $AB, CD$  are to one another as their bases, that is, that, as the base  $AE$  is to the base  $CF$ , so is the solid  $AB$  to the solid  $CD$ .



For let  $FH$  equal to  $AE$  be applied to  $FG$ ,  
and, on  $FH$  as base and with the same height as that of  $CD$ , let the parallelogram

fo  $\frac{1}{2}$  inch and 15 H.L., 1 ft and of the same height [XI 31]

And since the parallelepipedal solid  $CK$  is cut by the plane  $DG$  which is parallel to opposite planes,

let the straight line  $RT$  be produced in a straight line on the straight line  $RT$  and at the point  $R$  on it let constructed equal to the angle  $ALB$ ,

let  $RT$  be made equal to  $AL$  and

and let the base  $RH$  and the

Now since the two sides  $TR$   $RU$  are

and they are

therefore the parallelogram  $RH$

Since again  $AL$  is equal to

or

therefore the parallelogram

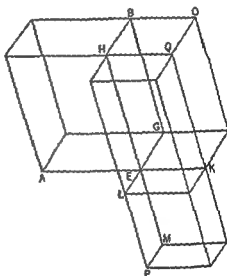
For the same reason

therefore the

parallelogram

But

is



solid  $DH$

[VI 23]

$\angle AB$  to the solid

$QED$

is the triplicate ratio of their

is,

responding to  $CF$ ,

$JD$  the ratio triplicate of that which

For let  $EA$   $EL$   $EM$  be produced in a straight line with  $AE$   $GE$   $HE$  let  $EA$  be made equal to  $CF$   $EL$  equal to  $FN$  and further  $EM$  equal to  $FR$  and let the parallelogram  $KL$  and the solid  $KP$  be completed

Now since the two sides  $KE$   $EL$  are equal to the two sides  $CF$   $FN$  while the angle  $KEL$  is also equal to the angle  $CFN$  inasmuch as the angle  $AEG$  is also equal to the angle  $CFN$  because of the similarity of the solids  $AB$   $CD$

therefore the parallelogram  $KL$  is equal and similar to the parallelogram  $CA$

For the same reason

the parallelogram  $KU$  is also equal and similar to  $CR$ ,

and further  $FP$  to  $DF$

therefore three parallelograms of the solid  $KP$  are equal and similar to three parallelograms of the solid  $CD$

But the former three parallelograms are equal and similar to their opposites, and the latter three to their opposites, [xi 24]  
 therefore the whole solid  $KP$  is equal and similar to the whole solid  $CD$  [xi Def 10]

with the same height as that

Then since owing to the similarity of the solids  $AB$ ,  $CD$ ,  
 as  $AE$  is to  $CF$ , so is  $EG$  to  $FN$ , and  $EH$  to  $FR$ ,  
 while  $CF$  is equal to  $EK$ ,  $FN$  to  $EL$ , and  $FR$  to  $EM$ ,  
 therefore, as  $AE$  is to  $EK$ , so is  $GE$  to  $EL$ , and  $HE$  to  $EM$ .

But, as  $AE$  is to  $EK$ , so is  $AG$  to the parallelogram  $GK$ ,

as  $GE$  is to  $EL$ , so is  $GK$  to  $KL$ ,

and, as  $HE$  is to  $EM$ , so is  $QE$  to  $KM$ , [vi 1]

therefore also, as the parallelogram  $AG$  is to  $GK$ , so is  $GK$  to  $KL$ , and  $QE$  to  $KM$

But, as  $AG$  is to  $GK$ , so is the solid  $AB$  to the solid  $EO$ ,

as  $GK$  is to  $KL$ , so is the solid  $OE$  to the solid  $QL$ ,

and, as  $QE$  is to  $KM$ , so is the solid  $QL$  to the solid  $KP$ , [xi 32]

therefore also, as the solid  $AB$  is to  $EO$ , so is  $EO$  to  $QL$ , and  $QL$  to  $KP$

$EO$

But, as  $AB$  is to  $EO$ , so is the parallelogram  $AG$  to  $GK$ , and the straight line  $AE$  to  $EK$  [vi 1],

hence the solid  $AB$  has also to  $KP$  the ratio triplicate of that which  $AE$  has to  $EK$

But the solid  $KP$  is equal to the solid  $CD$ ,

Therefore etc

Q E D

second inasmuch as the first has to the fourth the ratio triplicate of that which it has to the second

### PROPOSITION 34

*In equal parallelepipedal solids the bases are reciprocally proportional to the heights, and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal*

Let  $AB$ ,  $CD$  be equal parallelepipedal solids,

I say that in the parallelepipedal solids  $AB$ ,  $CD$  the bases are reciprocally proportional to the heights

that is as the base  $EIH$  is to the base  $NQ$ , so is the height of the solid  $CD$  to the height of the solid  $AB$

First, let the sides which stand up, namely  $AG$ ,  $EF$ ,  $LB$ ,  $HK$ ,  $CM$ ,  $NO$ ,  $PD$ ,

$QR$ , be at right angles to their bases,

I say that, as the base  $EH$  is to the base  $NQ$ , so is  $CM$  to  $AG$

If now the base  $EH$  is equal to the base  $NQ$ ,

while the solid  $AB$  is also equal to the solid  $CD$ ,

$CM$  will also be equal to  $AG$

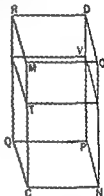
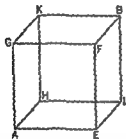
For parallelepipedal solids of the same height are to one another as the bases, [xi 32]

and as the base  $EH$  is to  $NQ$ , so will  $CM$  be to  $AG$ ,

and it is manifest that in the parallelepipedal solids  $AB$ ,  $CD$  the bases are reciprocally proportional to the heights

Next, let the base  $EH$  not be equal to the base  $NQ$ ,

but let  $EH$  be greater



Now the solid  $AB$  is equal to the solid  $CD$ ,

therefore  $CM$  is also greater than  $AG$

Let then  $CT$  be made equal to  $AG$ ,

and let the parallelepipedal solid  $VC$  be completed on  $NQ$  as base and with  $CT$  as height

Now, since the solid  $AB$  is equal to the solid  $CD$ ,

and  $CV$  is outside them

while equals have to the same the same ratio, [v 7]

therefore as the solid  $AB$  is to the solid  $CV$ , so is the solid  $CD$  to the solid  $CV$

But as the solid  $AB$  is to the solid  $CV$ , so is the base  $EH$  to the base  $NQ$

for the solids  $AB$ ,  $CV$  are of equal height, [xi 32]

and as the solid  $CD$  is to the solid  $CV$ , so is the base  $MQ$  to the base  $TQ$  [xi 25] and  $CM$  to  $CT$  [vi 1],

therefore also as the base  $EH$  is to the base  $NQ$  so is  $MC$  to  $CT$

But  $CT$  is equal to  $AG$ ,

therefore also as the base  $EH$  is to the base  $NQ$ , so is  $MC$  to  $AG$

Therefore in the parallelepipedal solids  $AB$ ,  $CD$  the bases are reciprocally proportional to the heights

Again in the parallelepipedal solids  $AB$ ,  $CD$  let the bases be reciprocally proportional to the heights that is, as the base  $EH$  is to the base  $NQ$ , so let the height of the solid  $CD$  be to the height of the solid  $AB$ ,

I say that the solid  $AB$  is equal to the solid  $CD$

Let the sides which stand up be again at right angles to the bases

Now, if the base  $EH$  is equal to the base  $NQ$ ,

and as the base  $EH$  is to the base  $NQ$ , so is the height of the solid  $CD$  to the height of the solid  $AB$ ,

therefore the height of the solid  $CD$  is also equal to the height of the solid  $AB$

But parallelepipedal solids on equal bases and of the same height are equal to one another,

[xi 31]

therefore the solid  $AB$  is equal to the solid  $CD$

the solid

$AB$ ,

that is,  $CM$  is greater than  $AG$

Let  $CT$  be again made equal to  $AG$ ,

and let the solid  $CV$  be similarly completed

Since as the base  $EH$  is to the base  $NQ$ , so is  $MC$  to  $AG$ ,

while  $AG$  is equal to  $CT$ ,

therefore

But as the

and as  $CM$  is to  $CT$ , so is the base  $MQ$  to the base  $QT$

[vi 1]

and the solid  $CD$  to the solid  $CV$

[xi 25]

Therefore also as the solid  $AB$  is to the solid  $CV$ , so is the solid  $CD$  to the solid  $CV$ ,

therefore each of the solids  $AB$ ,  $CD$  has to  $CV$  the same ratio

Therefore the solid  $AB$  is equal to the solid  $CD$

[v 9]

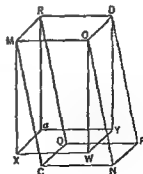
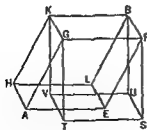
Now let the sides which stand up  $FE$ ,  $BL$ ,  $GA$ ,  $HK$ ,  $ON$ ,  $DP$ ,  $MC$ ,  $RQ$  not be at right angles to their bases

let perpendiculars be drawn from the points  $F$ ,  $G$ ,  $B$ ,  $K$ ,  $O$ ,  $M$ ,  $D$ ,  $R$  to the planes through  $EH$ ,  $NQ$ ,

and let them meet the planes at  $S$ ,  $T$ ,  $U$ ,  $V$ ,  $W$ ,  $X$ ,  $Y$ ,  $a$ ,

and let the solids  $FV$ ,  $Oa$  be completed,

I say that in this case too if the solids  $AB$ ,  $CD$  are equal the bases are reciprocal



while  $AB$  is equal to  $BT$ ,

for they are on the same base  $FA$  and of the same height

[xi 29 30]



and the solid  $CD$  is equal to  $DX$ ,

for they are again on the same base  $RO$  and of the same height, [id]  
therefore the solid  $BT$  is also equal to the solid  $DX$ .

Therefore, as the base  $FK$  is to the base  $OR$ , so is the height of the solid  $DY$   
to the height of the solid  $BT$  [Part 1]

But the base  $FK$  is equal to the base  $EH$ ,

and the base  $OR$  to the base  $NQ$ ,

therefore, as the base  $EH$  is to the base  $NQ$ , so is the height of the solid  $DX$  to  
the height of the solid  $BT$

But the solids  $DX$ ,  $BT$  and the solids  $DC$ ,  $BA$  have the same heights respectively,

therefore, as the base  $EH$  is to the base  $NQ$ , so is the height of the solid  $DC$  to  
the height of the solid  $AB$

Therefore in the parallelepipedal solids  $AB$ ,  $CD$  the bases are reciprocally  
proportional to the heights

Again, in the parallelepipedal solids  $AB$ ,  $CD$  let the bases be reciprocally  
proportional to the heights,

that is, as the base  $EH$  is to the base  $NQ$ , so let the height of the solid  $CD$  be to  
the height of the solid  $AB$ ,

I say that the solid  $AB$  is equal to the solid  $CD$

For, with the same construction,

since, as the base  $EH$  is to the base  $NQ$ , so is the height of the solid  $CD$  to the  
height of the solid  $AB$ ,

while the base  $EH$  is equal to the base  $FK$ ,

and  $NQ$  to  $OR$ ,

therefore as the base  $FK$  is to the base  $OR$ , so is the height of the solid  $CD$  to  
the height of the solid  $AB$

...

Therefore in the parallelepipedal solids  $BT$ ,  $DX$  the bases are reciprocally  
proportional to the heights,

therefore the solid  $BT$  is equal to the solid  $DX$  [Part 1]

But  $BT$  is equal to  $BA$ ,

for they are on the same base  $FK$  and of the same height, [xi 29, 30]

and the solid  $DX$  is equal to the solid  $DC$  [id]

Therefore the solid  $AB$  is also equal to the solid  $CD$  Q E D

### PROPOSITION 35

If there be two equal plane angles, and on their vertices there be set up elevated  
straight lines containing equal angles with the original straight lines respectively,  
if on the elevated straight lines points be taken at random and perpendiculars be  
drawn from them to the planes in which the original  
points so arising in the planes  
angles they will contain,

Let the angles  $BAC$ ,  $EDF$  be equal plane angles, and from the  
points  $A$ ,  $D$  let the elevated straight lines  $AG$ ,  $DM$  be set up containing with  
the original straight lines equal angles respectively, namely, the angle  $GAB$   
to the angle  $GAB$  and the angle  $MDF$  to the angle  $GAC$ ,

let points  $G, M$  be taken at random on  $AC, DM$ ,  
let  $GL, MN$  be drawn from the points  $G, M$  perpendicular to the planes through  
 $BA, AC$  and  $ED, DF$ , and let them meet the planes at  $L, N$ ,  
and let  $LA, ND$  be joined,

I say that the angle  $GAL$  is equal to the angle  $MDN$

Let  $AH$  be made equal to  $DM$ ,  
and let  $HK$  be drawn through the point  $H$  parallel to  $GL$ .

But  $GL$  is perpendicular to the plane through  $BA, AC$ , therefore  $HK$  is also perpendicular to the plane through  $BA, AC$  [XI 8]

From the points  $K, N$  let  $KC, NF, KB, NE$  be drawn perpendicular to the straight lines  $AC, DF, AB, DE$ .

and let  $HC, CB, MF, FE$  be joined

Since the square on  $HA$  is equal to the squares on  $HK$ ,  $KA$ ,

and the squares on  $KC$ ,  $CA$  are equal to the square on  $KA$ , [I 47]

therefore the square on  $HA$  is also equal to the squares on  $HK, KC, CA$

But the square on  $HC$  is equal to the squares on  $HK$ ,  $KC$ , (I. 47)

therefore the square on  $HA$  is equal to the squares on  $HC$ ,  $CA$

Therefore the angle  $HCA$  = right. [1 48]

For the same reason

the angle  $DFM$  is also right

Therefore the angle  $ACH$  is equal to the angle  $DFM$

respectively

[1 26]

Therefore  $AC$  is equal to  $DF$

Similarly we can prove that  $AB$  is also equal to  $DE$

Since then  $AC$  is equal to  $DF$ , and  $AB$  to  $DE$ ,

the two sides  $CA$ ,  $AB$  are equal to the two sides  $FD$   $DE$

But the angle  $CAB$  is also equal to the angle  $FDE$ ,

therefore the base  $BC$  is equal to the base  $EF$ , the triangle to the triangle and the same.

for the same reason

the angle  $CBK$  is also equal to the angle  $FEN$

an  
th

therefore they will also have the remaining sides equal to the remaining sides [1 26]

Therefore  $CK$  is equal to  $FN$ .

But  $AC$  is also equal to  $DF$ ,

therefore the two sides  $AC, CK$  are equal to the two sides  $DF, FN$ ,  
and they contain right angles

Therefore the base  $AK$  is equal to the base  $DN$  [1 4]

And since  $AH$  is equal to  $DV$ ,

the square on  $AH$  is also equal to the square on  $DM$

But the squares on  $AK, KH$  are equal to the square on  $AH$ ,  
for the angle  $AKH$  is right, [1 47]

and the squares on  $DN, NM$  are equal to the square on  $DM$   
for the angle  $DNM$  is right, [1 47]

therefore the squares on  $AK, KH$  are equal to the squares on  $DN, NM$ ,

and of these the square on  $AK$  is equal to the square on  $DV$ ,

therefore the remaining square on  $KH$  is equal to the square on  $NM$ ,

therefore  $HK$  is equal to  $MN$

And since the two sides  $HA, AK$  are equal to the two sides  $MD, DN$  respectively,

and the base  $HA$  was proved equal to the base  $MN$ ,  
therefore the angle  $HAK$  is equal to the angle  $MDN$  [1 8]

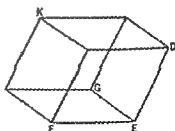
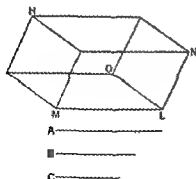
Therefore etc.

**PONISM** From this it is manifest that, if there be two equal plane angles, and if there be set up on them elevated straight lines which are equal and contain equal angles with the original straight lines respectively, the perpendiculars drawn from their extremities to the planes in which are the original angles are equal to one another Q E D

### PROPOSITION 36

If three straight lines be proportional the parallelepipedal solid formed out of the three is equal to the parallelepipedal solid on the mean which is equilateral but equiangular with the aforesaid solid

Let  $A, B, C$  be three straight lines in proportion, so that, as  $A$  is to  $B$ , so is  $B$  to  $C$ ,



I say that the solid formed out of  $A, B, C$  is equal to the solid on  $B$  which is equilateral but equiangular with the aforesaid solid

Let there be set out the solid angle at  $E$  contained by the angles  $DEG$ ,  $GEF$ ,  $FED$ ,

let each of the straight lines  $DE$ ,  $GE$ ,  $EF$  be made equal to  $B$ , and let the parallelepipedal solid  $EK$  be completed,

let  $LM$  be made equal to  $A$ ,

and on the straight line  $LM$ , and at the point  $L$  on it, let there be constructed a solid angle equal to the solid angle at  $E$ , namely that contained by  $NLO$ ,  $OLM$ ,  $MLN$ ,

let  $LO$  be made equal to  $B$ , and  $LN$  equal to  $C$

Now, since, as  $A$  is to  $B$ , so is  $B$  to  $C$ ,  
while  $A$  is equal to  $LM$ ,  $B$  to each of the straight lines  $LO$ ,  $ED$ , and  $C$  to  $LN$ ,  
therefore, as  $LM$  is to  $ED$ , so is  $DE$  to  $LN$

Thus the sides about the equal angles  $NLM$ ,  $DEF$  are reciprocally proportional,

therefore the perpendiculars drawn from the points  $G$ ,  $O$  to the planes through  $NL$ ,  $LM$  and  $DE$ ,  $EF$  are equal to one another, [xi 35 Por]

hence the solids  $LH$ ,  $EK$  are of the same height

But parallelepipedal solids on equal bases and of the same height are equal to one another, [xi 31]

therefore the solid  $HL$  is equal to the solid  $EK$

And  $LH$  is the solid formed out of  $A$ ,  $B$ ,  $C$ , and  $EK$  the solid on  $B$ ,  
therefore the parallelepipedal solid formed out of  $A$ ,  $B$ ,  $C$  is equal to the solid on  $B$  which is equilateral but equiangular with the aforesaid solid

Q E D

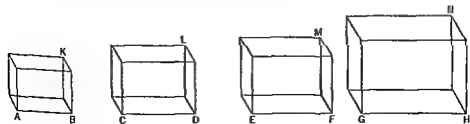
### PROPOSITION 37

If to equal parts be added equal parts, the wholes will be equal

straight lines will themselves also be proportional

Let  $AB$ ,  $CD$ ,  $EF$ ,  $GH$  be four straight lines in proportion, so that as  $AB$  is to  $CD$ , so is  $EF$  to  $GH$ ,

and let there be described on  $AB$ ,  $CD$ ,  $EF$ ,  $GH$  the similar and similarly situated



ated parallelepipedal solids  $KA$ ,  $LC$ ,  $ME$ ,  $NG$ ,

I say that, as  $KA$  is to  $LC$ , so is  $ME$  to  $NG$

For, since the parallelepipedal solid  $KA$  is similar to  $LC$ ,  
therefore  $KA$  has to  $LC$  the ratio triplicate of that which  $AB$  has to  $CD$  [xi 33]

For the same reason

$ME$  also has to  $NG$  the ratio triplicate of that which  $EF$  has to  $GH$  [id]

And, as  $AB$  is to  $CD$ , so is  $EF$  to  $GH$ .

Therefore also, as  $AK$  is to  $LC$ , so is  $ME$  to  $NG$

Next, as the solid  $AK$  is to the solid  $LC$ , so let the solid  $ME$  be to the solid  $NG$ ,

I say that, as the straight line  $AB$  is to  $CD$ , so is  $EF$  to  $GH$ .

For since, again,  $KA$  has to  $LC$  the ratio triplicate of that which  $AB$  has to  $CD$ , [xi 33]

and  $ME$  also has to  $NG$  the ratio triplicate of that which  $EF$  has to  $GH$ , [id]

and, as  $KA$  is to  $LC$ , so is  $ME$  to  $NG$ ,

therefore also, as  $AB$  is to  $CD$ , so is  $EF$  to  $GH$ .

Therefore etc

Q E D

### PROPOSITION 38

*If the sides of the opposite planes of a cube be bisected, and planes be carried through the points of section, the common section of the planes and the diameter of the cube bisect one another*

For let the sides of the opposite planes  $CF$ ,  $AH$  of the cube  $AF$  be bisected at the points  $K$ ,  $L$ ,  $M$ ,  $N$ ,  $O$ ,  $Q$ ,  $P$ ,  $R$ , and through the points of section let the planes  $KN$   $OR$  be carried,  
let  $US$  be the common section of the planes, and  $DG$  the diameter of the cube  $AF$

I say that  $UT$  is equal to  $TS$ , and  $DT$  to  $TG$

For let  $DU$ ,  $UE$ ,  $BS$ ,  $SG$  be joined

Then since  $DO$  is parallel to  $PE$ ,

the alternate angles  $DOU$ ,  $UPE$  are equal to one another [i 29]

And, since  $DO$  is equal to  $PE$ , and  $OU$  to  $UP$ ,

and they contain equal angles,  
therefore the base  $DU$  is equal to the base  $UE$ ,

the triangle  $DOU$  is equal to the triangle  $PUE$ ,

and the remaining angles are equal to the remaining angles, [i 4]  
therefore the angle  $ODU$  is equal to the angle  $PUE$

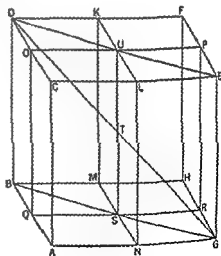
For this reason  $DUE$  is a straight line [i 14]

For the same reason  $BSG$  is also a straight line,

and  $BS$  is equal to  $SG$

Now, since  $CA$  is equal and parallel to  $DB$ ,

while  $CA$  is also equal and parallel to  $EG$ ,



therefore  $DB$  is also equal and parallel to  $EG$ .

[XI 9]

And the straight lines  $DE$ ,  $BG$  join their extremities;

therefore  $DE$  is parallel to  $BG$ .

[I 33]

Therefore the angle  $EDT$  is equal to the angle  $BGT$ ,

for they are alternate;

and the angle  $DTT'$ .

Therefore

and

and

therefore they will also have

[I 20]

Therefore  $DT$  is equal to  $TU$ , and  $UT$  to  $TS$ .

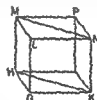
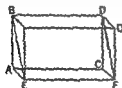
Therefore etc.

Q E D.

## PROPOSITION 39

If there be two prisms of equal height,  
other a triangle  
be equal

Let  $ABCDEF$  be two prisms of equal height,  
let one have the parallelogram  $AF$  as base, and the other the triangle  $GHK$ ,  
and let the parallelogram  $AF$  be double of the triangle  $GHK$ ,  
I say that the prism  $ABCDEF$  is equal to the prism  $GHKLMN$ .



For let the solids  $AO$ ,  $GP$  be completed

Since the parallelogram  $AF$  is double of the triangle  $GHK$ ,  
while the parallelogram  $HK$  is also double of the triangle  $GHK$ ,  
therefore the solids

But par  
are equal

[XI 31]

And the prism  $ABCDEF$  is half of the solid  $AO$   
and the prism  $GHKLMN$  is half of the solid  $GP$

[XI 28]

therefore the prisms

Therefore etc

Q E D

# BOOK TWELVE

## PROPOSITIONS

### PROPOSITION 1

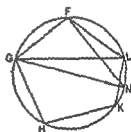
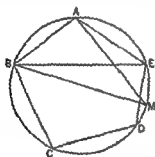
*Similar polygons inscribed in circles are to one another as the squares on the diameters*

Let  $ABC$ ,  $FGH$  be circles,  
let  $ABCDE$ ,  $FGHKL$  be similar polygons inscribed in them, and let  $BM$ ,  $GN$  be diameters of the circles,

I say that as the square on  $BM$  is to the square on  $GN$ , so is the polygon  $ABCDE$  to the polygon  $FGHKL$

For let  $BE$ ,  $AM$ ,  $GL$ ,  $FN$  be joined

Now, since the polygon  $ABCDE$  is similar to the polygon  $FGHKL$ ,



the angle  $BAE$  is equal to the angle  $GFL$ ,

and as  $BA$  is to  $AE$  so is  $GF$  to  $FL$

[vi Def 1]

Thus  $BAE$ ,  $GFL$  are two triangles which have one angle equal to one angle namely the angle  $BAE$  to the angle  $GFL$  and the sides about the equal angles proportional,

therefore the triangle  $ABE$  is equiangular with the triangle  $FGL$

[vi 6]

Therefore the angle  $AEB$  is equal to the angle  $FLG$

But the angle  $AEB$  is equal to the angle  $AMB$ ,

for they stand on the same circumference,

[iii 27]

and the angle  $FLG$  to the angle  $FNG$ ,

therefore the angle  $AMB$  is also equal to the angle  $FNG$

But the right angle  $BAM$  is also equal to the right angle  $GFN$ ,

[iii 31]

therefore the remaining angle is equal to the remaining angle

[i 32]

Therefore the triangle  $ABM$  is equiangular with the triangle  $FGN$

Therefore, proportionally as  $BM$  is to  $GN$ , so is  $BA$  to  $GF$

[vi 4]

But the ratio of the square on  $BM$  to the square on  $GN$  is duplicate of the ratio of  $BM$  to  $GN$ ,  
 and the ratio of the polygon  $ABCDE$  to the polygon  $FGHKL$  is as the ratio of  $BA$  to  $GF$ ,  
 therefore also, as the polygon  $ABCDE$  to the polygon  $FGHKL$ , so is the square on  $BM$  to the square on  $GN$ .  
 Therefore etc

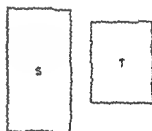
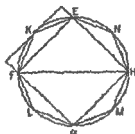
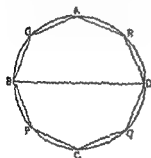
Q E D

## PROPOSITION 2

Circles are to one another as the squares on the diameters

Let  $ABCD$ ,  $EFGH$  be circles, and  $BD$ ,  $FH$  their diameters,

I say that, as the circle  $ABCD$  is to the circle  $EFGH$ , so is the square on  $BD$  to the square on  $FH$ .



For, if the square on  $BD$  is not to the square on  $FH$  as the circle  $ABCD$  is to the circle  $EFGH$ ,  
 then, as the square on  $BD$  is to the square on  $FH$ , so will the circle  $ABCD$  be either to some less area than the circle  $EFGH$ , or to a greater

First, let it be in that ratio to a less area  $S$

Let the square  $EFGH$  be inscribed in the circle  $EFGH$ ,  
 square is greater than  $S$

points  $E$ ,  $F$ ,  $G$ ,  $H$  we  
 square circumscribed  
 scribed square,  
 hence the inscribed

Let the circle  
 $N$ ,

and  $LG$ ,  $GM$ ,  $MH$ ,  $HN$ ,  $NE$  be joined,  
 therefore each of the triangles  $EKF$ ,  $FLG$ ,  $GMH$ ,  $HNE$  is also greater than  
 the half of the segment of the circle about it, inasmuch as the triangles  
 points  $K$ ,  $L$ ,  $M$ ,  $N$  we draw tangents to the circle at these points, and  
 parallelogram  $FLG$ ,  $GMH$ ,  
 and it is less than the parallelogram,

hence each of the triangles  $EKF$ ,  $FLG$ ,  $GMH$ ,  $HNE$  is greater than the half of  
 the segment of the circle about it

Thus by bisecting the remaining circumferences and joining straight lines,



and by doing this continually we shall leave some segments of the circle which will be less than the excess by which the circle  $EFGH$  exceeds the area  $S$

For it was proved in the first theorem of the tenth book that if two unequal magnitudes be set out and if from the greater there be subtracted a magnitude greater than the half and from that which is left a greater than the half and if this be done continually there will be left some magnitude which will be less than the lesser magnitude set out

Let segments be left such as described and let the segments of the circle  $EFGH$  on  $E\Lambda$   $KF$   $FL$   $LG$   $GM$   $NH$   $HN$   $\Lambda E$  be less than the excess by which the circle  $EFGH$  exceeds the area  $S$

Therefore the remainder the polygon  $E\Lambda FLGMHN$  is greater than the area  $S$

Let there be inscribed also in the circle  $ABCD$  the polygon  $AOBPCQDR$  similar to the polygon  $E\Lambda FLGMHN$

therefore as the square on  $BD$  is to the square on  $FH$  so is the polygon  $AOBPCQDR$  to the polygon  $E\Lambda FLGMHN$  (xi 1)

But as the square on  $BD$  is to the square on  $FH$  so also is the circle  $ABCD$  to the area  $S$

therefore also as the circle  $ABCD$  is to the area  $S$  so is the polygon  $AOBPCQDR$  to the polygon  $E\Lambda FLGMHN$  (v 11)

therefore alternately as the circle  $ABCD$  is to the polygon inscribed in it so is the area  $S$  to the polygon  $E\Lambda FLGMHN$  (v 16)

But the circle  $ABCD$  is greater than the polygon inscribed in it  
therefore the area  $S$  is also greater than the polygon  $E\Lambda FLGMHN$   
But it is also less

which is impossible

Therefore as the square on  $BD$  is to the square on  $FH$  so is not the circle  $ABCD$  to any area less than the circle  $EFGH$

Similarly we can prove that neither is the circle  $EFGH$  to any area less than the circle  $ABCD$  as the square on  $FH$  is to the square on  $BD$

I say next that neither is the circle  $ABCD$  to any area greater than the circle  $EFGH$  as the square on  $BD$  is to the square on  $FH$

For if possible let it be in that ratio to a greater area  $S$

Therefore inversely as the square on  $FH$  is to the square on  $BD$  so is the area  $S$  to the circle  $ABCD$

But as the area  $S$  is to the circle  $ABCD$  so is the circle  $EFGH$  to some area less than the circle  $ABCD$

therefore also as the square on  $FH$  is to the square on  $BD$  so is the circle  $EFGH$  to some area less than the circle  $ABCD$  (v 11)

which was proved impossible

Therefore as the square on  $BD$  is to the square on  $FH$  so is not the circle  $ABCD$  to any area greater than the circle  $EFGH$

And it was proved that neither is it in that ratio to any area less than the circle  $EFGH$

therefore as the square on  $BD$  is to the square on  $FH$  so is the circle  $ABCD$  to the circle  $EFGH$

Therefore etc

Q E D

## LEMMA

I say that, the area  $S$  being greater than the circle  $EFGH$ , as the area  $S$  is to the circle  $ABCD$ , so is the circle  $EFGH$  to some area less than the circle  $ABCD$

For let it be contrived that, as the area  $S$  is to the circle  $ABCD$ , so is the circle  $EFGH$  to the area  $T$

I say that the area  $T$  is less than the circle  $ABCD$

For since, as the area  $S$  is to the circle  $ABCD$ , so is the circle  $EFGH$  to the area  $T$ ,

therefore, alternately, as the area  $S$  is to the circle  $EFGH$ , so is the circle  $ABCD$  to the area  $T$ . [v. 16]

But the area  $S$  is greater than the circle  $EFGH$ ,

therefore the circle  $ABCD$  is also greater than the area  $T$

Hence, as the area  $S$  is to the circle  $ABCD$ , so is the circle  $EFGH$  to some area less than the circle  $ABCD$  Q E D

## PROPOSITION 3

Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another, similar to the whole and having triangular bases, and into two equal prisms, and the two prisms are greater than the half of the whole pyramid

Let there be a pyramid of which the triangle  $ABC$  is the base and the point  $D$  the vertex,

I say that the pyramid  $ABCD$  is divided into two pyramids equal to one another, having triangular bases and similar to the whole pyramid, and into two equal prisms, and the two prisms are greater than the half of the whole pyramid

For let  $AB$ ,  $BC$ ,  $CA$ ,  $AD$ ,  $DB$ ,  $DC$  be bisected at the points  $E$ ,  $F$ ,  $G$ ,  $H$ ,  $K$ ,  $L$ , and let  $HE$ ,  $EG$ ,  $GH$ ,  $HK$ ,  $KL$ ,  $LH$ ,  $KF$ ,  $FG$  be joined

Since  $AE$  is equal to  $EB$ , and  $AH$  to  $HD$ ,

therefore  $EH$  is parallel to  $DB$  [vi. 2]

For the same reason

$HK$  is also parallel to  $AB$

Therefore  $HEBK$  is a parallelogram,

therefore  $HK$  is equal to  $EB$

But  $EB$  is equal to  $EA$ ,

therefore  $AE$  is also equal to  $HK$

But  $AH$  is also equal to  $HD$ ,

therefore the two sides  $EA$ ,  $AH$  are equal to the two sides  $KH$ ,  $HD$  respectively,

and the angle  $EAH$  is equal to the angle  $KHD$ ,

therefore the base  $EH$  is equal to the base  $KD$

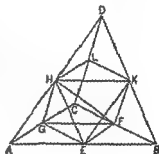
[i. 4]

Therefore the triangle  $AEH$  is equal and similar to the triangle  $HKD$

For the same reason

the triangle  $AHG$  is also equal and similar to the triangle  $HLD$

Now, since two straight lines  $EH$ ,  $HG$  meeting one another are parallel to two straight lines  $KD$ ,  $DL$  meeting one another, and are not in the same plane, they will contain equal angles [xi. 10]



Therefore the angle  $EHG$  is equal to the angle  $KDL$

And since the two straight lines  $EH, HG$  are equal to the two  $KD DL$  respectively,

and the angle  $EHG$  is equal to the angle  $KDL$ ,

therefore the base  $EG$  is equal to the base  $KL$ , [14]

therefore the triangle  $EHG$  is equal and similar to the triangle  $KDL$

For the same reason

the triangle  $AEG$  is also equal and similar to the triangle  $HKL$

Therefore the pyramid of which the triangle  $AEG$  is the base and the point  $H$  the vertex is equal and similar to the pyramid of which the triangle  $HKL$  is the base and the point  $D$  the vertex [xi Def 10]

And since  $HK$  has been drawn parallel to  $AB$  one of the sides of the triangle  $ADB$

the triangle  $ADB$  is equiangular to the triangle  $DHK$  [120]

and they have their sides proportional

therefore the triangle  $ADB$  is similar to the triangle  $DHK$  [xi Def 1]

For the same reason

the triangle  $DBC$  is also similar to the triangle  $DHL$  and the triangle  $ADC$  to the triangle  $DLH$

Now since the two straight lines  $BA AC$  meeting one another are parallel to the two straight lines  $KH HL$  meeting one another not in the same plane they will contain equal angles [xi 10]

Therefore the angle  $BAC$  is equal to the angle  $KHL$

And as  $BA$  is to  $AC$  so is  $KH$  to  $HL$

therefore the triangle  $ABC$  is similar to the triangle  $HKL$

Therefore also the pyramid of which the triangle  $ABC$  is the base and the point  $D$  the vertex is similar to the pyramid of which the triangle  $HKL$  is the base and the point  $D$  the vertex

But the pyramid of which the triangle  $HKL$  is the base and the point  $D$  the vertex was proved similar to the pyramid of which the triangle  $AEG$  is the base and the point  $H$  the vertex

Therefore each of the pyramids  $AEGH HKLD$  is similar to the whole pyramid  $ABCD$

Next since  $BF$  is equal to  $FC$

the parallelogram  $EBFG$  is double of the triangle  $GFC$

And since if there be two prisms of equal height and one have a parallelogram as base and the other a triangle and if the parallelogram be double of the triangle the prisms are equal [xi 39]

therefore the prism contained by the two triangles  $BAF EHG$  and the three parallelograms  $EBTG FBKH HKFG$  is equal to the prism contained by the two triangles  $GFC HKL$  and the three parallelograms  $KFCL LCGH HKFG$

And it is manifest that each of the prisms namely that in which the parallelogram  $EBFG$  is the base and the straight line  $HA$  is its opposite and that in which the triangle  $GFC$  is the base and the triangle  $HKL$  its opposite is greater than each of the pyramids of which the triangles  $ALG HKL$  are the bases and the points  $H D$  the vertices

inasmuch as if we join the straight lines  $EF FK$  the prism in which the parallelogram  $EBFG$  is the base and the straight line  $HK$  its opposite is greater than the pyramid of which the triangle  $EBF$  is the base and the point  $K$  the vertex

But the pyramid of which the triangle  $EBF$  is the base and the point  $K$  the vertex is equal to the pyramid of which the triangle  $AEG$  is the base and the point  $H$  the vertex,

for they are contained by equal and similar planes

Hence also the prism in which the parallelogram  $EBFG$  is the base and the straight line  $HK$  its opposite is greater than the pyramid of which the triangle  $AEG$  is the base and the point  $H$  the vertex.

But the prism in which the parallelogram  $EBFG$  is the base and the straight line  $HA$  its opposite is equal to the prism in which the parallelogram  $EBFG$  is the base and the triangle  $HKL$  its opposite, and the pyramid of which the triangle  $EBF$  is the base and the point  $K$  the vertex is equal to the pyramid of which the triangle  $HKL$  is the base and the point  $D$  the vertex.

Therefore the said two prisms are greater than the said two pyramids of which the triangles  $AEG$ ,  $HKL$  are the bases and the points  $H$ ,  $D$  the vertices.

Therefore the whole pyramid, of which the triangle  $ABC$  is the base and the point  $D$  the vertex, has been divided into two pyramids equal to one another and into two equal prisms, and the two prisms are greater than the half of the whole pyramid.

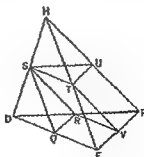
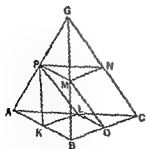
Q E D

#### PROPOSITION 4

If there be two pyramids of the same height which have triangular bases and each of them be divided into two pyramids equal to one another and similar to the whole, and into two equal prisms, then, as the base of the one pyramid is to the base of the other pyramid, so will all the prisms in the one pyramid be to all the prisms, being equal in multitude in the other pyramid.

Let there be two pyramids of the same height which have the triangular bases  $ABC$ ,  $DEF$  and vertices the points  $G$ ,  $H$ , and let each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms, [XII 3]

I say that, as the base  $ABC$  is to the base  $DEF$ , so are all the prisms in the pyramid  $ABCG$  to all the prisms, being equal in multitude, in the pyramid  $DEFH$ ,



For since  $BO$  is equal to  $OC$ , and  $AL$  to  $LC$ ,  
therefore  $LO$  is parallel to  $AB$ ,  
and the triangle  $ABC$  is similar to the triangle  $LOC$

For the same reason

the triangle  $DEF$  is also similar to the triangle  $RVF$

And, since  $BC$  is double of  $CO$ , and  $EF$  of  $FV$ ,

therefore, as  $BC$  is to  $CO$ , so is  $EF$  to  $FV$

And on  $BC$ ,  $CO$  are described the similar and similarly situated rectilinear figures  $ABC$ ,  $LOC$ ,

and on  $EF$ ,  $FV$  the similar and similarly situated figures  $DEF$ ,  $RVF$ ,  
therefore, as the triangle  $ABC$  is to the triangle  $LOC$ , so is the triangle  $DEF$  to the triangle  $RVF$ , (vi 22)

therefore alternately, as the triangle  $ABC$  is to the triangle  $DEF$ , so is the triangle  $LOC$  to the triangle  $RVF$  (v 16)

But, as the triangle  $LOC$  is to the triangle  $RVF$ , so is the prism in which the triangle  $LOC$  is the base and  $PMN$  its opposite, to the prism in which the triangle  $RVF$  is the base and  $STU$  its opposite, (Lemma following)

therefore also as the triangle  $ABC$  is to the triangle  $DEF$ , so is the prism in which the triangle  $LOC$  is the base and  $PMN$  its opposite, to the prism in which the triangle  $RVF$  is the base and  $STU$  its opposite

But as the said prisms are to one another so is the prism in which the parallelogram  $KBOL$  is the base and the straight line  $PM$  its opposite, to the prism in which the parallelogram  $QEV R$  is the base and the straight line  $ST$  its opposite (xi 39, cf xii 3)

Therefore also the two prisms, that in which the parallelogram  $KBOL$  is the base and  $PM$  its opposite and that in which the triangle  $LOC$  is the base and  $PMN$  its opposite are to the prisms in which  $QEV R$  is the base and the straight line  $ST$  its opposite and in which the triangle  $RVF$  is the base and  $STU$  its opposite in the same ratio (v 12)

Therefore also as the base  $ABC$  is to the base  $DEF$ , so are the said two prisms to the said two prisms

And similarly, if the pyramids  $PMNG$ ,  $STUH$  be divided into two prisms and two pyramids

as the base  $PMN$  is to the base  $STU$ , so will the two prisms in the pyramid  $PMNG$  be to the two prisms in the pyramid  $STUH$

But as the base  $PMN$  is to the base  $STU$ , so is the base  $ABC$  to the base  $DEF$ ,

for the triangles  $PMN$ ,  $STU$  are equal to the triangles  $LOC$ ,  $RVF$  respectively

Therefore also as the base  $ABC$  is to the base  $DEF$ , so are the four prisms to the four prisms

And similarly also if we divide the remaining pyramids into two pyramids and into two prisms then as the base  $ABC$  is to base the  $DEF$ , so will all the prisms in the pyramid  $ABCG$  be to all the prisms being equal in multitude, as the pyramid  $DEFH$  Q E D

#### LEMMA

But that as the triangle  $LOC$  is to the triangle  $RVF$  so is the prism in which the triangle  $LOC$  is the base and  $PMN$  its opposite, to the prism in which the triangle  $RVF$  is the base and  $STU$  its opposite, we must prove as follows

For in the same figure let perpendiculars be conceived drawn from  $G$ ,  $H$  to

the planes  $ABC$ ,  $DEF$ , these are of course equal because by hypothesis, the  
from  $G$  are cut  
they will be cut in the same ratios [XI 17]

And  $GC$  is bisected by the plane  $PMN$  at  $N$ ,  
therefore the perpendicular from  $G$  to the plane  $ABC$  will also be bisected by  
the plane  $PMN$

For the same reason  
the perpendicular from  $H$  to the plane  $DEF$  will also be bisected by the plane  
 $STU$

And the perpendiculars from  $G$ ,  $H$  to the planes  $ABC$ ,  $DEF$  are equal,  
therefore the perpendiculars from the triangles  $PMN$ ,  $STU$  to the planes  
 $ABC$ ,  $DEF$  are also equal

Therefore the prisms in which the triangles  $LOC$ ,  $RVF$  are bases, and  $PMN$ ,  
 $STU$  their opposites are of equal height

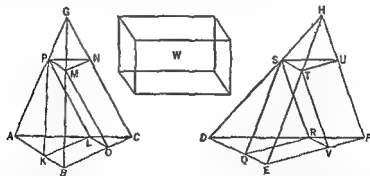
Hence also the parallelepipedal solids described from the said prisms are of  
equal height and are to one another as their bases, [XI 32]  
therefore their halves namely the said prisms, are to one another as the base  
 $LOC$  is to the base  $RVF$  Q E D

## PROPOSITION 5

*Pyramids which are of the same height and have triangular bases are to one another  
as the bases*

Let there be pyramids of the same height of which the triangles  $ABC$ ,  $DEF$   
are the bases and the points  $G$ ,  $H$  the vertices,

I say that as the base  $ABC$  is to the base  $DEF$ , so is the pyramid  $ABCG$  to  
the pyramid  $DEFH$



For, if the pyramid  $ABCG$  is not to the pyramid  $DEFH$  as the base  $ABC$  is  
to the base  $DEF$ ,

then as the base  $ABC$  is to the base  $DEF$  so will the pyramid  $ABCG$  be either  
to some solid less than the pyramid  $DEFH$  or to a greater

Let it first, be in that ratio to a less solid  $W$ , and let the pyramid  $DEFH$  be  
divided into two pyramids equal to one another and similar to the whole and  
into two equal prisms

then the two prisms are greater than the half of the whole pyramid [xii 3]

Again, let the pyramids arising from the division be similarly divided, and let this be done continually until there are left over from the pyramid  $DEFH$  some pyramids which are less than the excess by which the pyramid  $DEFH$  exceeds the solid  $W$  [x 1]

Let such be left, and let them be, for the sake of argument,  $DQRS$ ,  $STUH$ , therefore the remainders, the prisms in the pyramid  $DEFH$ , are greater than the solid  $W$

Let the pyramid  $ABCG$  also be divided similarly, and a similar number of times, with the pyramid  $DEFH$ , therefore, as the base  $ABC$  is to the base  $DEF$ , so are the prisms in the pyramid  $ABCG$  to the prisms in the pyramid  $DEFH$  [xii 4]

But, as the base  $ABC$  is to the base  $DEF$ , so also is the pyramid  $ABCG$  to the solid  $W$ ,

therefore also as the pyramid  $ABCG$  is to the solid  $W$ , so are the prisms in the pyramid  $ABCG$  to the prisms in the pyramid  $DEFH$ , [v 11]

therefore, alternately, as the pyramid  $ABCG$  is to the prisms in it, so is the solid  $W$  to the prisms in the pyramid  $DEFH$  [v 16]

But the pyramid  $ABCG$  is greater than the prisms in it, therefore the solid  $W$  is also greater than the prisms in the pyramid  $DEFH$  But it is also less

which is impossible

Therefore the prism  $ABCG$  is not to any solid less than the pyramid  $DEFH$  as the base  $ABC$  is to the base  $DEF$

Similarly it can be proved that neither is the pyramid  $DEFH$  to any solid less than the pyramid  $ABCG$  as the base  $DEF$  is to the base  $ABC$

I say next that neither is the pyramid  $ABCG$  to any solid greater than the pyramid  $DEFH$  as the base  $ABC$  is to the base  $DEF$

For if possible let it be in that ratio to a greater solid  $W$ , therefore inversely as the base  $DEF$  is to the base  $ABC$ , so is the solid  $W$  to the pyramid  $ABCG$

But, as the solid  $W$  is to the solid  $ABCG$ , so is the pyramid  $DEFH$  to some solid less than the pyramid  $ABCG$ , as was before proved, [xii 2 Lemma] therefore also as the base  $DEF$  is to the base  $ABC$ , so is the pyramid  $DEFH$  to some solid less than the pyramid  $ABCG$  [v 11]

which was proved absurd

Therefore the pyramid  $ABCG$  is not to any solid greater than the pyramid  $DEFH$  as the base  $ABC$  is to the base  $DEF$

But it was proved that neither is it in that ratio to a less solid

Therefore as the base  $ABC$  is to the base  $DEF$ , so is the pyramid  $ABCG$  to the pyramid  $DEFH$  Q E D

#### PROPOSITION 6

*Pyramids which are of the same height and have polygonal bases are to one another as the bases*

Let there be pyramids of the same height of which the polygons  $ABCDE$   $FGHKL$  are the bases and the points  $M$ ,  $N$  the vertices,

I say that as the base  $ABCDE$  is to the base  $FGHKL$ , so is the pyramid  $ABCDEM$  to the pyramid  $FGHKNL$

For let  $AC, AD, FH, FK$  be joined

Since then  $ABCM, ACDM$  are two pyramids which have triangular bases and equal height,

they are to one another as the bases, [XII 5]

therefore as the base  $ABC$  is to the base  $ACD$ , so is the pyramid  $ABCM$  to the pyramid  $ACDM$

And, *componendo* as the base  $ABCD$  is to the base  $ACD$ , so is the pyramid  $ABCDM$  to the pyramid  $ACDM$  [v 18]

But also as the base  $ACD$  is to the base  $ADE$  so is the pyramid  $ACDM$  to the pyramid  $ADEM$  [XII 5]

Therefore *ex aequali* as the base  $ABCD$  is to the base  $ADE$ , so is the pyramid  $ABCDM$  to the pyramid  $ADEM$  [v 22]

Py

so is the pyramid  $FGHKLN$  to the pyramid  $FGHN$

And since  $ADEM, FGHN$  are two pyramids which have triangular bases and equal height, therefore, as the base  $ADE$  is to the base  $FGH$ , so is the pyramid  $ADEM$  to the pyramid  $FGHN$  [XII 5]

But, as the base  $ADE$  is to the base  $ABCDE$ , so was the pyramid  $ADEM$  to the pyramid  $ABCDEM$

Therefore also *ex aequali* as the base  $ABCDE$  is to the base  $FGH$ , so is the pyramid  $ABCDEM$  to the pyramid  $FGHN$  [v 22]

But further, as the base  $FGH$  is to the base  $FGHKL$  so also was the pyramid  $FGHN$  to the pyramid  $FGHKLN$

Therefore also *ex aequali* as the base  $ABCDE$  is to the base  $FGHKL$  so is the pyramid  $ABCDEM$  to the pyramid  $FGHKLN$  [v 22]

Q E D

### PROPOSITION 7

Any prism which has a triangular base is divided into three pyramids equal to one another which have triangular bases

Let there be a prism in which the triangle  $ABC$  is the base and  $DEF$  its opposite

I say that the prism  $ABCDEF$  is divided into three pyramids equal to one another which have triangular bases

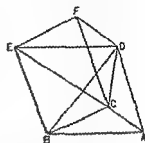
For let  $BD, EC, CD$  be joined



the point  $C$  the vertex

[XII 5]

But the pyramid of which the triangle  $DEB$  is the base and the point  $C$  the vertex is the same with the pyramid of which the triangle  $EBC$  is the base and the point  $D$  the vertex



for they are contained by the same planes

Therefore the pyramid of which the triangle  $ABD$  is the base and the point  $C$  the vertex is also equal to the pyramid of which the triangle  $EBC$  is the base and the point  $D$  the vertex

Again since  $FCBE$  is a parallelogram  
and  $CE$  is its diameter

the triangle  $CEF$  is equal to the triangle  $CBE$  [I 34]

Therefore also the pyramid of which the triangle  $BCE$  is the base and the point  $D$  the vertex is equal to the pyramid of which the triangle  $ECF$  is the base and the point  $D$  the vertex [XII 5]

But the pyramid of which the triangle  $BCE$  is the base and the point  $D$  the vertex was proved equal to the pyramid of which the triangle  $ABD$  is the base and the point  $C$  the vertex

therefore also the pyramid of which the triangle  $CEF$  is the base and the point  $D$  the vertex is equal to the pyramid of which the triangle  $ABD$  is the base and the point  $C$  the vertex

therefore the prism  $ABCDEF$  has been divided into three pyramids equal to one another which have triangular bases

And since the pyramid of which the triangle  $ABD$  is the base and the point  $C$  the vertex is the same with the pyramid of which the triangle  $CAB$  is the base and the point  $D$  the vertex

for they are contained by the same planes

while the pyramid of which the triangle  $ABD$  is the base and the point  $C$  the vertex was proved to be a third of the prism in which the triangle  $ABC$  is the base and  $DEF$  its opposite

therefore also the pyramid of which the triangle  $ABC$  is the base and the point  $D$  the vertex is a third of the prism which has the same base the triangle  $ABC$ , and  $DEF$  as its opposite

PROPOSITION 8 From this it is manifest that any pyramid is a third part of the prism which has the same base with it and equal height Q E D

### PROPOSITION 8

Similar pyramids which have triangular bases are in the triplicate ratio of their corresponding sides

Let there be similar and similarly situated pyramids of which the triangles  $ABC$   $DEF$  are the bases and the points  $G$   $H$  the vertices

I say that the pyramid  $ABCG$  has to the pyramid  $DEFH$  the ratio triplicate of that which  $BC$  has to  $EF$

For let the parallelepipedal solids  $BGML$   $EHQP$  be completed

Now since the pyramid  $ABCG$  is similar to the pyramid  $DEFH$ ,

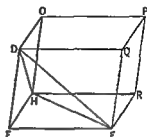
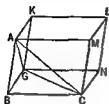
therefore the angle  $ABC$  is equal to the angle  $DEF$ ,

the angle  $GBC$  to the angle  $HFE$

and the angle  $ABG$  to the angle  $DEH$ ,

and, as  $AB$  is to  $DE$ , so is  $BC$  to  $EF$ , and  $BG$  to  $EH$

And since, as  $AB$  is to  $DE$ , so is  $BC$  to  $EF$ ,  
and the sides are proportional about equal angles,



therefore the parallelogram  $BM$  is similar to the parallelogram  $EQ$

For the same reason

$BN$  is also similar to  $ER$ , and  $BK$  to  $EO$ ,

therefore the three parallelograms  $MB$ ,  $BK$ ,  $BN$  are similar to the three  $EQ$ ,  $EO$ ,  $ER$

But the three parallelograms  $MB$ ,  $BK$ ,  $BN$  are equal and similar to their three opposites,

and the three  $EQ$ ,  $EO$ ,  $ER$  are equal and similar to their three opposites

[XI 24]

Therefore the solids  $BGML$ ,  $EHQP$  are contained by similar planes equal in multitude

Therefore the solid  $BGML$  is similar to the solid  $EHQP$

But similar parallelepipedal solids are in the triplicate ratio of their corresponding sides

[XI 33]

Therefore the solid  $BGML$  has to the solid  $EHQP$  the ratio triplicate of that which the corresponding side  $BC$  has to the corresponding side  $EF$

But, as the solid  $BGML$  is to the solid  $EHQP$ , so is the pyramid  $ABCG$  to the pyramid  $DEFH$ ,

inasmuch as the pyramid is a sixth part of the solid, because the prism which is

ing to the wholes

[VI 20]

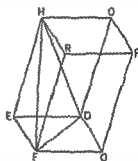
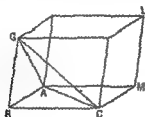
But the pyramid which has a triangular base is to the pyramid which has a triangular base in the triplicate ratio of the corresponding sides, therefore also the pyramid which has a polygonal base has to the pyramid which has a similar base the ratio triplicate of that which the side has to the side

### PROPOSITION 9

*In equal pyramids which have triangular bases the bases are reciprocally proportional to the heights and those pyramids in which the bases are reciprocally proportional to the heights are equal*

For let there be equal pyramids which have the triangular bases  $ABC$ ,  $DEF$  and vertices the points  $G$ ,  $H$ ,

I say that in the pyramids  $ABCG$ ,  $DEFH$  the bases are reciprocally proportional to the heights that is, as the base  $ABC$  is to the base  $DEF$ , so is the height of the pyramid  $DEFH$  to the height of the pyramid  $ABCG$



For let the parallelepipedal solids  $BGML$ ,  $EHQP$  be completed  
Now, since the pyramid  $ABCG$  is equal to the pyramid  $DEFH$ ,  
and the solid  $BGML$  is six times the pyramid  $ABCG$ ,  
and the solid  $EHQP$  six times the pyramid  $DEFH$ ,  
therefore the solid  $BGML$  is equal to the solid  $EHQP$

But in equal parallelepipedal solids the bases are reciprocally proportional to the heights [x 34]  
therefore as the base  $BM$  is to the base  $EQ$  so is the height of the solid  $EHQP$  to the height of the solid  $BGML$

But as the base  $BM$  is to  $IQ$  so is the triangle  $ABC$  to the triangle  $DEF$  [1 34]

Therefore also as the triangle  $ABC$  is to the triangle  $DEF$  so is the height of the solid  $IHPQ$  to the height of the solid  $BGML$  [1 11]

But the height of the solid  $EHQP$  is the same with the height of the pyramid  $DEFH$

and the height of the solid  $BGML$  is the same with the height of the pyramid  $ABCG$

therefore, as the base  $ABC$  is to the base  $DEF$  so is the height of the pyramid  $DEFH$  to the height of the pyramid  $ABCG$

Therefore in the pyramids  $ABCG$ ,  $DEFH$  the bases are reciprocally proportional to the heights

Next in the pyramids  $ABCG$ ,  $DEFH$  let the bases be reciprocally proportional to the heights,  
that is as the base  $ABC$  is to the base  $DEF$ , so let the height of the pyramid  $DEFH$  be to the height of the pyramid  $ABCG$ ,

I say that the pyramid  $ABCG$  is equal to the pyramid  $DEFH$

For with the same construction

and the height of the pyramid  $ABCG$  is the same with the height of the parallelepiped  $BGML$ ,  
therefore as the base  $BM$  is to the base  $EQ$  so is the height of the parallelepiped  $EHQP$  to the height of the parallelepiped  $BGML$

But those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal, [XI 34]  
therefore the parallelepipedal solid  $BGML$  is equal to the parallelepipedal solid  $EHQP$

And the pyramid  $ABCG$  is a sixth part of  $BGML$  and the pyramid  $DEFH$  a sixth part of the parallelepiped  $EHQP$ ,

therefore the pyramid  $ABCG$  is equal to the pyramid  $DEFH$

Therefore etc

Q E D

### PROPOSITION 10

Any cone is a third part of the cylinder which has the same base with it and equal height

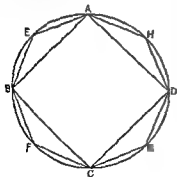
For let a cone have the same base namely the circle  $ABCD$ , with a cylinder and equal height,

I say that the cone is a third part of the cylinder, that is that the cylinder is triple of the cone

For if the cylinder is not triple of the cone, the cylinder will be either greater than triple or less than triple of the cone

First let it be greater than triple,  
and let the square  $ABCD$  be inscribed in the circle  $ABCD$ , [IV 6]  
then the square  $ABCD$  is greater than the half of the circle  $ABCD$

From the square  $ABCD$  let there be set up a prism of equal height with the cylinder



while parallelepipedal solids which are of the same height are to one another as their bases [xi 33]

therefore also the prism set up on the square  $ABCD$  is half of the prism set up from the square circumscribed about the circle  $ABCD$ ,

[cf xi 28 or xii 6 and 7, Por]

and the cylinder is less than the prism set up from the square circumscribed about the circle  $ABCD$

therefore the prism set up from the square  $ABCD$  and of equal height with the cylinder is greater than the half of the cylinder

Let the circumferences  $AB$   $BC$   $CD$   $DA$  be bisected at the points  $E$ ,  $F$ ,  $G$   $H$

and let  $AE$   $EB$   $BF$   $FC$   $CG$   $GD$   $DH$   $HA$  be joined,

then each of the triangles  $AEB$   $BFC$   $CGD$   $DHA$  is greater than the half of that segment of the circle  $ABCD$  which is about it as we proved before

[xii 2]

On each of the triangles  $AEB$   $BFC$   $CGD$   $DHA$  let prisms be set up of equal height with the cylinder

then each of the prisms so set up is greater than the half part of that segment of the cylinder which is about it

inasmuch as if we draw through the points  $E$   $F$   $G$   $H$  parallels to  $AB$ ,  $BC$ ,  $CD$   $DA$  complete the parallelograms on  $AB$   $BC$   $CD$   $DA$ , and set up from them parallelepipedal solids of equal height with the cylinder the prisms on the triangles  $AEB$   $BFC$   $CGD$   $DHA$  are halves of the several solids set up, and the segments of the cylinder are less than the parallelepipedal solids set up, hence also the prisms on the triangles  $AEB$   $BFC$   $CGD$   $DHA$  are greater than the half of the segments of the cylinder about them

Thus bisecting the circumferences that are left joining straight lines setting up on each of the triangles prisms of equal height with the cylinder,

and doing this continually

we shall leave some segments of the cylinder which will be less than the excess by which the cylinder exceeds the triple of the cone

[x 1]

Let such segments be left and let them be  $AE$   $EB$ ,  $BF$ ,  $FC$ ,  $CG$ ,  $GD$ ,  $DH$ ,  $HA$

therefore the remainder the prism of which the polygon  $AEBFCGDH$  is the base and the height is the same as that of the cylinder is greater than triple of the cone

But the prism of which the polygon  $AEBFCGDH$  is the base and the height the same as that of the cylinder is triple of the pyramid of which the polygon  $AEBFCGDH$  is the base and the vertex is the same as that of the cone

[xii 7, Por]

therefore also the pyramid of which the polygon  $AEBFCGDH$  is the base and the vertex is the same as that of the cone is greater than the cone which has the circle  $ABCD$  as base

But it is also less for it is enclosed by it

which is impossible

Therefore the cylinder is not greater than triple of the cone

I say next that neither is the cylinder less than triple of the cone,

For, if possible let the cylinder be less than triple of the cone, therefore inversely the cone is greater than a third part of the cylinder

Let the square  $ABCD$  be inscribed in the circle  $ABCD$ ,  
therefore the square  $ABCD$  is greater than the half of the circle  $ABCD$

Now let there be set up from the square  $ABCD$  a pyramid having the same vertex with the cone,

therefore the pyramid so set up is greater than the half part of the cone,  
seeing that, as we proved before, if we circumscribe a square about the circle,  
the square  $ABCD$  will be half of the square circumscribed about the circle,  
and if we set up from the squares parallelepipedal solids of equal height with  
the cone which are also called prisms the solid set up from the square  $ABCD$   
will be half of that set up from the square circumscribed about the circle,

for they are to one another as their bases [XI 32]

Hence also the thirds of them are in that ratio,  
therefore also the pyramid of which the square  $ABCD$  is the base is half of the  
pyramid set up from the square circumscribed about the circle

And the pyramid set up from the square about the circle is greater than the  
cone

for it encloses it

Therefore the pyramid of which the square  $ABCD$  is the base and the vertex  
is the same with that of the cone is greater than the half of the cone

Let the circumferences  $AB, BC, CD, DA$  be bisected at the points  $E, F, G, H$ ,

and let  $AE, EB, BF, FC, CG, GD, DH, HA$  be joined,  
therefore also the solid  $AEBFCGDH$  is greater than the  
cone which is about it

$CGD, DHA$  let pyramids be set up

therefore also each of the pyramids so set up is, in the same manner, greater  
than the half part of that segment of the cone which is about it

Thus by bisecting the circumferences that are left, joining straight lines,  
setting up on each of the triangles a pyramid which has the same vertex as the  
cone,

and doing this continually,

we shall leave some segments of the cone which will be less than the excess by  
which the cone exceeds the third part of the cylinder [X 1]

Let such be left and let them be the segments on  $AE, EB, BF, FC, CG, GD,$   
 $DH, HA$ ,

therefore the remainder, the pyramid of which the polygon  $AEBFCGDH$  is  
the base and the vertex the same with that of the cone, is greater than a third  
part of the cylinder

But the pyramid of which the polygon  $AEBFCGDH$  is the base and the ver-  
tex the same with that of the cone is a third part of the prism of which the  
polygon  $AEBFCGDH$  is the base and the height is the same with that of the  
cylinder,

therefore the prism of which the polygon  $AEBFCGDH$  is the base and the  
height is the same with that of the cylinder is greater than the cylinder of  
which the circle  $ABCD$  is the base

But it is also less, for it is enclosed by it

which is impossible

Therefore the cylinder is not less than triple of the cone

But it was proved that neither is it greater than triple,  
therefore the cylinder is triple of the cone;  
hence the cone is a third part of the cylinder.

Therefore etc

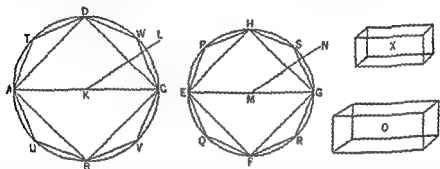
Q E D

### PROPOSITION II

*Cones and cylinders which are of the same height are to one another as their bases*

Let there be cones and cylinders of the same height,  
let the circles  $ABCD$ ,  $EFGH$  be their bases,  $KL$ ,  $MN$  their axes and  $AC$ ,  $EG$   
the diameters of their bases,

I say that as the circle  $ABCD$  is to the circle  $EFGH$ , so is the cone  $AL$  to  
the cone  $EN$



For, if not then as the circle  $ABCD$  is to the circle  $EFGH$ , so will the cone  
 $AL$  be either to some solid less than the cone  $EN$  or to a greater

First let it be in that ratio to a less solid  $O$ , and let the solid  $X$  be equal to  
that by which the solid  $O$  is less than the cone  $EN$ ,

therefore the cone  $EN$  is equal to the solids  $O$ ,  $X$ .

Let the square  $EFGH$  be inscribed in the circle  $EFGH$ ,

therefore the square is greater than the half of the circle

Let there be set up from the square  $EFGH$  a pyramid of equal height with  
the cone

therefore the pyramid so set up is greater than the half of the cone,  
inasmuch as if we circumscribe a square about the circle and set up from it a  
pyramid of equal height with the cone, the inscribed pyramid is half of the cir-  
cumscripted pyramid,

for they are to one another as their bases [XII 6]

Let the circ . . . R, S

Therefore each of the triangles  $HPL$ ,  $EQF$ ,  $FRG$ ,  $GSH$  is greater than the  
half of that segment of the circle which is about it

On each of the triangles  $HPL$ ,  $EQF$ ,  $FRG$ ,  $GSH$  let there be set up a pyra-  
mid of equal height with the cone,  
therefore, also each of the pyramids so set up is greater than the half of that  
segment of the cone which is about it

Thus, bisecting the circumferences which are left joining straight lines, set-

[x 1]

Let such be left, and let them be the segments on  $HP, PE, EQ, QF, FR, RG, GS, SH$ ;

sum

Si

$DT$   
 while, as the square on  $AC$  is to the square on  $EG$ , so is the circle  $ABCD$  to the circle  $EFGH$ , [xii 2]  
 therefore also, as the circle  $ABCD$  is to the circle  $EFGH$ , so is the polygon  $DTAUBVCW$  to the polygon  $HPEQFRGS$

But, as the circle  $ABCD$  is to the circle  $EFGH$ , so is the cone  $AL$  to the solid  $O$ ,

point  $N$  the vertex [xii 6]

Therefore also, as the cone  $AL$  is to the solid  $O$ , so is the pyramid of which the polygon  $DTAUBVCW$  is the base and the point  $L$  the vertex to the pyramid of which the polygon  $HPEQFRGS$  is the base and the point  $N$  the vertex, [v 11]

therefore, alternately, as the cone  $AL$  is to the pyramid in it, so is the solid  $O$  to the pyramid in the cone  $EN$  [v 16]

But the cone  $AL$  is greater than the pyramid in it,  
 therefore the solid  $O$  is also greater than the pyramid in the cone  $EN$

But it is also less

which is absurd

Therefore the cone  $AL$  is not to any solid less than the cone  $EN$  as the circle  $ABCD$  is to the circle  $EFGH$

Similarly we can prove that neither is the cone  $EN$  to any solid less than the cone  $AL$  as the circle  $EFGH$  is to the circle  $ABCD$

I say next that neither is the cone  $AL$  to any solid greater than the cone  $EN$  as the circle  $ABCD$  is to the circle  $EFGH$

But as the solid  $O$  is to the cone  $AL$ , so is the cone  $EN$  to some solid less than the cone  $AL$ ,

the



But it was proved that neither is it in this ratio to a less solid, therefore, as the circle  $ABCD$  is to the circle  $EFGH$ , so is the cone  $AL$  to the cone  $EN$

But as the cone is to the cone so is the cylinder to the cylinder, for each is triple of each, [AM 10]

Therefore also as the circle  $ABCD$  is to the circle  $EFGH$ , so are the cylinders on them which are of equal height

Therefore etc

Q E D

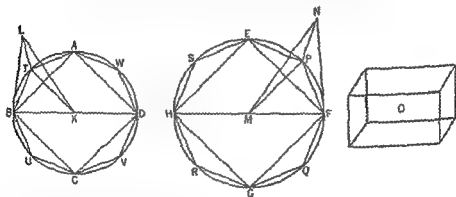
### PROPOSITION 12

*Similar cones and cylinders are to one another in the triplicate ratio of the diameters in their bases*

Let there be similar cones and cylinders

let the circles  $ABCD$ ,  $EFGH$  be their bases  $BD$   $FH$  the diameters of the bases, and  $KL$   $MN$  the axes of the cones and cylinders,

I say that the cone of which the circle  $ABCD$  is the base and the point  $L$  the vertex has to the cone of which the circle  $EFGH$  is the base and the point  $N$  the vertex the ratio triplicate of that which  $BD$  has to  $FH$



For if the cone  $ABCDL$  has not to the cone  $EFGHN$  the ratio triplicate of that which  $BD$  has to  $FH$  the cone  $ABCDL$  will have that triplicate ratio either to some solid less than the cone  $EFGHN$  or to a greater

First let it have that triplicate ratio to a less solid  $O$

Let the square  $FFGH$  be inscribed in the circle  $EFGH$  therefore the square  $FFGH$  is greater than the half of the circle  $EFGH$  [IV 6]

Now let there be set up on the square  $FFGH$  a pyramid having the same vertex with the cone

therefore the pyramid so set up is greater than the half part of the cone

Let the circumferences  $IF$   $IG$   $GH$   $HL$  be bisected at the points  $P$   $Q$   $R$   $S$  and let  $FP$   $PP$   $FQ$   $QG$   $GR$   $RH$   $HS$   $SE$  be joined

Therefore each of the triangles  $FPF$   $FQG$   $GRH$   $HSL$  is also greater than the half part of that segment of the circle  $LFGH$  which is about it

Now on each of the triangles  $FPF$   $FQG$   $GRH$ ,  $HSE$  let a pyramid be set up having the same vertex with the cone,

rt of

g up

we shall leave some segments in the cone which will be less than the excess by which the cone  $EFGHN$  exceeds the solid  $O$  [x 1]

Let such be left, and let them be the segments on  $LP$ ,  $PF$ ,  $FQ$ ,  $QG$ ,  $GR$ ,  $RH$ ,  $HS$ ,  $SE$ ,

the

ba.

similar and similarly situated to the polygon  $EPFQGRHS$ ,  
and let there be set up on the polygon  $ATBUCVDW$  a pyramid having the same vertex with the cone,

of the triangles containing the pyramid of which the polygon  $ATBUCVDW$  is the base and the point  $L$  the vertex let  $LBT$  be one,

and of the triangles containing the pyramid of which the polygon  $EPFQGRHS$  is the base and the point  $N$  the vertex let  $NFP$  be one,

and let  $KT$ ,  $MP$  be joined

Now, since the cone  $ABCDL$  is similar to the cone  $EFGHN$ ,  
therefore, as  $BD$  is to  $FH$ , so is the axis  $KL$  to the axis  $MN$  [xi Def 24]

But, as  $BD$  is to  $FH$ , so is  $BA$  to  $FM$ ,

therefore also as  $BK$  is to  $FM$ , so is  $KL$  to  $MN$

And, alternately, as  $BK$  is to  $KL$ , so is  $FM$  to  $MN$  [v 16]

And the sides are proportional about equal angles, namely the angles  $BAL$ ,  $FMN$ ,

therefore the triangle  $BKL$  is similar to the triangle  $FMN$  [vi 6]

Again, since, as  $BK$  is to  $KT$ , so is  $FM$  to  $MP$ ,

and they are about equal angles, namely the angles  $BKT$ ,  $FMP$ ,

inasmuch as, whatever part the angle  $BAT$  is of the four right angles at the centre  $K$ , the same part also is the angle  $FMP$  of the four right angles at the centre  $M$ ,

[vi 6]

while  $BK$  is equal to  $KT$ , and  $FM$  to  $PM$ ,

therefore, as  $TK$  is to  $KL$ , so is  $PM$  to  $MN$ ,

and the sides are proportional about equal angles, namely the angles  $TKL$ ,  $PMN$ , for they are right,

[vi 6]

and, owing to the similarity of the triangles  $BKT$ ,  $FMP$ ,

as  $KB$  is to  $BT$ , so is  $MF$  to  $FP$ ,

therefore, *ex aequali*, as  $LB$  is to  $BT$ , so is  $NF$  to  $FP$  [v 22]

Again, since, owing to the similarity of the triangles  $LTK$ ,  $NPM$ ,

as  $LT$  is to  $TK$ , so is  $NP$  to  $PM$ ,

and, owing to the similarity of the triangles  $TAB$ ,  $PMF$ ,

as  $KT$  is to  $TB$ , so is  $MP$  to  $PF$ ,

therefore, *ex aequali*, as  $LT$  is to  $TB$ , so is  $NP$  to  $PF$ . [v 22]

But it was also proved that, as  $TB$  is to  $BL$ , so is  $PF$  to  $FN$

Therefore, *ex aequali*, as  $TL$  is to  $LB$ , so is  $PN$  to  $NF$  [v 22]

Therefore in the triangles  $LTB$ ,  $NPF$  the sides are proportional,

therefore the triangles  $LTB$ ,  $NPF$  are equiangular, [vi 5]

hence they are also similar [vi Def 1]

Therefore the pyramid of which the triangle  $BKT$  is the base and the point  $L$  the vertex is also similar to the pyramid of which the triangle  $FMP$  is the base and the point  $N$  the vertex,

for they are contained by similar planes equal in multitude [xi Def 9]

But similar pyramids which have triangular bases are to one another in the triplicate ratio of their corresponding sides [xii 8]

Therefore the pyramid  $BKTL$  has to the pyramid  $FMPN$  the ratio triplicate of that which  $BA$  has to  $FM$

Similarly, by joining straight lines from  $A$ ,  $W$ ,  $D$ ,  $V$ ,  $C$ ,  $U$  to  $K$ , and from  $E$ ,  $S$ ,  $H$ ,  $R$ ,  $G$ ,  $Q$  to  $M$  and setting up on each of the triangles pyramids which have the same vertex with the cones,

we can prove that each of the similarly arranged pyramids will also have to each similarly arranged pyramid the ratio triplicate of that which the corresponding side  $BA$  has to the corresponding side  $FM$ , that is, which  $BD$  has to  $FH$

And as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [v 12]

therefore also as the pyramid  $BKTL$  is to the pyramid  $FMPN$ , so is the whole pyramid of which the polygon  $ATBUCVDW$  is the base and the point  $L$  the vertex to the whole pyramid of which the polygon  $EPFQGRHS$  is the base and the point  $N$  the vertex

hence also the pyramid of which  $ATBUCVDW$  is the base and the point  $L$  the vertex has to the pyramid of which the polygon  $EPFQGRHS$  is the base and the point  $N$  the vertex the ratio triplicate of that which  $BD$  has to  $FH$

But by hypothesis the cone of which the circle  $ABCD$  is the base and the point  $L$  the vertex has also to the solid  $O$  the ratio triplicate of that which  $BD$  has to  $FH$ ,

therefore as the cone of which the circle  $ABCD$  is the base and the point  $L$  the vertex is to the solid  $O$  so is the pyramid of which the polygon  $ATBUCVDW$  is the base and  $L$  the vertex to the pyramid of which the polygon  $EPFQGRHS$  is the base and the point  $N$  the vertex,

therefore, alternately as the cone of which the circle  $ABCD$  is the base and  $L$  the vertex is to the pyramid contained in it of which the polygon  $ATBUCVDW$  is the base and  $L$  the vertex so is the solid  $O$  to the pyramid of which the polygon  $EPFQGRHS$  is the base and  $N$  the vertex [v 16]

But the said cone is greater than the pyramid in it,

for it encloses it

Therefore the solid  $O$  is also greater than the pyramid of which the polygon  $EPFQGRHS$  is the base and  $N$  the vertex

But it is also less

which is impossible

Therefore the cone of which the circle  $ABCD$  is the base and  $L$  the vertex

has not to any solid less than the cone of which the circle  $EFGH$  is the base and the point  $N$  the vertex the ratio triplicate of that which  $BD$  has to  $FH$   
 solid less  
 $BD$   
 than the

But, as the solid  $O$  is to the cone  $ABCDL$ , so is the cone  $EFGHN$  to some solid less than the cone  $ABCDL$

Therefore the cone  $EFGHN$  also has to some solid less than the cone  $ABCDL$  the ratio triplicate of that which  $FH$  has to  $BD$

which was proved impossible

Therefore the cone  $ABCDL$  has not to any solid greater than the cone  $EFGHN$  the ratio triplicate of that which  $BD$  has to  $FH$

But it was proved that neither has it this ratio to a less solid than the cone  $EFGHN$

Therefore the cone  $ABCDL$  has to the cone  $EFGHN$  the ratio triplicate of that which  $BD$  has to  $FH$

But, as the cone is to the cone, so is the cylinder to the cylinder, for the cylinder which is on the same base as the cone and of equal height with it is triple of the cone, [XII 10]

therefore the cylinder also has to the cylinder the ratio triplicate of that which  $BD$  has to  $FH$

Therefore etc

Q E D

### PROPOSITION 13

If a cylinder be cut by a plane which is parallel to its opposite planes, then, as the cylinder is to the cylinder, so will the axis be to the axis

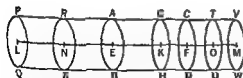
For let the cylinder  $AD$  be cut by the plane  $GH$  which is parallel to the opposite planes  $AB, CD$ ,

and let the plane  $GH$  meet the axis at the point  $K$ ,

I say that, as the cylinder  $BG$  is to the cylinder  $GD$ , so is the axis  $EK$  to the axis  $KF$

For let the axis  $EF$  be produced in both directions to the points  $L, M$ ,

and let there be set out any number whatever of axes  $EN, NL$  equal to the axis  $EK$ , and any number whatever  $FO, OM$  equal to  $FK$ ,



and let the cylinder  $PW$  on the axis  $LM$  be conceived of which the circles  $PQ, VW$  are the bases

Let planes be carried through the points  $N, O$  parallel to  $AB, CD$  and to the bases of the cylinder  $PW$ ,

and let them produce the circles  $RS, TU$  about the centres  $N, O$

Then, since the axes  $LN, NE, EK$  are equal to one another,

therefore <sup>11</sup>

[XII 11]

But

the cylinders  $QR, RB, BG$  are also equal to one another

Since then the axes  $LN, NE, EK$  are equal to one another,

and the cylinders  $QR, RB, BG$  are also equal to one another

and the multitude of the former is equal to the <sup>12</sup>

therefore, whatever multiple <sup>13</sup>

also will the cylinder  $QG$

For the same reason, <sup>14</sup> multiple the axis  $MK$  is of the axis  $KF$ , the same multiple also is the cylinder  $WG$  of the cylinder  $GD$ .

And, if the axis  $KL$  is equal to the axis  $KM$ , the cylinder  $QG$  will also be equal to the cylinder  $GW$ ,

if the axis is greater than the axis, the cylinder will also be greater than the cylinder,

and if less, less

Thus, there being four magnitudes, the axes  $EK, KF$  and the cylinders  $BG, GD$ ,

there have been taken equimultiples of the axis  $EK$  and of the cylinder  $BG$ , namely the axis  $LK$  and the cylinder  $QG$ ,

and equimultiples of the axis  $KF$  and of the cylinder  $GD$ , namely the axis  $AM$  and the cylinder  $GW$ ,

and it has been proved that,

if the axis  $KL$  is in excess of the axis  $KM$  the cylinder  $QG$  is also in excess of the cylinder  $GW$ ,

if equal, equal,

and if less, less

Therefore, as the axis  $EK$  is to the axis  $KF$ , so is the cylinder  $BG$  to the cylinder  $GD$

[V Def 5]

Q.E.D.

#### PROPOSITION 14

*Cones and cylinders which are on equal bases are to one another as their heights*

For let  $EB, FD$  be cylinders on equal bases, the circles  $AB, CD$ ,

I say that as the cylinder  $EB$  is to the cylinder  $FD$ , so is the axis  $GH$  to the axis  $KL$

For let the axis  $KL$  be produced to the point  $N$ ,

let  $LN$  be made equal to the axis  $GH$ ,  
and let the cylinder  $CM$  be conceived about  $LN$  as axis

Since then the cylinders  $EB, CM$  are of the same height, they are to one another as their bases

[XII 11]

But the bases are equal to one another

therefore the cylinders  $EB, CM$  are also equal

And, since the cylinder  $FM$  has been cut by the plane  $CD$  which is parallel to its opposite planes

therefore, as the cylinder  $CM$  is to the cylinder  $FD$ , so is the axis  $LN$  to the axis  $KL$

[XII 13]



But the cylinder  $CM$  is equal to the cylinder  $EB$ ,  
and the axis  $LN$  to the axis  $GH$ ,  
therefore as the cylinder  $EB$  is to the cylinder  $FD$ , so is the axis  $GH$  to the axis  
 $AL$ .

But, as the cylinder  $EB$  is to the cylinder  $FD$ , so is the cone  $ABG$  to the cone  $CDK$  [XII. 10]

Therefore also, as the axis  $GH$  is to the axis  $KL$ , so is the cone  $ABG$  to the cone  $CDK$  and the cylinder  $EB$  to the cylinder  $FD$  Q E D

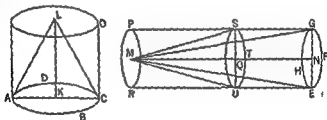
### PROPOSITION 15

*In equal cones and cylinders the bases are reciprocally proportional to the heights, and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal*

Let there be equal cones and cylinders of which the circles  $ABCD$ ,  $EFGH$  are the bases.

let  $AC, EG$  be the diameters of the bases,  
and  $KL, MN$  the axes, which are also the heights of the cones or cylinders,  
let the cylinders  $AO, EP$  be completed

I say that in the cylinders  $AO, EP$  the bases are reciprocally proportional to the heights,  
that is, as the base  $ABCD$  is to the base  $EFGH$ , so is the height  $MN$  to the height  $KL$ .



For the height  $LK$  is either equal to the height  $MN$  or not equal

First, let it be equal

Now the cylinder  $AO$  is also equal to the cylinder  $EP$

But cones and cylinders which are of the same height are to one another as their bases. [xii 11]

therefore the base  $ABCD$  is also equal to the base  $EFGH$

to the base  $EFGH$ , so in the

from the height  $MA$  let  $QN$  be cut off equal to  $KL$ ,  
through the point  $Q$  let the cylinder  $EP$  be cut by the plane  $TUS$  parallel to  
the planes of the circles  $EPGH$ ,  $RP$   
and let the cylinder  $ES$  be conceived erected from the circle  $EFGH$  as base and  
with height  $NQ$

Now, since the cylinder  $AO$  is equal to the cylinder  $EP$ ,  
therefore as the cylinder  $AO$  is to the cylinder  $ES$ , so is the cylinder  $EP$  to the  
cylinder  $ES$ .

But as the cylinder  $AO$  is to the cylinder  $ES$  so is the base  $ABCD$  to the base  $EFGH$ ,

for the

[xi 11]

and as

to the height

$QV$

for the cylinder  $AO$  has been cut by a plane which is parallel to its opposite planes

[xii 13]

Therefore also as the base  $ABCD$  is to the base  $EFGH$ , so is the height  $MY$  to the height  $QV$

[v 11]

But the height  $QV$  is equal to the height  $KL$ , therefore as the base  $ABCD$  is to the base  $EFGH$  so is the height  $MY$  to the height  $KL$

Therefore in the cylinders  $AO$   $EP$  the bases are reciprocally proportional to the heights

Next in the cylinders  $AO$   $EP$  let the bases be reciprocally proportional to the heights

that is as the base  $ABCD$  is to the base  $EFGH$  so let the height  $MY$  be to the height  $KL$

I say that the cylinder  $AO$  is equal to the cylinder  $EP$

For with the same construction

since as the base  $ABCD$  is to the base  $EFGH$  so is the height  $MY$  to the height  $KL$

while the height  $KL$  is equal to the height  $QV$

therefore as the base  $ABCD$  is to the base  $EFGH$  so is the height  $MY$  to the height  $QV$

But as the base  $ABCD$  is to the base  $EFGH$  so is the cylinder  $AO$  to the cylinder  $ES$

for they are of the same height

[xi 11]

and as the height  $MY$  is to  $QV$  so is the cylinder  $EP$  to the cylinder  $ES$

[xi 11]

therefore as the cylinder  $AO$  is to the cylinder  $ES$  so is the cylinder  $EP$  to the cylinder  $ES$

[v 11]

Therefore the cylinder  $AO$  is equal to the cylinder  $EP$

[v 9]

And the same is true for the cones also

Q E D

### PROPOSITION 16

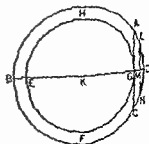
Given two circles about the same centre to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle

Let  $ABCD$   $EFGH$  be the two given circles about the same centre  $K$

thus it is required to inscribe in the greater circle  $ABCD$  an equilateral polygon with an even number of sides which does not touch the circle  $EFGH$

For let the straight line  $BKD$  be drawn through the centre  $K$

and from the point  $G$  let  $GA$  be drawn at right angles to the straight line  $BD$  and carried through to  $C$ ,



therefore  $AC$  touches the circle  $EFGH$  [III 16, Por.]

Then, bisecting the circumference  $BAD$ , bisecting the half of it, and doing this continually, we shall leave a circumference less than  $AD$  [5.1]

Let such be left, and let it be  $LD$ ,  
from  $L$  let  $LM$  be drawn perpendicular to  $BD$  and carried through to  $N$ ,  
and let  $LD$ ,  $DN$  be joined,  
therefore  $LD$  is equal to  $DN$  [III 3, 1 4]

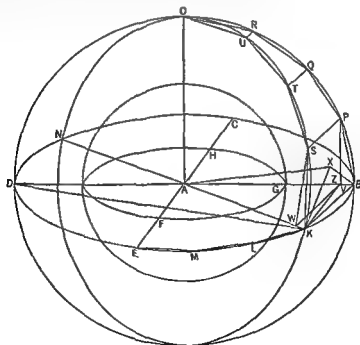
Now, since  $LN$  is parallel to  $AC$ ,  
and  $AC$  touches the circle  $EFGH$ ,  
therefore  $LN$  does not touch the circle  $EFGH$ ,  
therefore  $LD, DN$  are far from touching the circle  $EFGH$

If then we fit into the circle  $ABCD$  straight lines equal to the straight line  $LD$  and placed continuously, there will be inscribed in the circle  $ABCD$  an equilateral polygon with an even number of sides which does not touch the lesser circle  $EFGH$  Q E F

### PROPOSITION 17

Given two spheres about the same centre, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface

Let two spheres be conceived about the same centre  $A$ , thus it is required to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface



Let the spheres be cut by any plane through the centre,  
then the sections will be circles.



inasmuch as the sphere was produced by the diameter remaining fixed and the semicircle being carried round it, [xi Def 14]  
hence in whatever position we conceive the semicircle to be, the plane carried through it will produce a circle on the circumference of the sphere

And it is manifest that this circle is the greatest possible,  
inasmuch as the diameter of the sphere which is of course the diameter both of the semicircle and of the circle is greater than all the straight lines drawn across in the circle or the sphere

Let then  $BCDE$  be the circle in the greater sphere,  
and  $FGH$  the circle in the lesser sphere,  
let two diameters in them  $BD$   $CE$  be drawn at right angles to one another,  
then given the two circles  $BCDE$ ,  $FGH$  about the same centre, let there be inscribed in the greater circle  $BCDE$  an equilateral polygon with an even number of sides which does not touch the lesser circle  $FGH$

let  $BA$   $AL$   $LM$   $ME$  be its sides in the quadrant  $BE$ ,  
let  $A$  be joined and carried through to  $N$   
let  $AO$  be set up from the point  $A$  at right angles to the plane of the circle  $BCDE$  and let it meet the surface of the sphere at  $O$   
and through  $O$  and each of the straight lines  $BD$   $AN$  let planes be carried they will then make greatest circles on the surface of the sphere for the reason stated

Let them make such  
and in them let  $BOD$   $AOV$  be the semicircles on  $BD$   $AN$   
Now since  $OA$  is at right angles to the plane of the circle  $BCDE$ ,  
therefore all the planes through  $OA$  are also at right angles to the plane of the circle  $BCDE$  [xi 18]  
hence the semicircles  $BOD$   $AOV$  are also at right angles to the plane of the circle  $BCDE$

And since the semicircles  $BED$   $BOD$   $AON$  are equal  
for they are on the equal diameters  $BD$   $AN$   
therefore the quadrants  $BE$   $BO$   $AO$  are also equal to one another  
Therefore there are as many straight lines in the quadrants  $BO$   $AO$  equal to the straight lines  $BA$   $AL$   $LM$   $ME$  as there are sides of the polygon in the quadrant  $BE$

Let them be inscribed and let them be  $BP$   $PQ$   $QR$   $RO$  and  $AS$ ,  $ST$   $TU$   $UO$

let  $SP$   $TQ$   $UR$  be joined  
and from  $P$  let perpendiculars be drawn to the plane of the circle  $BCDE$  [xi 11]

these will fall on  $BD$   $AV$  the common sections of the planes  
inasmuch as the planes of  $BOD$   $AOV$  are also at right angles to the plane of the circle  $BCDE$  [xi Def 4]

Let them be  $PI$   $SH$  and let them be  $PI$   $SH$   
and let  $BI$  be joined

Now since in the equal semicircles  $BOD$   $AOV$  equal straight lines  $BP$ ,  $KS$  have been cut off

and the perpendiculars  $PI$   $SH$  have been drawn  
therefore  $BI$  is equal to  $SH$  and  $BI$  to  $AK$  [iii 27 i 26]  
But the whole  $BI$  is also equal to the whole  $AK$

therefore the remainder  $VA$  is also equal to the remainder  $WA$ ,

therefore, as  $BV$  is to  $VA$ , so is  $KW$  to  $WA$ ;

therefore  $WV$  is parallel to  $KB$  [xi 2]

And, since each of the straight lines  $PV$ ,  $SW$  is at right angles to the plane of the circle  $BCDE$ ,

therefore  $PV$  is parallel to  $SW$  [xi 6]

But it was also proved equal to it;

therefore  $WV$ ,  $SP$  are also equal and parallel [i 33]

And, since  $WV$  is parallel to  $SP$ ,

while  $WV$  is parallel to  $KB$ ,

therefore  $SP$  is also parallel to  $KB$  [xi 9]

And  $BP$ ,  $KS$  join their extremities;

therefore the quadrilateral  $KBPS$  is in one plane,

inasmuch as, if two straight lines be parallel, and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallels [xi 7]

For the same reason

each of the quadrilaterals  $SPQT$ ,  $TQRU$  is also in one plane

But the triangle  $URO$  is also in one plane [xi 2]

vertex

And, if we make the same construction in the case of each of the sides  $KL$ ,  $LM$ ,  $ME$  in the case of  $BK$ , and further, in the case of the remaining three quadrants,

on which the circle  $FGH$  is

Let  $AX$  be drawn from the point  $A$  perpendicular to the plane of the quadrilateral  $KBPS$ , and let it meet the plane at the point  $X$ , [xi 11]

let  $XB$ ,  $KK$  be joined

Then

And, since  $AB$  is equal to  $AK$ ,

the square on  $AB$  is also equal to the square on  $AK$

And the squares on  $AX$ ,  $XB$  are equal to the square on  $AB$ ,

[17]

[1]

Let the square on  $AX$  be subtracted from each,  
therefore the remainder the square on  $BX$ , is equal to the remainder, the square on  $AK$ ,

therefore  $BX$  is equal to  $AK$

Similarly we can prove that the straight lines joined from  $X$  to  $P, S$  are equal to each of the straight lines  $B\lambda, XK$

Therefore the circle described with centre  $X$  and distance one of the straight lines  $XB, XK$  will pass through  $P, S$  also,

and  $KBPS$  will be a quadrilateral in a circle

Now, since  $KB$  is greater than  $BV$ ,

while  $BV$  is equal to  $SP$ ,

therefore  $KB$  is greater than  $SP$

But  $KB$  is equal to each of the straight lines  $KS, BP$ ,

therefore each of the straight lines  $KS, BP$  is greater than  $SP$ .

And, since  $KBPS$  is a quadrilateral in a circle,

and  $KB, BP, KS$  are equal, and  $PS$  less,

and  $BX$  is the radius of the circle,

therefore the square on  $KB$  is greater than double of the square on  $BX$ .

Let  $KZ$  be drawn from  $K$  perpendicular to  $BV$

Then, since  $BD$  is less than double of  $DZ$ ,

and, as  $BD$  is to  $DZ$ , so is the rectangle  $DB, BZ$  to the rectangle  $DZ, ZB$ ,

if a square be described upon  $BZ$  and the parallelogram on  $ZD$  be completed, then the rectangle  $DB, BZ$  is also less than double of the rectangle  $DZ, ZB$

And, if  $KD$  be joined,

the rectangle  $DB, BZ$  is equal to the square on  $BK$ ,

and the rectangle  $DZ, ZB$  equal to the square on  $KZ$ , [III 31, VI 8 and Por.]

therefore the square on  $KB$  is less than double of the square on  $KZ$

But the square on  $KB$  is greater than double of the square on  $BX$ ,

therefore the square on  $KZ$  is greater than the square on  $BX$

And since  $BA$  is equal to  $KA$ ,

the square on  $BA$  is equal to the square on  $KA$

And the squares on  $B\lambda, XA$  are equal to the square on  $BA$ ,

and the squares on  $KZ, ZA$  equal to the square on  $KA$ , [I 47]

therefore the squares on  $BX, XA$  are equal to the squares on  $KZ, ZA$

and of these the square on  $KZ$  is greater than the square on  $BX$ ,

therefore the remainder, the square on  $ZA$ , is less than the square on  $XA$

Therefore  $AA$  is greater than  $AZ$ ,

therefore  $AA$  is much greater than  $AG$

And  $AX$  is the perpendicular on one base of the polyhedron,

and  $AG$  on the surface of the lesser sphere,

hence the polyhedron will not touch the lesser sphere on its surface

Therefore given two spheres about the same centre, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere at its surface

Q. E. F.

PROPOSITION But if in another sphere also a polyhedral solid be inscribed similar to the solid in the sphere  $BCDE$ ,

the polyhedral solid in the sphere  $BCDE$  has to the polyhedral solid in the other sphere the ratio triplicate of that which the diameter of the sphere  $BCDE$  has to the diameter of the other sphere

For, the solids being divided into their pyramids similar in multitude and arrangement the pyramids will be similar

But similar pyramids are to one another in the triplicate ratio of their corresponding sides

[XI 8, Por.]

therefore the ratio of the whole polyhedral solid in the sphere about  $A$  as centre has to the radius of the other sphere

Similarly

each similar  
cate of that

And, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [v 12]  
hence the whole polyhedral solid in the sphere about  $A$  as centre has to the whole polyhedral solid in the other sphere the ratio triplicate of that which  $AB$  has to the radius of the other sphere, that is, of that which the diameter  $BD$  has to the diameter of the other sphere Q E D

## PROPOSITION 18

Spheres are to one another in the triplicate ratio of their respective diameters

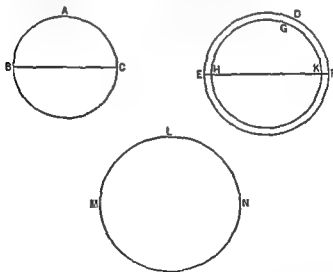
Let the spheres  $ABC$ ,  $DEF$  be conceived,

and let  $BC$ ,  $EF$  be their diameters,

I say that the ratio of the sphere  $ABC$  to the sphere  $DEF$  is triplicate of that which  $BC$  has to  $EF$ .

First, let it have that ratio to a less sphere  $GHI$ ,  
then the sphere  $ABC$  will have either to some less sphere than the sphere  $DEF$ ,  
or to a greater, the ratio triplicate of that which  $BC$  has to  $EF$

First, let it have that ratio to a less sphere  $GHI$ ,



Let the sphere  $ABC$  have the same ratio with  $GHI$ ,

the whole polyhedral solid which does

[xii 17]

and let there also be inscribed in the sphere  $ABC$  a polyhedral solid similar to the polyhedral solid in the sphere  $DEF$ ,  
therefore the polyhedral solid in  $ABC$  has to the polyhedral solid in  $DEF$  the ratio triplicate of that which  $BC$  has to  $EF$  [XII 17 Por]

But the sphere  $ABC$  also has to the sphere  $GHK$  the ratio triplicate of that which  $BC$  has to  $EF$ ,  
therefore as the sphere  $ABC$  is to the sphere  $GHK$ , so is the polyhedral solid in the sphere  $ABC$  to the polyhedral solid in the sphere  $DEF$ ,  
and, alternately as the sphere  $ABC$  is to the polyhedron in it, so is the sphere  $GHA$  to the polyhedral solid in the sphere  $DEF$  [v 16]

But the sphere  $ABC$  is greater than the polyhedron in it,  
therefore the sphere  $GHA$  is also greater than the polyhedron in the sphere  $DEF$

But it is also less,

for it is enclosed by it

Therefore the sphere  $ABC$  has not to a less sphere than the sphere  $DEF$  the ratio triplicate of that which the diameter  $BC$  has to  $EF$

Similarly we can prove that neither has the sphere  $DEF$  to a less sphere than the sphere  $ABC$  the ratio triplicate of that which  $EF$  has to  $BC$

I say next that neither has the sphere  $ABC$  to any greater sphere than the sphere  $DEF$  the ratio triplicate of that which  $BC$  has to  $EF$

For if possible let it have that ratio to a greater  $LMN$ ,  
therefore inversely, the sphere  $LMN$  has to the sphere  $ABC$  the ratio triplicate of that which the diameter  $EF$  has to the diameter  $BC$

But inasmuch as  $LMN$  is greater than  $DEF$   
therefore as the sphere  $LMA$  is to the sphere  $ABC$  so is the sphere  $DEF$  to some less sphere than the sphere  $ABC$  as was before proved [XII 2 Lemma]

Therefore the sphere  $DEF$  also has to some less sphere than the sphere  $ABC$  the ratio triplicate of that which  $EF$  has to  $BC$

which was proved impossible

Therefore the sphere  $ABC$  has not to any sphere greater than the sphere  $DEF$  the ratio triplicate of that which  $BC$  has to  $EF$

But it was proved that neither has it that ratio to a less sphere  
Therefore the sphere  $ABC$  has to the sphere  $DEF$  the ratio triplicate of that which  $BC$  has to  $EF$  Q E D

# BOOK THIRTEEN

## PROPOSITIONS

### PROPOSITION 1

*If a straight line be cut in extreme and mean ratio, the square on the greater segment added to the half of the whole is five times the square on the half*

For let the straight line  $AB$  be cut in extreme and mean ratio at the point  $C$ ,  
and let  $AC$  be the greater segment,

let the straight line  $AD$  be produced in a straight line with  $CA$ ,  
and let  $AD$  be made half of  $AB$ ,

I say that the square on  $CD$  is five times the square on  $AD$

For let the squares  $AE$ ,  $DF$  be described on  $AB$ ,  $DC$ ,

and let the figure in  $DF$  be drawn,

let  $FC$  be carried through to  $G$

Now, since  $AB$  has been cut in extreme and mean ratio at  $C$ ,

therefore the rectangle  $AB$   $BC$  is equal to the square on  $AC$  [vi Def 3, vi 17]

And  $CE$  is the rectangle  $AB$   $BC$  and  $FH$  the square on  $AC$ ,

therefore  $CE$  is equal to  $FH$

And since  $BA$  is double of  $AD$ ,

while  $BA$  is equal to  $KA$ , and  $AD$  to  $AH$ ,

therefore  $KA$  is also double of  $AH$

But, as  $KA$  is to  $AH$ , so is  $CK$  to  $CH$ , [vi 1]

therefore  $CK$  is double of  $CH$

But  $LH$ ,  $HC$  are also double of  $CH$

Therefore  $KC$  is equal to  $LH$ ,  $HC$

But  $CE$  was also proved equal to  $HF$ ,

therefore the whole square  $AE$  is equal to the gnomon  $MNO$

And, since  $BA$  is double of  $AD$ ,

the square on  $BA$  is quadruple of the square on  $AD$ ,

that is,  $AE$  is quadruple of  $DH$

P . . .

And  $DF$  is the square on  $DC$ , and  $AP$  the square on  $DA$ ,

therefore the square on  $CD$  is five times the square on  $DA$

Therefore etc

Q E D

### PROPOSITION 2

If the square on a straight line be five times the square on a segment of it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line

For let the square on the straight line  $AB$  be five times the square on the segment  $AC$  of it,

and let  $CD$  be double of  $AC$ .

I say that, when  $CD$  is cut in extreme and mean ratio, the greater segment is  $CB$

Let the squares  $AF$ ,  $CG$  be described on  $AB$ ,  $CD$  respectively,

let the figure in  $AF$  be drawn,

and let  $BE$  be drawn through

Now, since the square on  $BA$  is five times the square on  $AC$ ,

$AF$  is five times  $AH$

Therefore the gnomon  $MNO$  is quadruple of  $AH$

And, since  $DC$  is double of  $CA$ ,  
therefore the square on  $DC$  is quadruple of the  
square on  $CA$ , that is,  $CQ$  is quadruple of  $AH$

But the gnomon  $MNO$  was also proved quadruple of  $AH$ .

therefore the gnomon  $MNO$  is equal to  $CG$

And, since  $DC$  is double of  $CA$ ,

while  $DC$  is equal to  $CH$ , and  $AC$  to  $CH$ ,

therefore  $KB$  is also double of  $BH$

But  $LH, HB$  are also double of  $HB$ .

therefore  $KB$  is equal to  $LH$ ,  $HB$

But the whole gnomon  $MNO$  was also proved equal to the whole  $CG$ ,

therefore the remainder  $HF$  is equal to  $BG$

And  $BG$  is the rectangle  $\angle D, DB$

for  $CD$  is equal to  $DG$ .

and  $HF$  is the square on  $CB$ .

therefore the rectangle  $CD, DB$  is equal to the square on  $CB$

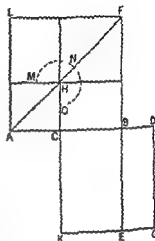
Therefore as  $DC$  is to  $CB$  so is  $CB$  to  $BD$

But  $DC$  is greater than  $CB$ .

therefore  $\angle B$  is also greater than  $\angle D$

Therefore when the straight line  $CD$  is cut in extreme and mean ratio,  $CB$  is the greater segment.

Therefore etc



[VI 1]

Q X D

### LEMMA

That the double of  $AC$  is greater than  $BC$  is to be proved thus

If not let  $BC$  be if possible, double of  $CA$

Therefore the square on  $BC$  is quadruple of the square on  $CA$ .

therefore the squares on  $BC$ ,  $CA$  are five times the square on  $CA$

But, by hypothesis the square on  $BA$  is also five times the square on  $CA$ ,

therefore the square on  $BA$  is equal to the squares on  $BC$ ,  $CA$  ·  
which is impossible

(II 4)

Therefore  $CB$  is not double of  $AC$

Similarly we can prove that neither is a straight line less than  $CB$  double of  $CA$ ,

for the absurdity is much greater.

Therefore the double of  $AC$  is greater than  $CB$

Q E D

## PROPOSITION 3

*If a straight line be cut in extreme and mean ratio, the square on the lesser segment added to the half of the greater segment is five times the square on the half of the greater segment*

For let any straight line  $AB$  be cut in extreme and mean ratio at the point  $C$ ,  
let  $AC$  be the greater segment,  
and let  $AC$  be bisected at  $D$ ,

I say that the square on  $BD$  is five times the square on  $DC$

For let the square  $AE$  be described on  $AB$ ,  
and let the figure be drawn double

Since  $AC$  is double of  $DC$ ,  
therefore the square on  $AC$  is quadruple of the square on  $DC$ ,

that is,  $RS$  is quadruple of  $FG$

And, since the rectangle  $AB$ ,  $BC$  is equal to the square on  $AC$ ,

and  $CE$  is the rectangle  $AB$ ,  $BC$ ,  
therefore  $CE$  is equal to  $RS$

But  $RS$  is quadruple of  $FG$ ,  
therefore  $CE$  is also quadruple of  $FG$

Again, since  $AD$  is equal to  $DC$ ,  
 $HK$  is also equal to  $KF$

Hence the square  $GF$  is also equal to the square  $HL$

Therefore  $GK$  is equal to  $KL$ , that is  $MN$  to  $NE$ ,

hence  $MF$  is also equal to  $FE$

But  $MF$  is equal to  $CG$ ,  
therefore  $CG$  is also equal to  $FE$

Let  $CN$  be added to each,  
therefore the gnomon  $OPQ$  is equal to  $CE$

But  $CE$  was proved quadruple of  $GF$ ,  
therefore the gnomon  $OPQ$  is also quadruple of  $GF$

And  $DN$  is the square on  $DB$ , and  $GF$  the square on  $DC$   
Therefore the square on  $DB$  is five times the square on  $DC$  Q E D

## PROPOSITION 4

*If a straight line be cut in extreme and mean ratio the square on the whole and the square on the lesser segment together are triple of the square on the greater segment*

Let  $AB$  be a straight line,



let it be cut in extreme and mean ratio at  $C$ , and let  $AC$  be the greater segment,

I say that the squares on  $AB$ ,  $BC$  are triple of the square on  $CA$

For let the square  $ADEB$  be described on  $AB$ ,  
and let the figure be drawn

Since, then,  $AB$  has been cut in extreme and mean ratio at  $C$ ,

and  $AC$  is the greater segment,  
therefore the rectangle  $AB$ ,  $BC$  is equal to the square on  $AC$  [vi Def 3, vi 17]

And  $AK$  is the rectangle  $AB$ ,  $BC$ , and  $HG$  the square on  $AC$ ,

therefore  $AK$  is equal to  $HG$

And, since  $AF$  is equal to  $FE$ ,

let  $CK$  be added to each,

therefore the whole  $AK$  is equal to the whole  $CE$ ,

therefore  $AK$ ,  $CE$  are double of  $AK$

But  $AK$ ,  $CE$  are the gnomon  $LMN$  and the square  $CK$ ,

therefore the gnomon  $LMN$  and the square  $CK$  are double of  $AK$

But further  $AK$  was also proved equal to  $HG$ ,

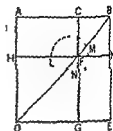
therefore the gnomon  $LMN$  and the squares  $CK$ ,  $HG$  are triple of the square  $HG$

And the gnomon  $LMN$  and the squares  $CK$ ,  $HG$  are the whole square  $AE$  and  $CK$ , which are the squares on  $AB$ ,  $BC$ ,

while  $HG$  is the square on  $AC$

Therefore the squares on  $AB$ ,  $BC$  are triple of the square on  $AC$

Q E D



### PROPOSITION 5

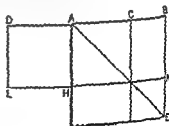
If a straight line be cut in extreme and mean ratio and there be added to it a straight line equal to the greater segment the whole straight line has been cut in extreme and mean ratio, and the original straight line is the greater segment

For let the straight line  $AB$  be cut in extreme and mean ratio at the point  $C$ ,  
let  $AC$  be the greater segment, and let  $AD$   
be equal to  $AC$

I say that the straight line  $DB$  has been cut in extreme and mean ratio at  $A$  and the original straight line  $AB$  is the greater segment

For let the square  $AE$  be described on  $AB$ ,  
and let the figure be drawn

Since  $AB$  has been cut in extreme and mean ratio at  $C$ ,  
therefore the rectangle  $AB$ ,  $BC$  is equal to the square on  $AC$  [vi Def 3 vi 17]



And  $CE$  is the rectangle  $AB$ ,  $BC$  and  $CH$  the square on  $AC$ ;

therefore  $CE$  is equal to  $HC$

But  $HE$  is equal to  $CE$ ,

and  $DH$  is equal to  $HC$ ,

therefore  $DH$  is also equal to  $HE$ .

Therefore the whole  $DK$  is equal to the whole  $AE$ .

And  $DK$  is the rectangle  $BD, DA$ ,

for  $AD$  is equal to  $DL$ ,

and  $AE$  is the square on  $AB$ ,

therefore the rectangle  $BD, DA$  is equal to the square on  $AB$

Therefore, as  $DB$  is to  $BA$ , so is  $BA$  to  $AD$

[vi 17]

And  $DB$  is greater than  $BA$ ;

therefore  $BA$  is also greater than  $AD$

[v 14]

Therefore  $DB$  has been cut in extreme and mean ratio at  $A$ , and  $AB$  is the greater segment

Q E D

### PROPOSITION 6

If a rational straight line be cut in extreme and mean ratio, each of the segments is the irrational straight line called apotome

Let  $AB$  be a rational straight line,

let it be cut in extreme and mean ratio at  $C$ ,

D

For

Since, then the straight line  $AB$  has been cut in extreme and mean ratio,

and to the greater segment  $AC$  is added  $AD$  which is half of  $AB$ ,

therefore the square on  $CD$  is five times the square on  $DA$  [xiii 1]

Therefore the square on  $CD$  has to the square on  $DA$  the ratio which a number has to a number,

therefore the square on  $CD$  is commensurable with the square on  $DA$  [x 6]

But the square on  $DA$  is rational,

for  $DA$  is rational, being half of  $AB$  which is rational,

therefore the square on  $CD$  is also rational, [x Def 4]

therefore  $CD$  is also rational

And, since the square on  $CD$  has not to the square on  $DA$  the ratio which a square number has to a square number,

therefore  $CD$  is incommensurable in length with  $DA$ , [x 9]

therefore  $CD, DA$  are rational straight lines commensurable in square only,

therefore  $AC$  is an apotome [x 73]

Again, since  $AB$  has been cut in extreme and mean ratio,

3, vi 17]

1 straight

line  $AB$ , produces  $BC$  as breadth

But the square on an apotome, if applied to a rational straight line, produces as breadth a first apotome,

[x 97]

therefore  $CB$  is a first apotome

And  $CA$  was also proved to be an apotome

Therefore etc

Q E D

## PROPOSITION 7

If three angles of an equilateral pentagon, taken either in order or not in order, be equal, the pentagon will be equiangular

For in the equilateral pentagon  $ABCDE$  let, first, three angles taken in order, those at  $A, B, C$ , be equal to one another,

I say that the pentagon  $ABCDE$  is equiangular

For let  $AC, BE, FD$  be joined

Now, since the two sides  $CB, BA$  are equal to the two sides  $BA, AE$  respectively,

and the angle  $CBA$  is equal to the angle  $BAE$ ,

therefore the base  $AC$  is equal to the base  $BE$ ,

the triangle  $ABC$  is equal to the triangle  $ABE$ ,

and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend, [1 4]

that is the angle  $BCA$  to the angle  $BEA$ , and the angle  $ABE$  to the angle  $CAB$ , [1 6]

hence the side  $AF$  is also equal to the side  $BF$

But the whole  $AC$  was also proved equal to the whole  $BE$ ,

therefore the remainder  $FC$  is also equal to the remainder  $FE$

But  $CD$  is also equal to  $DE$

Therefore the two sides  $FC, CD$  are equal to the two sides  $FE, ED$ ,

and the base  $FD$  is common to them,

therefore the angle  $PCD$  is equal to the angle  $FED$  [1 8]

But the angle  $BCA$  was also proved equal to the angle  $AEB$ ,  
therefore the whole angle  $BCD$  is also equal to the whole angle  $AED$

But, by hypothesis, the angle  $BCD$  is equal to the angles at  $A, B$ ,

therefore the angle  $AED$  is also equal to the angles at  $A, B$

Similarly we can prove that the angle  $CDE$  is also equal to the angles at  $A, B, C$ ,

therefore the pentagon  $ABCDE$  is equiangular

Next let the given equal angles not be angles taken in order, but let the angles at the points  $A, C, D$  be equal,

I say that in this case too the pentagon  $ABCDE$  is equiangular

For let  $BD$  be joined

Then since the two sides  $BA, AE$  are equal to the two sides  $BC, CD$ ,

and they contain equal angles

therefore the base  $BE$  is equal to the base  $BD$ ,

the triangle  $ABE$  is equal to the triangle  $BCD$ ,

and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend, [1 4]

therefore the angle  $AEB$  is equal to the angle  $CDB$

But the angle  $BED$  is also equal to the angle  $BDE$ , [1 5]

since the side  $BE$  is also equal to the side  $BD$

Therefore the whole angle  $AED$  is equal to the whole angle  $CDE$

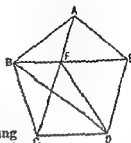
But the angle  $CDE$  is by hypothesis equal to the angles at  $A, C$ ,

therefore the angle  $AED$  is also equal to the angles at  $A, C$

For the same reason

the angle  $ABC$  is also equal to the angles at  $A, C, D$

Therefore the pentagon  $ABCDE$  is equiangular

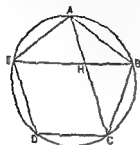


## PROPOSITION 8

If in an equilateral and equiangular pentagon straight lines subtend two angles taken in order, they cut one another in extreme and mean ratio, and their greater segments are equal to the side of the pentagon

For in the equilateral and equiangular pentagon  $ABCDE$  let the straight lines  $AC$ ,  $BE$ , cutting one another at the point  $H$ , subtend two angles taken in order, the angles at  $A$ ,  $B$ ,

I say that each of them has been cut in extreme



the pentagon  $ABCDE$  [IV 14]

Then, since the two straight lines  $EA$ ,  $AB$  are equal to the two  $AB$ ,  $BC$ ,

tively,  
[I 4]

angle  $BAH$  [I 32]

the circumference  $CB$ ,  
[III 33]

therefore, as  $BE$  is to  $EH$ , so is  $EH$  to  $HB$

And  $BE$  is greater than  $EH$ ,

therefore  $EH$  is also greater than  $HB$  [V 14]

Therefore  $BE$  has been cut in extreme and mean ratio at  $H$  and the greater

and mean ratio  
Q E D

## PROPOSITION 9

If the side of the hexagon and that of the decagon inscribed in the same circle be added together, the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon



squares on the side of the hexagon and on that of the decagon inscribed in the circle  $ABCDE$

For let the centre of the circle, the point  $F$ , be taken,

let  $AF$  be joined and carried through to the point  $G$ ,

let  $FB$  be joined,

let  $FH$  be drawn from  $F$  perpendicular to  $AB$  and be carried through to  $K$ ,

let  $AK, KB$  be joined,

let  $FL$  be again drawn from  $F$  perpendicular to  $AK$ , and be carried through to  $M$ ,

and let  $KN$  be joined

Since the circumference  $ABCG$  is equal to the circumference  $AEDG$ ,

and in them  $ABC$  is equal to  $AED$ , therefore the remainder, the circumference

$CG$ , is equal to the remainder  $GD$

But  $CD$  belongs to a pentagon,

therefore  $CG$  belongs to a decagon

And, since  $FA$  is equal to  $FB$ ,

and  $FH$  is perpendicular,

therefore the angle  $AFK$  is also equal to the angle  $KFB$  [I 5, I 26]

Hence the circumference  $AK$  is also equal to  $KB$ , [III 26]

therefore the circumference  $AB$  is double of the circumference  $BK$ ,

therefore the straight line  $AK$  is a side of a decagon

For the same reason

$AK$  is also double of  $KM$

Now, since the circumference  $AB$  is double of the circumference  $BK$ ,

while the circumference  $CD$  is equal to the circumference  $AB$ ,

therefore the circumference  $CD$  is also double of the circumference  $BK$

But the circumference  $CD$  is also double of  $CG$ ,

therefore the circumference  $CG$  is equal to the circumference  $BK$

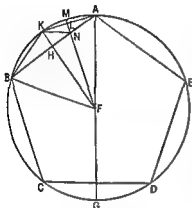
But  $BK$  is double of  $KM$ , since  $KA$  is so also,

therefore  $CG$  is also double of  $KM$

But, further, the circumference  $CB$  is also double of the circumference  $BK$ ,

for the circumference  $CB$  is equal to  $BA$

Therefore the whole circumference  $GB$  is also double of  $BM$ ,



therefore the rectangle  $AB, BN$  is equal to the square on  $BF$  [VI 17]

Again, since  $AL$  is equal to  $LK$ ,

while  $LN$  is common and at right angles,  
therefore the base  $KN$  = equal to the base  $AN$ ; [I 4]

therefore the angle  $LKN$  is also equal to the angle  $LAN$

But the angle  $LAN$  is equal to the angle  $KBN$ ;

therefore the angle  $LKN$  is also equal to the angle  $KBN$

And the angle at  $A$  is common to the two triangles  $AKB$  and  $AKN$

Therefore the remaining angle  $AKB$  is equal to the remaining angle  $KNA$ , [I 32]

therefore the triangle  $KBA$  is equiangular with the triangle  $KNA$

Therefore, proportionally, as the straight line  $BA$  is to  $AK$ , so is  $KA$  to  $AN$ , [VI 4]

therefore the rectangle  $BA, AN$  is equal to the square on  $AK$ . [VI 17]

But the rectangle  $AB, BN$  was also proved equal to the square on  $BF$ ,  
therefore the rectangle  $AB, BN$  together with the rectangle  $BA, AN$ , that is  
the square on  $BA$  [II 3], is equal to the square on  $BF$  together with the square  
on  $AK$

And  $BA$  is = side of the pentagon,  $BF$  of the hexagon [IV 15, Por], and  $AK$   
of the decagon

Therefore etc

Q E D

### PROPOSITION 11

*If in a circle which has its diameter rational an equilateral pentagon be inscribed,  
the side of the pentagon is the irrational straight line called minor*

For in the circle  $ABCDE$  which has its diameter rational let the equilateral  
pentagon  $ABCDE$  be inscribed,

I say that the side of the pentagon = the irrational straight line called minor

For let the centre of the circle, the point  $F$ , be taken,  
let  $AF, FB$  be joined and carried

through to the points,  $G, H$ ,

let  $AC$  be joined,

and let  $FK$  be made a fourth  
part of  $AF$

Now  $AF$  is rational,

therefore  $FK$  is also rational

But  $BF$  is also rational

therefore the whole  $BK$  is rational

And, since the circumference  
 $ACG$  is equal to the circumfer-  
ence  $ADG$ ,

and in them  $ABC$  is equal to  
 $AED$ ,

therefore the remainder  $CO$  is  
equal to the remainder  $GD$

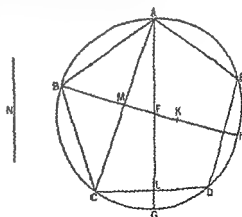
And, if we join  $AD$ , we conclude that the angles at  $L$  are right,  
and  $CD$  is double of  $CL$

For the same reason

the angles at  $M$  are also right,

and  $AC$  is double of  $CM$

Since then the angle  $ALL$  is equal to the angle  $AMF$ ,



and the angle  $LAC$  is common to the two triangles  $ACL$  and  $AMF$ ,  
therefore the remaining angle  $ACL$  is equal to the remaining angle  $MFA$ , [I 32]

therefore the triangle  $ACL$  is equiangular with the triangle  $AMF$ ,

therefore, proportionally, as  $LC$  is to  $CA$ , so is  $MF$  to  $FA$

And the doubles of the antecedents may be taken,

therefore, as the double of  $LC$  is to  $CA$ , so is the double of  $MF$  to  $FA$

But, as the double of  $MF$  is to  $FA$ , so is  $MF$  to the half of  $FA$ ,

therefore also, as the double of  $LC$  is to  $CA$ , so is  $MF$  to the half of  $FA$

And the halves of the consequents may be taken;

therefore, as the double of  $LC$  is to the half of  $CA$ , so is  $MF$  to the fourth of  $FA$ .

And  $DC$  is double of  $LC$ ,  $CM$  is half of  $CA$ , and  $FK$  a fourth part of  $FA$ ,

therefore, as  $DC$  is to  $CM$ , so is  $MF$  to  $FK$

Componendo also, as the sum of  $DC$ ,  $CM$  is to  $CM$ , so is  $MA$  to  $KF$ , [v 18]

therefore also, as the square on the sum of  $DC$ ,  $CM$  is to the square on  $CM$ , so is the square on  $MA$  to the square on  $KF$

And since, when the straight line subtending two sides of the pentagon, as  $AC$ , is cut in extreme and mean ratio, the greater segment is equal to the side of the pentagon, that is, to  $DC$ , [xiii 8]

while the square on the greater segment added to the half of the whole is five times the square on the half of the whole, [xiii 1]

and  $CM$  is half of the whole  $AC$ ,

therefore the square on  $DC$ ,  $CM$  taken as one straight line is five times the square on  $CM$

But it was proved that as the square on  $DC$ ,  $CM$  taken as one straight line is to the square on  $CM$ , so is the square on  $MA$  to the square on  $KF$ ,

therefore the square on  $MA$  is five times the square on  $KF$

But the square on  $AF$  is rational,

for the diameter is rational,

therefore the square on  $MA$  is also rational,

therefore  $MA$  is rational

And since  $BF$  is quadruple of  $FK$ ,

therefore  $BK$  is five times  $KF$ ,

therefore the square on  $BA$  is twenty-five times the square on  $KF$

But the square on  $MA$  is five times the square on  $AF$ ,

therefore the square on  $BA$  is five times the square on  $AM$ ,

therefore the square on  $BK$  has not to the square on  $KM$  the ratio which a square number has to a square number,

therefore  $BK$  is incommensurable in length with  $AM$  [x 9]

And each of them is rational

Therefore  $BK$ ,  $AM$  are rational straight lines commensurable in square only

But, if from a rational straight line there be subtracted a rational straight line which is commensurable with the whole in square only, the remainder is irrational, namely an apotome,

therefore  $MB$  is an apotome and  $MA$  the annex to it. [x 73]

I say next that  $MB$  is also a fourth apotome

Let the square on  $N$  be equal to that by which the square on  $BK$  is greater than the square on  $KM$ ,

therefore the square on  $BK$  is greater than the square on  $KM$  by the square on  $N$ .



And, since  $KF$  is commensurable with  $FB$ ,

*componendo* also,  $KB$  is commensurable with  $FB$  [x 15]

But  $BF$  is commensurable with  $BH$ ,

therefore  $BK$  is also commensurable with  $BH$  [x 12]

And, since the square on  $BK$  is five times the square on  $KM$ ,

therefore the square on  $BK$  has to the square on  $KM$  the ratio which 5 has to 1

Therefore, *convertendo*, the square on  $BK$  has to the square on  $N$  the ratio which 5 has to 4 [v 19, Por], and this is not the ratio which a square number has to a square number;

therefore  $BK$  is incommensurable with  $N$ ; [x 9]

therefore the square on  $BK$  is greater than the square on  $KM$  by the square on a straight line incommensurable with  $BK$

Since then the square on the whole  $BK$  is greater than the square on the annex  $KM$  by the square on a straight line incommensurable with  $BK$ , and the whole  $BA$  is commensurable with the rational straight line,  $BH$ , set out,

therefore  $MB$  is a fourth apotome [x Def iii 4]

But the rectangle contained by a rational straight line and a fourth apotome is irrational,

and its square root is irrational, and is called minor [x 94]

But the square on  $AB$  is equal to the rectangle  $HB$ ,  $BM$ , because when  $AH$  is joined, the triangle  $ABH$  is equiangular with the triangle  $ABM$ , and, as  $HB$  is to  $BA$ , so is  $AB$  to  $BM$

Therefore the side  $AB$  of the pentagon is the irrational straight line called minor Q E D

### PROPOSITION 12

If an equilateral triangle be inscribed in a circle, the square on the side of the triangle is triple of the square on the radius of the circle

Let  $ABC$  be a circle,

and let the equilateral triangle  $ABC$  be inscribed in it,

I say that the square on one side of the triangle

$ABC$  is triple of the square on the radius of the circle

For let the centre  $D$  of the circle  $ABC$  be taken,

let  $AD$  be joined and carried through to  $E$ ,

and let  $BC$  be joined

Then since the triangle  $ABC$  is equilateral, therefore the circumference  $BEC$  is a third part of the circumference of the circle  $ABC$

Therefore the circumference  $BE$  is a sixth part of the circumference of the circle,

therefore the straight line  $BE$  belongs to a hexagon,

therefore it is equal to the radius  $DE$  [iv 15, Por]

And since  $AE$  is double of  $DE$  the square on  $AE$  is quadruple of the square on  $ED$ , that is, of the square on  $BE$

But the square on  $AE$  is equal to the squares on  $AB$ ,  $BE$ , [iii 31, 1 47] therefore the squares on  $AB$ ,  $BE$  are quadruple of the square on  $BE$

Therefore, *separando*, the square on  $AB$  is triple of the square on  $BE$



But  $BE$  is equal to  $DE$ ,

therefore the square on  $AB$  is triple of the square on  $DE$

Therefore the square on the side of the triangle is triple of the square on the radius

Q E D

### PROPOSITION 13

*To construct a pyramid, to comprehend it in a given sphere, and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid*

Let the diameter  $AB$  of the given sphere be set out,

and let it be cut at the point  $C$  so that  $AC$  is double of  $CB$ ,

let the semicircle  $ADB$  be described on  $AB$ ,

let  $CD$  be drawn from the point  $C$  at right angles to  $AB$ ,

and let  $DA$  be joined,

let the circle  $EFG$  which has its radius equal to  $DC$  be set out,

let the equilateral triangle  $EFG$  be inscribed in the circle  $EFG$ , [IV 2]

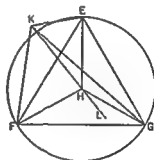
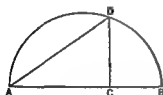
let the centre of the circle, the point  $H$ , be taken, [XII 1]

let  $CH$ ,  $HF$ ,  $HG$  be joined,

from the point  $H$  let  $HK$  be set up at right angles to the plane of the circle  $EFG$ , [XI 12]

let  $HK$  equal to the straight line  $AC$  be cut off from  $HK$ ,

and let  $KE$ ,  $KF$ ,  $KG$  be joined



Now, since  $KH$  is at right angles to the plane of the circle  $EFG$ , therefore it will also make right angles with all the straight lines which meet it and are in the plane of the circle  $EFG$  [XI Def 3]

and they contain right angles,

therefore the base  $DA$  is equal to the base  $KE$

[I 4]

For the same reason

each of the straight lines  $KF$ ,  $KG$  is also equal to  $DA$ ,

therefore the three straight lines  $KE$ ,  $KF$ ,  $KG$  are equal to one another

And, since  $AC$  is double of  $CB$ ,

therefore  $AB$  is triple of  $BC$

But, as  $AB$  is to  $BC$ , so is the square on  $AD$  to the square on  $DC$ , as will be proved afterwards

Therefore the square on  $AD$  is triple of the square on  $DC$

But the square on  $FE$  is also triple of the square on  $EH$ , [XIII 12]  
and  $DC$  is equal to  $EH$ ,

therefore  $DA$  is also equal to  $EF$

But  $DA$  was proved equal to each of the straight lines  $KE, KF, KG$ ,  
therefore each of the straight lines  $EF, FG, GE$  is also equal to each of the straight lines  $KE, KF, KG$ ,

therefore the four triangles  $EFG, KEF, KFG, KEG$  are equilateral

Therefore a pyramid has been constructed out of four equilateral triangles the triangle  $EFG$  being its base and the point  $K$  its vertex

It is next required to comprehend it in the given sphere and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

For let the straight line  $HL$  be produced in a straight line with  $KH$ ,  
and let  $HL$  be made equal to  $CB$

Now, since, as  $AC$  is to  $CD$ , so is  $CD$  to  $CB$ , [VI 8, Por.]

while  $AC$  is equal to  $KH$ ,  $CD$  to  $HE$ , and  $CB$  to  $HL$ ,

therefore, as  $KH$  is to  $HE$ , so is  $EH$  to  $HL$ ,

therefore the rectangle  $KH, HL$  is equal to the square on  $EH$  [VI 17]

And each of the angles  $KHE, EHL$  is right,

therefore the semicircle described on  $KL$  will pass through  $E$  also

[cf VI 8, III 31]

If then,  $KL$  remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through the points  $F, G$ ,

since, if  $FL, LG$  be joined the angles at  $F, G$  similarly become right angles,

and the pyramid will be comprehended in the given sphere

For  $KL$ , the diameter of the sphere, is equal to the diameter  $AB$  of the given sphere, inasmuch as  $KH$  was made equal to  $AC$ , and  $HL$  to  $CB$

I say next that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

For, since  $AC$  is double of  $CB$ ,

therefore  $AB$  is triple of  $BC$ ,

and *convertendo*  $BA$  is one and a half times  $AC$

But as  $BA$  is to  $AC$ , so is the square on  $BA$  to the square on  $AC$

Therefore the square on  $BA$  is also one and a half times the square on  $AC$

And  $BA$  is the diameter of the given sphere, and  $AD$  is equal to the side of the pyramid

Therefore the square on the diameter of the sphere is one and a half times the square on the side of the pyramid Q E D

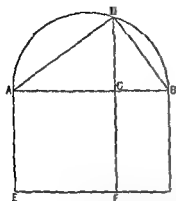
#### LEMMA

It is to be proved that, as  $AB$  is to  $BC$ , so is the square on  $AD$  to the square on  $DC$

For let the figure of the semicircle be set out,

let  $DB$  be joined,

let the square  $EC$  be described on  $AC$ ,



base, because the angle  $ADB$  is right

Therefore, as  $AB$  is to  $BC$ , so is the square on  $AD$  to the square on  $DC$ .

Q E D

#### PROPOSITION 14

To construct an octahedron and comprehend it in a sphere, as in the preceding case, and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron

Let the diameter  $AB$  of the given sphere be set out,

and let it be bisected at  $C$ ,

let the semicircle  $ADB$  be described on  $AB$ ,

let  $CD$  be drawn from  $C$  at right angles to  $AB$ ,

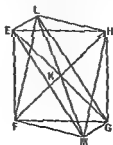
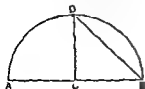
let  $DB$  be joined,

let the square  $EFGH$ , having each of its sides equal to  $DB$ , be set out,

let  $HF$ ,  $EG$  be joined,

from the point  $K$  let the straight line  $KL$  be set up at right angles to the plane of the square  $EFGH$  [xi 12], and let it be carried through to the other side of the plane, as  $KM$ ,

from the straight lines  $KL$ ,  $KM$  let  $KL$ ,  $KM$  be respectively cut off equal to one of the straight lines  $EK$ ,  $FK$ ,  $GK$ ,  $HK$ ,



and let  $LE$ ,  $LF$ ,  $LG$ ,  $LH$ ,  $ME$ ,  $MF$ ,  $MG$ ,  $MH$  be joined

Then, since  $KE$  is equal to  $KH$ , and the angle  $EKH$  is right, therefore the square on  $HE$  is double of the square on  $EK$  [i 47]

Again, since  $LK$  is equal to  $KE$ ,

and the angle  $LKE$  is right,

therefore the square on  $EL$  is double of the square on  $EK$ . [1d]

But the square on  $HE$  was also proved double of the square on  $EK$ ;

therefore the square on  $LE$  is equal to the square on  $EH$ ,

therefore  $LE$  is equal to  $EH$

For the same reason

$LH$  is also equal to  $HE$ ,

therefore the triangle  $LEH$  is equilateral

Similarly we can prove that each of the remaining triangles of which the sides of the square  $EFGH$  are the bases, and the points  $L, M$  the vertices, is equilateral,

therefore an octahedron has been constructed which is contained by eight equilateral triangles

It is next required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron

For, since the three straight lines  $LK, KM, KE$  are equal to one another,

therefore the semicircle described on  $LM$  will also pass through  $E$

And for the same reason,

if,  $LM$  remaining fixed the semicircle be carried round and restored to the same position from which it began to be moved,

it will also pass through the points  $F, G, H$ ,

and the octahedron will have been comprehended in a sphere

I say next that it is also comprehended in the given sphere

For, since  $LK$  is equal to  $KM$ ,

while  $KE$  is common,

and they contain right angles,

therefore the base  $LE$  is equal to the base  $EM$  [1 4]

And, since the angle  $LEM$  is right, for it is in a semicircle, [III 31]

therefore the square on  $LM$  is double of the square on  $LE$  [1 47]

Again, since  $AC$  is equal to  $CB$ ,

$AB$  is double of  $BC$

But as  $AB$  is to  $BC$ , so is the square on  $AB$  to the square on  $BC$ ,

therefore the square on  $AB$  is double of the square on  $BC$

But the square on  $LM$  was also proved double of the square on  $LE$

And the square on  $DB$  is equal to the square on  $LE$ , for  $EH$  was made equal to  $DB$

Therefore the square on  $AB$  is also equal to the square on  $LM$ ,

therefore  $AB$  is equal to  $LM$

And  $AB$  is the diameter of the given sphere,

therefore  $LM$  is equal to the diameter of the given sphere

Therefore the octahedron has been comprehended in the given sphere, and it has been demonstrated at the same time that the square on the diameter of the sphere is double of the square on the side of the octahedron Q E D

#### PROPOSITION 15

To construct a cube and comprehend it in a sphere like the pyramid, and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube

Let the diameter  $AB$  of the given sphere be set out,

and let it be cut at  $C$  so that  $AC$  be double of  $CB$ ;

let the semicircle  $ADB$  be described on  $AB$ ,

let  $CD$  be drawn from  $C$  at right angles to  $AB$ ,

and let  $DB$  be joined,

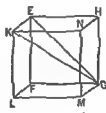
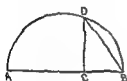
let the square  $EFGH$  having its side equal to  $DB$  be set out,

from  $E, F, G, H$  let  $EK, FL, GM, HN$  be drawn at right angles to the plane of the square  $EFGH$ ,

from  $EK, FL, GM, HN$  let  $EK, FL, GM, HN$  respectively be cut off equal to one of the straight lines  $EF, FG, GH, HE$ ,

and let  $KL, LM, MN, NK$  be joined;

therefore the cube  $FN$  has been constructed which is contained by six equal squares.



It is then required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube

For let  $KG, EG$  be joined

Then, since the angle  $KEG$  is right, because  $KE$  is also at right angles to the plane  $EG$

[XI Def 3]

and of course to the straight line  $EG$  also,  
therefore the sem

Again, since  $G$ ,

hence also, if we join  $KA, GA$  will be at right angles to  $KA$ ,  
and for this reason again the semicircle described on  $GK$  will also pass through  $F$

Similarly it will also pass through the remaining angular points of the cube

I say next that it is also comprehended in the given sphere

For, since  $GF$  is equal to  $FE$ ,

and the angle at  $F$  is right,

therefore the square on  $EG$  is double of the square on  $EF$ .

But  $EF$  is equal to  $EK$ ;

therefore the square on  $EG$  is double of the square on  $EK$ ,

hence the squares on  $GE, EK$ , that is the square on  $GK$  [I 47], is triple of the square on  $EK$

And since  $AB$  is triple of  $BC$ ,

while, as  $AB$  is to  $BC$ , so is the square on  $AB$  to the square on  $BC$

therefore the square on  $AB$  is triple of the square on  $BC$

But the square on  $GK$  was also proved triple of the square on  $KE$

And  $KE$  was made equal to  $DB$ ,

therefore  $GK$  is also equal to  $AB$

And  $AB$  is the diameter of the given sphere,

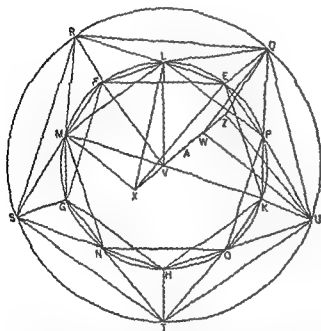
therefore  $GK$  is also equal to the diameter of the given sphere

Therefore the cube has been comprehended in the given sphere, and it has been demonstrated at the same time that the square on the diameter of the sphere is triple of the square on the side of the cube Q E D

### PROPOSITION 16

To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures, and to prove that the side of the icosahedron is the irrational straight line called minor

Let the diameter  $AB$  of the given sphere be set out  
 and let it be cut at  $C$  so that  $AC$  is quadruple of  $CB$ ,  
 let the semicircle  $ADB$  be described on  $AB$ ,  
 let the straight line  $CD$  be drawn from  $C$  at right angles to  $AB$ ,  
 and let  $DB$  be joined,



let the circle  $EFGHA$  be set out and let its radius be equal to  $DB$ ,  
 let the equilateral and equiangular pentagon  $EFGHA$  be inscribed in the circle  $EFGHA$   
 let the circumferences  $EF$ ,  $FG$ ,  $GH$ ,  $HK$ ,  $AE$  be bisected at the points  $L$ ,  $M$ ,  
 $N$ ,  $O$ ,  $P$

and let  $LM$ ,  $MN$ ,  $NO$ ,  $OP$ ,  $PL$ ,  $EP$  be joined

Therefore the pentagon  $LMNOP$  is also equilateral,

and the straight line  $EP$  belongs to a decagon

Now, since each of the straight lines  $EQ$ ,  $KU$  is at right angles to the same plane,

therefore  $EQ$  is parallel to  $KU$  [XI 6]

But it is also equal to it,

∴  $EQ$  is equal and parallel to  $KU$  [I 33]

But  $EK$  belongs to an equilateral pentagon,  
therefore  $QU$  also belongs to the equilateral pentagon inscribed in the circle  $EFGHK$

For the same reason  
each of the straight lines  $QR$ ,  $RS$ ,  $ST$ ,  $TU$  also belongs to the equilateral pentagon inscribed in the circle  $EFGHK$ ,

therefore the pentagon  $QRSTU$  is equilateral

And, since  $QE$  belongs to a hexagon,

and  $EP$  to a decagon,

and the angle  $QEP$  is right,

circle [XIII 10]

For the same reason

$PU$  is also a side of a pentagon

But  $QU$  also belongs to a pentagon,

therefore the triangle  $QPU$  is equilateral

For the same reason

each of the triangles  $QLR$ ,  $RMS$ ,  $SNT$ ,  $TOU$  is also equilateral

And, since each of the straight lines  $QL$ ,  $QP$  was proved to belong to a pentagon,

and  $LP$  also belongs to a pentagon,

therefore the triangle  $QLP$  is equilateral

For the same reason

each of the triangles  $LRM$ ,  $MSN$ ,  $NTO$ ,  $OUP$  is also equilateral

Let the centre of the circle  $EFGHK$ , the point  $V$ , be taken,

from  $V$  draw straight lines

to the lines

and be joined

Now since each of the straight lines  $VW$ ,  $QE$  is at right angles to the plane of the circle,

therefore  $VW$  is parallel to  $QE$  [XI 6]

But they are also equal,

therefore  $EV$ ,  $QW$  are also equal and parallel [I 33]

But  $EV$  belongs to a hexagon,

therefore  $QW$  also belongs to a hexagon

And, since  $QW$



therefore  $QZ$  belongs to a pentagon

[xiii 10]

For the same reason

$UZ$  also belongs to a pentagon,

inasmuch as, if we join  $VK$ ,  $WU$ , they will be equal and opposite, and  $VK$ , being a radius, belongs to a hexagon,

[iv 15, Por]

therefore  $WU$  also belongs to a hexagon

But  $WZ$  belongs to a decagon,

and the angle  $UWZ$  is right,

therefore  $UZ$  belongs to a pentagon

[xiii 10]

But  $QU$  also belongs to a pentagon,

therefore the triangle  $QUZ$  is equilateral

For the same reason

each of the remaining triangles of which the straight lines  $QR$ ,  $RS$ ,  $ST$ ,  $TU$  are the bases and the point  $Z$  the vertex, is also equilateral

Again, since  $VL$  belongs to a hexagon,

and  $VX$  to a decagon,

and the angle  $LVX$  is right,

therefore  $LX$  belongs to a pentagon

[xiii 10]

For the same reason,

if we join  $MV$ , which belongs to a hexagon,

$MX$  is also inferred to belong to a pentagon

But  $LM$  also belongs to a pentagon,

therefore the triangle  $LMX$  is equilateral

Similarly it can be proved that each of the remaining triangles of which  $MN$ ,  $NO$ ,  $OP$ ,  $PL$  are the bases, and the point  $X$  the vertex, is also equilateral

Therefore an icosahedron has been constructed which is contained by twenty equilateral triangles

It is next required to comprehend it in the given sphere and to prove that the side of the icosahedron is the irrational straight line called minor

For since  $VW$  belongs to a hexagon,

and  $WZ$  to a decagon

therefore  $VZ$  has been cut in extreme and mean ratio at  $W$ ,

and  $VW$  is its greater segment,

[xiii 9]

therefore as  $ZV$  is to  $VW$  so is  $VW$  to  $WZ$

But  $VW$  is equal to  $VE$  and  $WZ$  to  $VX$ ,

therefore as  $ZV$  is to  $VE$ , so is  $EV$  to  $VX$

And the angles  $ZVE$ ,  $EVX$  are right,

therefore if we join the straight line  $EZ$ , the angle  $XEZ$  will be right because of the similarity of the triangles  $XEZ$ ,  $VEZ$

For the same reason

since as  $ZV$  is to  $VW$  so is  $VW$  to  $WZ$ ,

and  $ZV$  is equal to  $XW$ , and  $VW$  to  $WQ$ ,

therefore as  $XW$  is to  $WQ$ , so is  $QW$  to  $WZ$

And for this reason again,

if we join  $QX$ , the angle at  $Q$  will be right,

[vi 8]

therefore the semicircle described on  $XZ$  will also pass through  $Q$

[iii 31]

And if,  $XZ$  remaining fixed the semicircle be carried round and restored to the same position from which it began to be moved it will also pass through  $Q$  and the remaining angular points of the icosahedron,

and the icosahedron will have been comprehended in a sphere  
 I say next that it is also comprehended in the given sphere  
 For let  $VW$  be bisected at  $A'$   
 Then, since the straight line  $VZ$  has been cut in extreme and mean ratio at  
 $W$ ,

and  $ZW$  is its lesser segment,

therefore the square on  $ZW$  added to the half of the greater segment, that is  
 $WA'$ , is five times the square on the half of the greater segment, [XIII 3]

therefore the square on  $ZA'$  is five times the square on  $A'W$

And  $ZX$  is double of  $ZA'$ , and  $VW$  double of  $A'W$ ,

therefore the square on  $ZX$  is five times the square on  $VW$

And, since  $AC$  is quadruple of  $CB$ ,

therefore  $AB$  is five times  $BC$

But, as  $AB$  is to  $BC$ , so is the square on  $AB$  to the square on  $BD$ ,  
 [VI 8, v Def 9]

therefore the square on  $AB$  is five times the square on  $BD$

But the square on  $ZX$  was also proved to be five times the square on  $VW$

And  $DB$  is equal to  $VW$ ,

for each of them is equal to the radius of the circle  $EFGHK$ ;

therefore  $AB$  is also equal to  $\lambda Z$

And  $AB$  is the diameter of the given sphere,

therefore  $\lambda Z$  is also equal to the diameter of the given sphere

Therefore the icosahedron has been comprehended in the given sphere

I say next that the side of the icosahedron is the irrational straight line  
 called minor

For, since the diameter of the sphere is rational,  
 and the square on it is five times the square on the radius of the circle  $EFGHK$ ,  
 therefore the radius of the circle  $EFGHK$  is also rational,

hence its diameter is also rational

But if an equilateral pentagon be inscribed in a circle which has its diameter  
 rational, the side of the pentagon is the irrational straight line called minor  
 [XIII 11]

And the side of the pentagon  $EFGHK$  is the side of the icosahedron

Therefore the side of the icosahedron is the irrational straight line called  
 minor

**PORISM** From this it is manifest that the square on the diameter of the  
 sphere is five times the square on the radius of the circle  $EFGHK$ .

same circle

Q E D

### PROPOSITION 17

To construct a dodecahedron and comprehend it in a sphere, like the aforesaid  
 figures, and to prove that the side of the dodecahedron is the irrational straight line  
 called apotome

Let  $ABCD$ ,  $CBEF$ , two planes of the aforesaid cube at right angles to one  
 another, be set out,  
 let the sides  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ ,  $EF$ ,  $EB$ ,  $FC$  be bisected at  $G$ ,  $H$ ,  $K$ ,  $L$ ,  $M$ ,  $N$ ,  $O$   
 respectively,

let  $GK, HL, MH, NO$  be joined

let the straight lines  $NP, PO, HQ$  be cut in extreme and mean ratio at the points  $R, S, T$  respectively,

and let  $RP, PS, TQ$  be their greater segments,

from the points  $R, S, T$  let  $RU, SV, TW$  be set up at right angles to the planes of the cube towards the outside of the cube,

let them be made equal to  $RP, PS, TQ$ ,

and let  $UB, BW, WC, CV, VU$  be joined

I say that the pentagon  $UBWCV$  is equilateral, and in one plane, and is further equiangular

For let  $RB, SB, VB$  be joined

Then, since the straight line  $NP$  has been cut in extreme and mean ratio at  $R$ ,

and  $RP$  is the greater segment, therefore the squares on  $PN, NR$  are triple of the square on  $RP$  [xiii. 4]

But  $PN$  is equal to  $NB$ , and  $PR$  to  $RU$ , therefore the squares on  $BN, NR$  are triple of the square on  $RU$

But the square on  $BR$  is equal to the squares on  $BN, NR$ , [i. 47] therefore the square on  $BR$  is triple of the square on  $RU$ ,

hence the squares on  $BR, RU$  are quadruple of the square on  $RU$ .

But the square on  $BU$  is equal to the squares on  $BR, RU$ ,

therefore the square on  $BU$  is quadruple of the square on  $RU$ , therefore  $BU$  is double of  $RU$

But  $VU$  is also double of  $UR$ ,

inasmuch as  $SR$  is also double of  $PR$ , that is, of  $RU$ ,

therefore  $BU$  is equal to  $UV$

Similarly it can be proved that each of the straight lines  $BW, WC, CV$  is also equal to each of the straight lines  $BU, UV$

Therefore the pentagon  $BUVCW$  is equilateral

I say next that it is also in one plane

For let  $PX$  be drawn from  $P$  towards the outside of the cube

I say

For, since  $HQ$  has been cut in extreme and mean ratio at  $T$ , and  $QT$  is its greater segment,

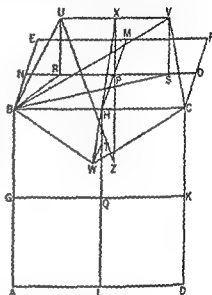
But  $HQ$

is equal to  $PX$ ,

And  $HP$  is parallel to  $TW$ ,

for each of them is at right angles to the plane  $BD$ , and  $TH$  is parallel to  $PX$ ,

[xi. 6]



for each of them is at right angles to the plane  $BF$  [id]

∴ two sides proportional to  
corresponding sides are

the remaining straight lines will be in a straight line, [vi 32]

therefore  $\lambda H$  is in a straight line with  $HW$

But every straight line is in one plane, [xi 1]

therefore the pentagon  $UBWCV$  is in one plane

I say next that it is also equiangular

For since the straight line  $NP$  has been cut in extreme and mean ratio at  $R$ , and  $PR$  is the greater segment

while  $PR$  is equal to  $PS$ ,

therefore  $NS$  has also been cut in extreme and mean ratio at  $P$ ,

and  $NP$  is the greater segment, [xiii 5]

therefore the squares on  $NS$ ,  $SP$  are triple of the square on  $NP$  [xiii 4]

But  $NP$  is equal to  $NB$ , and  $PS$  to  $SV$ ,

therefore the squares on  $NS$ ,  $SV$  are triple of the square on  $NB$ ,

hence the squares on  $VS$ ,  $SN$ ,  $NB$  are quadruple of the square on  $NB$

But the square on  $SB$  is equal to the squares on  $SN$ ,  $NB$ ,

therefore the squares on  $BS$ ,  $SV$ , that is, the square on  $BV$ —for the angle  $\angle SB$  is right—is quadruple of the square on  $NB$ ,

therefore  $BV$  is double of  $BN$

But  $BC$  is also double of  $BN$ ,

therefore  $BV$  is equal to  $BC$

And since the two sides  $BU$ ,  $UV$  are equal to the two sides  $BW$ ,  $WC$ ,

and the base  $BV$  is equal to the base  $BC$ ,

therefore the angle  $BUV$  is equal to the angle  $BWC$  [i 8]

Similarly we can prove that the angle  $UVC$  is also equal to the angle  $BWC$ ,  
therefore the three angles  $BWC$ ,  $BUV$ ,  $UVC$  are equal to one another

But if in an equilateral pentagon three angles are equal to one another the pentagon will be equiangular, [xiii 7]

therefore the pentagon  $BUVCW$  is equiangular

And it was also proved equilateral,

therefore the pentagon  $BUVCW$  is equilateral and equiangular and it is on one side  $BC$  of the cube

Therefore if we make the same construction in the case of each of the twelve sides of the cube

a solid figure will have been constructed which is contained by twelve equilateral and equiangular pentagons and which is called a dodecahedron

It is then manifest that the cube is contained by the dodecahedron

∴

therefore  $PZ$  meets the diameter of the cube, and they bisect one another, for this has been proved in the last theorem but one of the eleventh book

[xi 38]

Let them cut at  $Z$ ,

therefore  $Z$  is the centre of the sphere which comprehends the cube,

and  $ZP$  is half of the side of the cube

Let  $UZ$  be joined

Now, since the straight line  $NS$  has been cut in extreme and mean ratio at  $P$ ,

and  $NP$  is its greater segment,

therefore the squares on  $NS$ ,  $SP$  are triple of the square on  $NP$ . [xiii 4]

But  $NS$  is equal to  $XZ$ ,

inasmuch as  $NP$  is also equal to  $PZ$ , and  $XP$  to  $PS$

But further,  $PS$  is also equal to  $XU$ ,

since it is also equal to  $RP$ ,

therefore the squares on  $ZX$ ,  $XU$  are triple of the square on  $NP$

But the square on  $UZ$  is equal to the squares on  $ZX$ ,  $XU$ ;

therefore the square on  $UZ$  is triple of the square on  $NP$

But the square on the radius of the sphere which comprehends the cube is also triple of the square on the half of the side of the cube, for it has previously been shown how to construct a cube and comprehend it in a sphere, and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube [xiii 15]

But, if whole is so related to whole, so is half to half also;

and  $NP$  is half of the side of the cube;

therefore  $UZ$  is equal to the radius of the sphere which comprehends the cube

And  $Z$  is the centre of the sphere which comprehends the cube;

therefore the point  $U$  is on the surface of the sphere

Similarly we can prove that each of the remaining angles of the dodecahedron is also on the surface of the sphere,

therefore the dodecahedron has been comprehended in the given sphere

I say next that the side of the dodecahedron is the irrational straight line called apotome

For since when  $NP$  has been cut in extreme and mean ratio,  $RP$  is the greater segment,

and, when  $PO$  has been cut in extreme and mean ratio,  $PS$  is the greater segment,

therefore when the whole  $NO$  is cut in extreme and mean ratio,  $RS$  is the greater segment

[Thus since as  $NP$  is to  $PR$  so is  $PR$  to  $RN$ ,

the same is true of the doubles also,

for parts have the same ratio as their equimultiples, [v 15]

therefore as  $NO$  is to  $RS$ , so is  $RS$  to the sum of  $NR$ ,  $SO$

But  $NO$  is greater than  $RS$ ,

therefore  $RS$  is also greater than the sum of  $NR$ ,  $SO$ ,

therefore  $NO$  has been cut in extreme and mean ratio,

and  $RS$  is its greater segment }

But  $RS$  is equal to  $UV$ ,

therefore when  $NO$  is cut in extreme and mean ratio,  $UV$  is the greater segment.

And since the diameter of the sphere is rational,

and the square on it is triple of the square on the side of the cube,

therefore  $NO$  being a side of the cube, is rational

[But if a rational line be cut in extreme and mean ratio, each of the segments is an irrational apotome]

Therefore  $UV$ , being a side of the dodecahedron, is an irrational apotome [xiii 6]

**PORISM** From this it is manifest that, when the side of the cube is cut in extreme and mean ratio, the greater segment is the side of the dodecahedron

Q E D

### PROPOSITION 18

*To set out the sides of the five figures and to compare them with one another*

Let  $AB$ , the diameter of the given sphere, be set out, and let it be cut at  $C$  so that  $AC$  is equal to  $CB$ , and at  $D$  so that  $AD$  is double of  $DB$ , let the semicircle  $AEB$  be described on  $AB$ , from  $C, D$  let  $CE, DF$  be drawn at right angles to  $AB$ .

and let  $AF, FB, EB$  be joined

Then, since  $AD$  is double of  $DB$ ,

therefore  $AB$  is triple of  $BD$

Converting, therefore,  $BA$  is one and a half times  $AD$ .

But, as  $BA \parallel AD$ , so is the square on  $BA$  to the square on  $AF$ , [3 Def 9, 11 Sp] for the triangle  $AFB \parallel$  equiangular with the triangle  $AFD$ .

therefore the square on  $BA$  is one and a half times the square on  $AF$

But the square on the diameter of the sphere is also one and a half times the square on the side of the pyramid. [XIII 13]

[XIII 13]

And  $AB$  is the diameter of the sphere.

therefore  $AF$  is equal to the side of the pyramid

Again, since  $AD$  is double of  $DB$ ,

therefore  $AB$  is triple of  $BD$

But, as  $AB$  is to  $BD$ , so is the square on  $AB$  to the square on  $BD$ .

[VI 8. v Def 9]

therefore the square on  $AB$  is triple of the square on  $BF$

But the square on the diameter of the sphere is also triple of the square on the side of the cube

[xiii 15]

And  $AB$  is the diameter of the sphere.

therefore  $BF$  is the side of the cube

And since  $AC$  is equal to  $CB$ ,

therefore  $AB$  is double of  $BC$

But as  $AB$  is to  $BC$ , so is the square on  $AB$  to the square on  $BE$ .

therefore the square on  $AB$  is double of the square on  $BE$

But the square on the diameter of the sphere is also double of the square on the side of the octahedron. [VII. 14]

[VIII 14]

And  $AB$  is the diameter of the given sphere,

therefore  $BE$  is the side of the octahedron

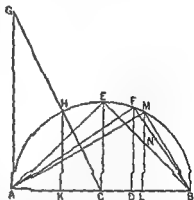
Next, let  $AG$  be drawn from the point  $A$  at right angles to the straight line  $AB$ .

let  $AG$  be made equal to  $AB$ .

let  $GC$  be joined.

and from  $H$  let  $HK$  be drawn perpendicular to  $AB$

Then, since  $GA$  is double of  $AC$ ,



for  $GA$  is equal to  $AB$ ,  
 and, as  $GA$  is to  $AC$ , so is  $HK$  to  $KC$ ,  
 therefore  $HK$  is also double of  $KC$

Therefore the square on  $HK$  is quadruple of the square on  $KC$ ,  
 therefore the squares on  $HK$ ,  $KC$ , that is, the square on  $HC$ ,  $\equiv$  five times the square on  $KC$

But  $HC \equiv$  equal to  $CB$ ,

therefore the square on  $BC$  is five times the square on  $CK$

And, since  $AB$  is double of  $CB$ ,

and, in them,  $AD$  is double of  $DB$ ,

therefore the remainder  $BD$  is double of the remainder  $DC$

Therefore  $BC$  is triple of  $CD$ ,

therefore the square on  $BC$  is nine times the square on  $CD$

But the square on  $BC$  is five times the square on  $CK$ ,

therefore the square on  $CK$  is greater than the square on  $CD$ ,

therefore  $CK$  is greater than  $CD$

Let  $CL$  be made equal to  $CK$ ,

from  $L$  let  $LM$  be drawn at right angles to  $AB$ ,

and let  $MB$  be joined

Now, since the square on  $BC$  is five times the square on  $CK$ ,

and  $AB$  is double of  $BC$ , and  $KL$  double of  $CK$ ,

therefore the square on  $AB \equiv$  five times the square on  $KL$

But the square on the diameter of the sphere is also five times the square on the radius of the circle from which the icosahedron has been described

[xiii 16, Por.]

And  $AB \equiv$  the diameter of the sphere,

therefore  $KL$  is the radius of the circle from which the icosahedron has been described,

therefore  $KL$  is a side of the hexagon in the said circle [iv 15 Por.]

And since the diameter of the sphere is made up of the side of the hexagon and two of the sides of the decagon inscribed in the same circle,

[xiii 16, Por.]

and  $AB$  is the diameter of the sphere,

while  $KL$  is a side of the hexagon,

and  $AK$  is equal to  $LB$ ,

therefore each of the straight lines  $AK$ ,  $LB$  is a side of the decagon inscribed in the circle from which the icosahedron has been described

And since  $LB$  belongs to a decagon, and  $ML$  to a hexagon,  
 for  $ML$  is equal to  $KL$ , since it is also equal to  $HK$ , being the same distance from the centre, and each of the straight lines  $HK$ ,  $KL$  is double of  $KC$ ,

therefore  $MB$  belongs to a pentagon

[xiii 10]

But the side of the pentagon is the side of the icosahedron,

[xiii 16]

therefore  $MB$  belongs to the icosahedron

Now, since  $FB$  is a side of the cube,

let it be cut in extreme and mean ratio at  $N$ ,

and let  $NB$  be the greater segment,

therefore  $NB$  is a side of the dodecahedron [xiii 17, Por.]

And, since the square on the diameter of the sphere was proved to be one and  $\frac{1}{2}$  times the square on the side  $AF$  of the pyramid, double of the square on

the side  $BE$  of the octahedron and triple of the side  $FB$  of the cube, therefore of parts of which the square on the diameter of the sphere contains six, the square on the side of the pyramid contains four, the square on the side of the octahedron three, and the square on the side of the cube two

Therefore the square on the side of the pyramid is four-thirds of the square on the side of the octahedron, and double of the square on the side of the cube, and the square on the side of the octahedron is one and a half times the square on the side of the cube

The said sides, therefore, of the three figures, I mean the pyramid, the octahedron and the cube, are to one another in rational ratios

But the remaining two, I mean the side of the icosahedron and the side of the dodecahedron, are not in rational ratios either to one another or to the aforesaid sides,

for they are irrational, the one being minor [XIII 16] and the other an apotome [XIII 17]

That the side  $MB$  of the icosahedron is greater than the side  $NB$  of the dodecahedron we can prove thus

triangle  $FAB$ , [VI 8]  
to  $BA$  [VI 4]

as the first is to the third, so is the square on the first to the square on the second, [v Def 9, VI 20, Por]

therefore, as  $DB$  is to  $BA$ , so is the square on  $DB$  to the square on  $BF$ , therefore, inversely, as  $AB$  is to  $BD$ , so is the square on  $FB$  to the square on  $BD$

But  $AB$  is triple of  $BD$ ,

therefore the square on  $FB$  is triple of the square on  $BD$

But the square on  $AD$  is also quadruple of the square on  $DB$ ,

for  $AD$  is double of  $DB$ ,

therefore the square on  $AD$  is greater than the square on  $FB$ ,

therefore  $AD$  is greater than  $FB$ ,

therefore  $AL$  is by far greater than  $FB$

And when  $AL$  is cut in extreme and mean ratio,

$KL$  is the greater segment,

inasmuch as  $LK$  belongs to a hexagon and  $KA$  to a decagon, [XIII 9]

and, when  $FB$  is cut in extreme and mean ratio,  $NB$  is the greater segment,

therefore  $KL$  is greater than  $NB$

But  $KL$  is equal to  $LM$ ,

therefore  $LM$  is greater than  $NB$

Therefore  $MB$ , which is a side of the icosahedron, is by far greater than  $NB$  which is a side of the dodecahedron Q E D

I say next that no other figure, besides the said five figures, can be constructed which is contained by equilateral and equiangular figures equal to one another

For a solid angle cannot be constructed with two triangles, or indeed planes

With three triangles the angle of the pyramid is constructed, with four the angle of the octahedron and with five the angle of the icosahedron,



but a solid angle cannot be formed by six equilateral and equiangular triangles placed together at one point,  
 for, the angle of the equilateral triangle being two-thirds of a right angle, the six will be equal to four right angles  
 which is impossible, for any solid angle is contained by angles less than four right angles [XI 21]

For the same reason, neither can a solid angle be constructed by more than six plane angles

By three squares the angle of the cube is contained, but by four it is impossible for a solid angle to be contained,  
 for they will again be four right angles

By three equilateral and equiangular pentagons the angle of the dodecahedron is contained,  
 but by four such it is impossible for any solid angle to be contained,  
 for, the angle of the equilateral pentagon being a right angle and a fifth, the four angles will be greater than four right angles  
 which is impossible

Neither again will a solid angle be contained by other polygonal figures by reason of the same absurdity

Therefore etc

Q E D

#### LEMMA

But that the angle of the equilateral and equiangular pentagon is a right angle and a fifth we must prove thus

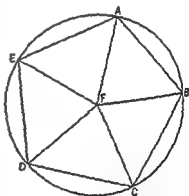
Let  $ABCDE$  be an equilateral and equiangular pentagon,  
 let the circle  $ABCDE$  be circumscribed  
 about it,

let its centre  $F$  be taken,  
 and let  $FA, FB, FC, FD, FE$  be joined  
 Therefore they bisect the angles of the pentagon at  $A, B, C, D, E$

And, since the angles at  $F$  are equal to four right angles and are equal,  
 therefore one of them, as the angle  $AFB$ , is one right angle less a fifth,  
 therefore the remaining angles  $FAB, ABF$  consist of one right angle and a fifth

But the angle  $FAB$  is equal to the angle  $FBC$ ,

therefore the whole angle  $ABC$  of the pentagon consists of one right angle and a fifth



Q E D

**THE WORKS OF ARCHIMEDES  
INCLUDING THE METHOD**



## BIOGRAPHICAL NOTE

ARCHIMEDES, ■ 287-212 B C.

which was dedicated to Gelo, Archimedes speaks of his father, Pheidias, as an astronomer who investigated the sizes and distances of the sun and moon

friendship of Conon of Samos and Eratosthenes To Conon he was in the habit

to be credited that associates Archimedes with this problem After the death of

*more a given weight by a given force*, he boasted to King Hiero "Give me a place to stand on and I can move the earth " Asked for a practical demonstration, he contrived a machine by which with the use of only one arm he drew out of the dock a large ship, laden with passengers and goods which the combined strength of the Syracusans could scarcely move From that day Hiero ordered that "Archimedes was to be believed in everything he might say " At the king's request Archimedes then made for him catapults, battering rams cranes, and many other engines of war, which were later used with such success in the defense of Syracuse against the Romans that they were unable to take the city except by treachery There is also a story in Lucian that Archimedes set fire to the Roman ships by an arrangement of burning glasses

Although Archimedes acquired by his mechanical inventions "the renown

lost

po

ch

sufficiently accurate to show the eclipses of the sun and the moon Except for

lonius he wrote no textbooks. Of his writings, although some have been lost the most important have survived.

The absorption of Archimedes in his mathematical investigations was so great that he forgot his food and neglected his person, and when carried by force to the bath, Plutarch records, "he used to trace geometrical figures in the ashes of the fire and diagrams in the oil on his body." Asked by Hiero to discover whether a goldsmith had alloyed with silver the gold of his crown, Archimedes found the answer while bathing by considering the water displaced by his body, whereupon he is reported to have run home in his excitement without his clothes, shouting, "Eureka" (I have found it).

Archimedes' preoccupation with mathematics is even said to have been the cause of his death. In the general massacre which followed the capture of Syracuse by Marcellus in 212 B. C., Archimedes was so intent upon a mathematical diagram that he took no notice, and when ordered by a soldier to attend the victorious general, he refused until he should have solved his problem, whereupon he was slain by the enraged soldier. No blame attaches to the Roman general, Marcellus, since he had given orders to spare the house and person of the mathematician, and in the midst of his triumph he lamented the death of Archimedes, provided him with an honorable burial, and befriended his surviving relatives. In accordance with the expressed desire of Archimedes, his family and friends interred him in a vaulted tomb, the entrance to which was in the shape of a sphere and the ceiling was flat.

The neglected and forgotten tomb of Archimedes near the Agrigentine Gate and piously restored it.

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# ON THE SPHERE AND CYLINDER

## BOOK ONE

ARCHIMEDES to DOSITHEUS greeting

On a former occasion I sent you the investigations which I had up to that time completed including the proofs showing that any segment bounded by a straight line and a section of a right-angled cone [a parabola] is four thirds of the triangle which has the same base with the segment and equal height. Since then certain theorems not hitherto demonstrated have occurred to me and I have worked out the proofs of them. They are these: first, that the surface of

surface also [including its bases] is half as large again as the surface of the sphere. Now these properties were all along naturally inherent in the figures referred to but remained unknown to those who were before my time engaged in the study of geometry. Having however now discovered that the properties are true of these figures I cannot feel any hesitation in setting them side by side both with my former investigations and with those of the theorems of

possess the requisite ability to examine these discoveries of mine. They ought to have been published while Conon was still alive for I should conceive that

out which it will be open to mathematicians to examine. Farewell.

I first set out the axioms and the assumptions which I have used for the proofs of my propositions.



## DEFINITIONS

1 "There are in a plane certain terminated bent lines, which either lie wholly on the same side of the straight lines joining their extremities, or have no part of them on the other side "

2 "I apply the term *concave in the same direction* to a line such that, if any two points on it are taken, either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself, but none on the other side "

3 "Similarly also there are certain terminated surfaces, not themselves being in a plane but having their extremities in a plane, and such that they will either be wholly on the same side of the plane containing their extremities, or have no part of them on the other side "

4 "I apply the term *concave in the same direction* to surfaces such that, if any two points on them are taken, the straight lines connecting the points either all fall on the same side of the surface, or some fall on one and the same side of it while some fall upon it, but none on the other side "

5 "I use the term *solid sector*, when a cone cuts a sphere, and has its apex at the centre of the sphere, to denote the figure comprehended by the surface of the cone and the surface of the sphere included within the cone "

6 "I apply the term *solid rhombus*, when two cones with the same base have their apices on opposite sides of the plane of the base in such a position that their axes lie in a straight line, to denote the solid figure made up of both the cones "

## ASSUMPTIONS

1 "Of all lines which have the same extremities the straight line is the least "

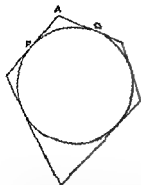
2 "Of other lines in a plane and having the same extremities, [any two] such are unequal whenever both are concave in the same direction and one of them is either wholly included between the other and the straight line which has the same extremities with it, or is partly included by, and is partly common with, the other and that [line] which is included is the lesser [of the two] "

3 "Similarly, of surfaces which have the same extremities, if those extremities are in a plane the plane is the least [in area] "

4 "Of other surfaces with the same extremities, the extremities being in a plane [any two] such are unequal whenever both are concave in the same direction and one surface is either wholly included between the other and the plane which has the same extremities with it, or is partly included by, and partly common with the other, and that [surface] which is included is the lesser [of the two in area] "

5 "Further of unequal lines unequal surfaces and unequal solids, the greater exceeds the less by such a magnitude as when added to itself can be made to exceed any assigned magnitude among those which are comparable with it and with one another "

"These things being premised if a polygon be inscribed in a circle, it is plain that the perimeter of the inscribed polygon is less than the circumference of the circle, for each of the sides of the polygon is less than that part of the circumference of the circle which is cut off by it "



## PROPOSITION 1

If a polygon be circumscribed about a circle, the perimeter of the circumscribed polygon is greater than the perimeter of the circle

Let any two adjacent sides, meeting in  $A$ , touch the circle at  $P, Q$  respectively

Then [Assumptions, 2]

$$PA + AQ > (\text{arc } PQ)$$

A similar inequality holds for each angle of the polygon, and, by addition, the required result follows

## PROPOSITION 2

Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less

Let  $AB, D$  represent the two unequal magnitudes  $AB$  being the greater

Suppose  $BC$  measured along  $BA$  equal to  $D$ , and let  $GH$  be any straight line

Then if  $CA$  be added to itself a sufficient number of times, the sum will exceed  $D$ . Let  $AF$  be this sum and take  $E$  on  $GH$  produced such that  $GH$  is the same multiple of  $HE$  that  $AF$  is of  $AC$

$$\text{Thus } EH : HG = AC : AF$$

But, since  $AF > D$  (or  $CB$ ),

$$AC : AF < AC : CB$$

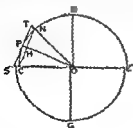
Therefore *componendo*,

$$EG : GH < AB : D$$

Hence  $EG, GH$  are two lines satisfying the given condition

## PROPOSITION 3

Given two unequal magnitudes and a circle, it is possible to inscribe a polygon in the circle and to describe another about it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than that of the greater magnitude to the less



Let  $A, B$  represent the given magnitudes  $A$  being the greater

Find [Prop 2] two straight lines  $F, KL$  of which  $F$  is the greater, such that

$$F : KL < A : B$$

(1)

ing  
m.

Join  $NC$ , which (by the construction) will be the side of a regular polygon inscribed in the circle. Let  $OP$  be the radius of the circle bisecting the angle  $NOC$  (and therefore bisecting  $NC$  at right angles, in  $H$ , say), and let the tangent at  $P$  meet  $OC$ ,  $ON$  produced in  $S$ ,  $T$  respectively

Now, since  $\angle CON < 2\angle LKM$ ,  
 $\angle HOC < \angle LKM$ ,

and the angles at  $H$ ,  $L$  are right,

therefore  $MK > OC$   $OH$   
 $> OP$   $OH$

Hence  $ST < CN$   $MK > LK$   
 $< F$   $LK$ ,

therefore, *a fortiori*, by (1),

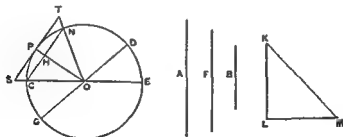
$ST < CN < A B$

Thus two polygons are found satisfying the given condition

#### PROPOSITION 4

Again, given two unequal magnitudes and a sector, it is possible to describe a polygon about the sector and to inscribe another in it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than the greater magnitude has to the less

[The "inscribed polygon" found in this proposition is one which has for two sides the two radii bounding the sector, while the remaining sides (the number of which is, by construction, some power of 2) subtend equal parts of the arc of the sector, the "circumscribed polygon" is formed by the tangents parallel to the sides of the inscribed polygon and by the two bounding radii produced]

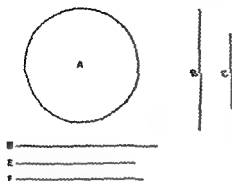


In the case of the two unequal magnitudes, the ratio of the sides of the polygons is less than the ratio of the magnitudes, except between exactly

similar to the preceding one

#### PROPOSITION 5

Given a circle and two unequal magnitudes, to describe a polygon about the circle and inscribe another in it, so that the circumscribed polygon may have to the inscribed a ratio less than the greater magnitude has to the less



Let  $A$  be the given circle and  $B$ ,  $C$  the given magnitudes,  $B$  being the greater

Take two unequal straight lines  $D$ ,  $E$ , of which  $D$  is the greater, such that  $D : E < B : C$  [Prop 2], and let  $F$  be a mean proportional between  $D$ ,  $E$  so that  $D$  is also greater than  $F$

Describe (in the manner of Prop 3) one polygon about the circle, and inscribe another in it, so that the side of the former has to the side

of the latter a ratio less than the ratio  $D : F$

Thus the duplicate ratio of the side of the former polygon to the side of the latter is less than the ratio  $D^2 : F^2$

But the said duplicate ratio of the sides is equal to the ratio of the areas of the polygons, since they are similar,

therefore the area of the circumscribed polygon has to the area of the inscribed polygon a ratio less than the ratio  $D^2 : F^2$ , or  $D : E$ , and a fortiori less than the ratio  $B : C$

#### PROPOSITION 6

"Similarly we can show that, given two unequal magnitudes and a sector, it is possible to circumscribe a polygon about the sector and inscribe in it another similar one so that the circumscribed may have to the inscribed a ratio less than the greater magnitude has to the less

"And it is likewise clear that, if a circle or a sector, as well as a certain area, be given it is possible, by inscribing regular polygons in the circle or sector, and by continually inscribing such in the remaining segments, to leave segments of the circle or sector which are [together] less than the given area. For this is proved in the Elements [Eucl XII 2]

"But it is yet to be proved that, given a circle or sector and an area, it is possible to describe a polygon about the circle or sector, such that the area remaining between the circumference and the circumscribed figure is less than the given area"

The proof for the circle (which, as Archimedes says, can be equally applied to a sector) is as follows

Let  $A$  be the given circle and  $B$  the given area

Now, there being two unequal magnitudes  $A + B$  and  $A$ , let a polygon ( $C$ ) be circumscribed about the circle and a polygon ( $I$ ) inscribed in it [as in Prop 5],



so that

$$C : I < A + B : A$$

(1)

The circumscribed polygon ( $C$ ) shall be that required

For the circle ( $A$ ) is greater than the inscribed polygon ( $I$ )

Therefore, from (1), *a fortiori*,

$$C \quad A < A+B : A,$$

whence

$$C < A+B,$$

or

$$C-A < B$$

### PROPOSITION 7

If in an isosceles cone [i.e. a right circular cone] a pyramid be inscribed having an equilateral base, the surface of the pyramid excluding the base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the perpendicular drawn from the apex on one side of the base

Since the sides of the base of the pyramid are equal, it follows that the perpendiculars from the apex to all the sides of the base are equal, and the proof of the proposition is obvious

### PROPOSITION 8

If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding its base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the side [i.e. a generator] of the cone

of  
of  
these perpendiculars, being generators of the cone, are equal, whence the proposition follows immediately

### PROPOSITION 9

If in the circular base of an isosceles cone a chord be placed, and from its extremities straight lines be drawn to the apex of the cone, the triangle so formed will be less than the portion of the surface of the cone intercepted between the lines drawn to the apex

Let  $ABC$  be the circular base of the cone, and  $O$  its apex

Draw a chord  $AB$  in the circle, and join  $OA$ ,  $OB$ . Bisect the arc  $ACB$  in  $C$ , and join  $AC$ ,  $BC$ ,  $OC$

Then

$$\triangle OAC + \triangle OBC > \triangle OAB$$

Let the excess of the sum of the first two triangles over the third be equal to the area  $D$

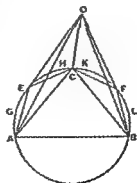
Then  $D$  is either less than the sum of the segments  $AEC$ ,  $CFB$ , or not less

I Let  $D$  be not less than the sum of the segments referred to

We have now two surfaces

(1) that consisting of the portion  $OAEC$  of the surface of the cone together with the segment  $AEC$ , and

(2) the triangle  $OAC$ ,



and, since the two surfaces have the same extremities (the perimeter of the triangle  $OAC$ ), the former surface is greater than the latter, which is included by it [Assumptions, 3 or 4]

Hence (surface  $OAEC$ ) + (segment  $AEC$ ) >  $\triangle OAC$

Similarly (surface  $OCFB$ ) + (segment  $CFB$ ) >  $\triangle OBC$

Therefore, since  $D$  is not less than the sum of the segments, we have, by addition,

$$\begin{aligned} (\text{surface } OAECFB) + D &> \triangle OAC + \triangle OBC \\ &> \triangle OAB + D, \text{ by hypothesis} \end{aligned}$$

Taking away the common part  $D$ , we have the required result

II Let  $D$  be less than the sum of the segments  $AEC$ ,  $CFB$

If now we bisect the arcs  $AC$ ,  $CB$ , then bisect the halves, and so on, we shall ultimately leave segments which are together less than  $D$  [Prop 6]

Let  $AGE$ ,  $EHC$ ,  $CKF$ ,  $FLB$  be those segments, and join  $OE$ ,  $OF$

Then, as before,

$$(\text{surface } OAGE) + (\text{segment } AGE) > \triangle OAE$$

and (surface  $OEHC$ ) + (segment  $EHC$ ) >  $\triangle OEC$

Therefore (surface  $OAGHC$ ) + (segments  $AGE$ ,  $EHC$ )

$$> \triangle OAE + \triangle OEC$$

$$> \triangle OAC, \text{ a fortiori}$$

Similarly for the part of the surface of the cone bounded by  $OC$ ,  $OB$  and the arc  $CFB$

Hence, by addition,

$$(\text{surface } OAGEHCKFLB) + (\text{segments } AGE, EHC, CKF, FLB)$$

$$> \triangle OAC + \triangle OBC$$

$$> \triangle OAB + D, \text{ by hypothesis}$$

But the sum of the segments is less than  $D$ , and the required result follows

#### PROPOSITION 10

If in the plane of the circular base of an isosceles cone two tangents be drawn to the circle meeting in a point, and the points of contact and the point of concurrence of the tangents be respectively joined to the apex of the cone, the sum of the two triangles formed by the joining lines and the two tangents are together greater than the included portion of the surface of the cone

Let  $ABC$  be the circular base of the cone,  $O$  its apex,  $AD$ ,  $BD$  the two tangents to the circle meeting in  $D$ . Join  $OA$ ,  $OB$ ,  $OD$

Let  $ECF$  be drawn touching the circle at  $C$ , the middle point of the arc  $ACB$ , and therefore parallel to  $AB$ . Join  $OE$ ,  $OF$

Then  $ED + DF > EF$ ,

and, adding  $AE + FB$  to each side,

$$AD + DB > AE + EF + FB$$

Now  $OA$ ,  $OC$ ,  $OB$ , being generators of the cone, are equal, and they are respectively perpendicular to the tangents at  $A$ ,  $C$ ,  $B$

It follows that

$$\triangle OAD + \triangle ODB > \triangle OAE + \triangle OEF + \triangle OFB$$

Let the area  $G$  be equal to the excess of the first sum over the second.  $G$  is then either less, or not less than the sum of the spaces  $EAHC$ ,  $FCKB$  remaining between the circle and the tangents, which sum we will call  $L$

I Let  $G$  be not less than  $L$

We have now two surfaces

- (1) that of the pyramid with apex  $O$  and base  $AEFB$ , excluding the face  $OAB$ ,
- (2) that consisting of the part  $OACB$  of the surface of the cone together with the segment  $ACB$

These two surfaces have the same extremities, viz the perimeter of the triangle  $OAB$ , and, since the former includes the latter, the former is the greater [Assumptions, 4]

That is, the surface of the pyramid exclusive of the face  $OAB$  is greater than the sum of the surface  $OACB$  and the segment  $ACB$

Taking away the segment from each sum, we have

$$\triangle OAE + \triangle OEF + \triangle OFB + L > \text{the surface } OAHCKB$$

And  $G$  is not less than  $L$

It follows that

$$\triangle OAE + \triangle OEF + \triangle OFB + G,$$

which is by hypothesis equal to  $\triangle OAD + \triangle ODB$ , is greater than the same surface

II Let  $G$  be less than  $L$

If we bisect the arcs  $AC$ ,  $CB$  and draw tangents at their middle points, then bisect the halves and draw tangents, and so on, we shall lastly arrive at a polygon such that the sum of the parts remaining between the sides of the polygon and the circumference of the segment is less than  $G$

Let the remainders be those between the segment and the polygon  $APQRSB$ , and let their sum be  $M$   
Join  $OP$ ,  $OQ$  etc

Then, as before,

$$\triangle OAE + \triangle OEF + \triangle OFB > \triangle OAP + \triangle OPQ + \dots + \triangle OSB$$

Also as before,

(surface of pyramid  $OAPQRSB$  exclusive of the face  $OAB$ ) > the part  $OACB$  of the surface of the cone together with the segment  $ACB$

Taking away the segment from each sum,

$$\triangle OAP + \triangle OPQ + \dots + M > \text{the part } OACB \text{ of the surface of the cone}$$

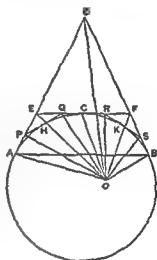
Hence, *a fortiori*

$$\triangle OAE + \triangle OEF + \triangle OFB + G,$$

which is by hypothesis equal to

$$\triangle OAD + \triangle ODB,$$

is greater than the part  $OACB$  of the surface of the cone



## PROPOSITION 11

If a plane parallel to the axis of a right cylinder cut the cylinder, the part of the surface of the cylinder cut off by the plane is greater than the area of the parallelogram in which the plane cuts it

## PROPOSITION 12

If at the extremities of two generators of any right cylinder tangents be drawn to the circular bases in the planes of those bases respectively, and if the pairs of tangents meet, the parallelograms formed by each generator and the two corresponding tangents respectively are together greater than the included portion of the surface of the cylinder between the two generators

[The proofs of these two propositions follow exactly the methods of Props 9, 10 respectively, and it is therefore unnecessary to reproduce them]

"From the properties thus proved it is clear (1) that, if a pyramid be inscribed in an isosceles cone, the surface of the pyramid excluding the base is less than the surface of the cone [excluding the base], and (2) that, if a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding the base is greater than the surface of the cone excluding the base

"It is also clear from what has been proved both (1) that, if a prism be inscribed in a right cylinder, the surface of the prism made up of its parallelograms [i.e. excluding its bases] is less than the surface of the cylinder excluding its bases, and (2) that, if a prism be circumscribed about a right cylinder, the surface of the prism made up of its parallelograms is greater than the surface of the cylinder excluding its bases"

## PROPOSITION 13

The surface of any right cylinder excluding the bases is equal to a circle whose radius is a mean proportional between the side [i.e. a generator] of the cylinder and the diameter of its base

Let the base of the cylinder be the circle *A*, and make *CD* equal to the diameter of this circle, and *EF* equal to the height of the cylinder

Let *H* be a mean proportional between *CD*, *EF*, and *B* a circle with radius equal to *H*

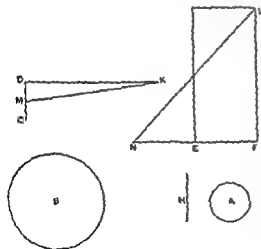
Then the circle *B* shall be equal to the surface of the cylinder (excluding the bases), which we will call *S*

For, if not, *B* must be either greater or less than *S*

I Suppose *B* < *S*

Then it is possible to circumscribe a regular polygon about *B*, and to inscribe another in it, such that the ratio of the former to the latter is less than the ratio *S* : *B*

Suppose this done, and circumscribe about *A* a polygon



Let *KD*, perpendicular to *CD*, and *FL*, perpendicular to *EF*, be each equal to the perimeter of the polygon about *A*. Bisect *CD* in *M*, and join *MK*



Then  $\triangle KDM =$  the polygon about  $A$

Also  $\square EL =$  surface of prism (excluding bases)

Produce  $FE$  to  $N$  so that  $FE = EN$ , and join  $NL$

Now the polygons about  $A$ ,  $B$ , being similar, are in the duplicate ratio of the radii of  $A$ ,  $B$

Thus

$$\begin{aligned}\triangle KDM \text{ (polygon about } B) &= MD^2 : H^2 \\ &= MD^2 : CD : EF \\ &= MD : NF \\ &= \triangle KDM : \triangle LFN\end{aligned}$$

(since  $DK = FL$ )

Therefore (polygon about  $B$ )  $= \triangle LFN$

$$= \square EL$$

$=$  (surface of prism about  $A$ ),

from above

But (polygon about  $B$ ) (polygon in  $B$ )  $< S : B$

Therefore

(surface of prism about  $A$ ) (polygon in  $B$ )  $< S : B$ ,

and, alternately,

(surface of prism about  $A$ )  $S < (\text{polygon in } B) : B$ ,

which is impossible, since the surface of the prism is greater than  $S$ , while the polygon inscribed in  $B$  is less than  $B$

Therefore  $B \nless S$

II Suppose  $B > S$

Let a regular polygon be circumscribed about  $B$  and another inscribed in it so that

a prism on

Again, let  $DK$ ,  $EL$ , drawn as before, be each equal to the perimeter of the polygon inscribed in  $A$

Then, in this case

$$\triangle KDM > (\text{polygon inscribed in } A)$$

(since the perpendicular from the centre on a side of the polygon is less than the radius of  $A$ )

Also  $\triangle LFN = \square EL =$  surface of prism (excluding bases)

Now

$$\begin{aligned}(\text{polygon in } A) : (\text{polygon in } B) &= MD^2 : H^2, \\ &= \triangle KDM : \triangle LFN, \text{ as before}\end{aligned}$$

And

$$\triangle KDM > (\text{polygon in } A)$$

Therefore

$$\triangle LFN \text{ or (surface of prism)} > (\text{polygon in } B)$$

But this is impossible, because

$$(\text{polygon about } B) : (\text{polygon in } B) < B : S$$

$$< (\text{polygon about } B) : S, \text{ a fortiori,}$$

so that

$$(\text{polygon in } B) > S,$$

$$> (\text{surface of prism}), \text{ a fortiori}$$

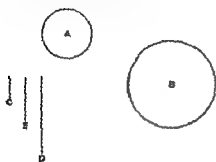
Hence  $B$  is neither greater nor less than  $S$ , and therefore

$$B = S$$

## PROPOSITION 14

The surface of any isosceles cone excluding the base is equal to a circle whose radius is a mean proportional between the side of the cone [a generator] and the radius of the circle which is the base of the cone

Let the circle  $A$  be the base of the cone, draw  $C$  equal to the radius of the circle, and  $D$  equal to the side of the cone, and let  $E$  be a mean proportional between  $C$ ,  $D$



Draw a circle  $B$  with radius equal to  $E$

Then shall  $B$  be equal to the surface of the cone (excluding the base), which we will call  $S$

If not,  $B$  must be either greater or less than  $S$

I Suppose  $B < S$

Let a regular polygon be described about  $B$  and a similar one inscribed in it such that the former has to the latter a ratio less than the ratio  $S : B$

Describe about  $A$  another similar polygon, and on it set up a pyramid with apex the same as that of the cone

Then (polygon about  $A$ ) (polygon about  $B$ )

$$= C^2 : E^2$$

$$= C : D$$

$$= (\text{polygon about } A) : (\text{surface of pyramid excluding base})$$

Therefore

$$(\text{surface of pyramid}) = (\text{polygon about } B)$$

$$\text{Now } (\text{polygon about } B) : (\text{polygon in } B) < S : B$$

Therefore

$$(\text{surface of pyramid}) : (\text{polygon in } B) < S : B,$$

which is impossible (because the surface of the pyramid is greater than  $S$ , while the polygon in  $B$  is less than  $B$ )

$$\text{Hence } B < S$$

II Suppose  $B > S$

Take regular polygons circumscribed and inscribed to  $B$  such that the ratio of the former to the latter is less than the ratio  $B : S$

Inscribe in  $A$  a similar polygon to that inscribed in  $B$ , and erect a pyramid on the polygon inscribed in  $A$  with apex the same as that of the cone

In this case

$$(\text{polygon in } A) : (\text{polygon in } B) = C^2 : E^2 \\ = C : D$$

$$> (\text{polygon in } A) : (\text{surface of pyramid excluding base})$$

This is clear because the ratio of  $C$  to  $D$  is greater than the ratio of the perpendicular from the centre of  $A$  on a side of the polygon to the perpendicular from the apex of the cone on the same side

Therefore

$$(\text{surface of pyramid}) > (\text{polygon in } B)$$

But

$$(\text{polygon about } B) : (\text{polygon in } B) < B : S$$

Therefore *a fortiori*,

(polygon about  $B$ ) (surface of pyramid)  $< B \ S$ ,

which is impossible

Since therefore  $B$  is neither greater nor less than  $S$ ,

$$B = S$$

#### PROPOSITION 15

The surface of any isosceles cone has the same ratio to its base as the side of the cone has to the radius of the base

By Prop 14, the surface of the cone is equal to  $\pi$  circle whose radius is a mean proportional between the side of the cone and the radius of the base

Hence, since circles are to one another as the squares of their radii, the proposition follows

#### PROPOSITION 16

If an isosceles cone be cut by a plane parallel to the base the portion of the surface of the cone between the parallel planes is equal to a circle whose radius is a mean proportional between (1) the portion of the side of the cone intercepted by the parallel planes and (2) the line which is equal to the sum of the radii of the circles in the parallel planes

Let  $OAB$  be a triangle through the axis of a cone,  $DE$  its intersection with the plane cutting off the frustum and  $OFC$  the axis of the cone

Then the surface of the cone  $OAB$  is equal to  $\pi$  circle whose radius is equal to  $\sqrt{OA \ AC}$  [Prop 14]

Similarly the surface of the cone  $ODE$  is equal to a circle whose radius is equal to  $\sqrt{OD \ DF}$

And the surface of the frustum is equal to the difference between the two circles

Now

$$OA \ AC - OD \ DF = DA \ AC + OD \ AC - OD \ DF$$

But

$$OD \ AC = OA \ DF$$

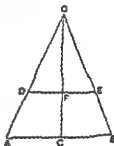
since

$$OA \ AC = OD \ DF$$

Hence

$$OA \ AC - OD \ DF = DA \ AC + DA \ DF \\ = DA \ (AC + DF)$$

And  $\pi$  circle whose radius is  $\sqrt{DA \ (AC + DF)}$



Therefore the surface of the frustum is equal to this circle

#### LEMMAS

"1 Cones having equal height have the same ratio as their bases and those having equal bases have the same ratio as their heights"

2 If a cylinder be cut by a plane parallel to the base then as the cylinder is to the cylinder so is the axis to the axis"

3 The cones which have the same bases as the cylinders [and equal height] are in the same ratio as the cylinders

4 Also the bases of equal cones are reciprocally proportional to their heights, and those cones whose bases are reciprocally proportional to their heights are equal<sup>1</sup>

5 Also the cones, the diameters of whose bases have the same ratio as their axes, are to one another in the triplicate ratio of the diameters of the bases<sup>2</sup>

And all these propositions have been proved by earlier geometers "

### PROPOSITION 17

If there be two isosceles cones, and the surface of one cone be equal to the base of the other, while the perpendicular from the centre of the base [of the first cone] on the side of that cone is equal to the height [of the second], the cones will be equal

Let  $OAB$ ,  $DEF$  be triangles through the axes of two cones respectively,  $C$ ,  $G$  the centres of the respective bases,  $GH$  the perpendicular from  $G$  on  $FD$ , and suppose that the base of the cone  $OAB$  is equal to the surface of the cone  $DEF$ , and that  $OC = GH$

Then, since the base of  $OAB$  is equal to the surface of  $DEF$ ,  
(base of cone  $OAB$ ) (base of cone  $DEF$ )

$=$  (surface of  $DEF$ ) (base of  $DEF$ )  
 $= DF \cdot FG$  [Prop 15]  
 $= DG \cdot GH$ , by similar triangles  
 $= DG \cdot OC$

Therefore the bases of the cones are reciprocally proportional to their heights, whence the cones are equal [Lemma 4]

### PROPOSITION 18

Any solid rhombus consisting of isosceles cones is equal to the cone which has its base equal to the surface of one of the cones composing the rhombus and its height equal to the perpendicular drawn from the apex of the second cone to one side of the first cone

Let the rhombus be  $OABD$  consisting of two cones with apices  $O$ ,  $D$  and with a common base (the circle about  $AB$  as diameter)

Let  $FHK$  be another cone with base equal to the surface of the cone  $OAB$  and height  $FG$  equal to  $DE$ , the perpendicular from  $D$  on  $OB$

Then shall the cone  $FHK$  be equal to the rhombus

Construct a third cone  $LMA$  with base (the circle about  $MN$ ) equal to the base of  $OAB$  and height  $LP$  equal to  $OD$

<sup>1</sup>Euclid XII 15 "The bases of equal cones and cylinders are reciprocally proportional to their heights and those cones and cylinders whose bases are reciprocally proportional to their heights are equal."

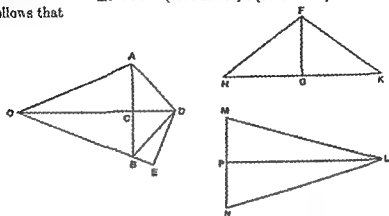
<sup>2</sup>Euclid XII 12 "Similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases"

Then, since

$$LP = OD, \\ LP \cdot CD = OD \cdot CD.$$

But [Lemma 1]  $OD : CD = (\text{rhombus } OADB) : (\text{cone } DAB),$   
and  $LP : CD = (\text{cone } LMN) : (\text{cone } DAB).$

It follows that



$$(\text{rhombus } OADB) = (\text{cone } LMN) \quad (1)$$

Again, since  $AB = MN$ , and

$$\begin{aligned} (\text{surface of } OAB) &= (\text{base of } FHK), \\ (\text{base of } FHK) (\text{base of } LMN) &= (\text{surface of } OAB) : (\text{base of } OAB) \\ &= OB \cdot BC \quad [\text{Prop 15}] \\ &= OD \cdot DE, \text{ by similar triangles,} \\ &= LP \cdot FG, \text{ by hypothesis} \end{aligned}$$

Thus, in the cones  $FHK$ ,  $LMN$ , the bases are reciprocally proportional to the heights

Therefore the cones  $FHK$ ,  $LMN$  are equal,

and hence, by (1) the cone  $FHK$  is equal to the given solid rhombus

#### PROPOSITION 19

If an isosceles cone be cut by a plane parallel to the base, and on the resulting circular section a cone be described having as its apex the centre of the base [of the first cone], and if the rhombus so formed be taken away from the whole cone, the part remaining will be equal to the cone with base equal to the surface of the portion of the first cone between the parallel planes and with height equal to the perpendicular drawn from the centre of the base of the first cone on one side of that cone

Let the cone  $OAB$  be cut by a plane parallel to the base in the circle on  $DE$  as diameter. Let  $C$  be the centre of the base of the cone, and with  $C$  as apex and the circle about  $DE$  as base describe a cone, making with the cone  $ODE$  the rhombus  $ODCE$

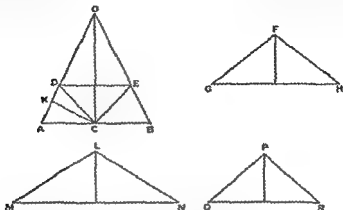
Take a cone  $FGH$  with base equal to the surface of the frustum  $DABE$  and height equal to the perpendicular ( $CK$ ) from  $C$  on  $AO$

Then shall the cone  $FGH$  be equal to the difference between the cone  $OAB$  and the rhombus  $ODCE$

Take (1) a cone  $LMN$  with base equal to the surface of the cone  $OAB$ , and height equal to  $CK$ ,

(2) a cone  $PQR$  with base equal to the surface of the cone  $ODE$  and height equal to  $CK$ .

Now, since the surface of the cone  $OAB$  is equal to the surface of the cone  $ODE$  together with that of the frustum  $DABE$ , we have, by the construction,



(base of  $LMN$ ) = (base of  $FGH$ ) + (base of  $PQR$ )

and, since the heights of the three cones are equal,

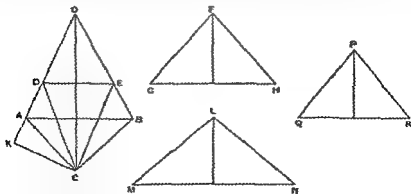
(cone  $LMN$ ) = (cone  $FGH$ ) + (cone  $PQR$ )

But the cone  $LMN$  is equal to the cone  $OAB$  [Prop 17], and the cone  $PQR$  is equal to the rhombus  $ODCE$  [Prop 18]

Therefore (cone  $OAB$ ) = (cone  $FGH$ ) + (rhombus  $ODCE$ ), and the proposition is proved

### PROPOSITION 20

If one of the two isosceles cones forming a rhombus be cut by a plane parallel to the base and on the resulting circular section a cone be described having the same apex as the second cone, and if the resulting rhombus be taken from the whole rhombus, the remainder will be equal to the cone with base equal to the surface of the portion of the cone between the parallel planes and with height equal to the perpendicular drawn from the apex of the second cone to the side of the first cone



Let the rhombus be  $OACB$  and let the cone  $OAB$  be cut by a plane parallel to its base in the circle about  $DE$  as diameter. With this circle as base and  $C$

as apex describe a cone, which therefore with  $ODE$  forms the rhombus  $ODCE$

Take a cone  $FGH$  with base equal to the surface of the frustum  $DABE$  and height equal to the perpendicular ( $CK$ ) from  $C$  on  $OA$

The cone  $FGH$  shall be equal to the difference between the rhombi  $OACB$ ,  $ODCE$

For take (1) a cone  $LMN$  with base equal to the surface of  $OAB$  and height equal to  $CK$ ,

(2) a cone  $PQR$ , with base equal to the surface of  $ODE$ , and height equal to  $CK$

Then, since the surface of  $OAB$  is equal to the surface of  $ODE$  together with that of the frustum  $DABE$ , we have, by construction,

$$(\text{base of } LMN) = (\text{base of } PQR) + (\text{base of } FGH),$$

and the three cones are of equal height,

therefore  $(\text{cone } LMN) = (\text{cone } PQR) + (\text{cone } FGH)$

But the cone  $LMN$  is equal to the rhombus  $OACB$ , and the cone  $PQR$  is equal to the rhombus  $ODCE$  [Prop 18]

Hence the cone  $FGH$  is equal to the difference between the two rhombi  $OACB$ ,  $ODCE$

### PROPOSITION 21

A regular polygon of an even number of sides being inscribed in a circle, as  $ABC \dots A' \dots C'B'A$ , so that  $AA'$  is a diameter, if two angular points next but one to each other, as  $B, B'$ , be joined, and the other lines parallel to  $BB'$  and joining pairs of angular points be drawn, as  $CC', DD', \dots$ , then

$(BB' + CC' + \dots) AA' = A'B \cdot BA$

Let  $BB', CC', DD', \dots$  meet  $AA'$

in  $F, G, H, \dots$ , and let  $CB', DC', \dots$

be joined meeting  $AA'$  in  $K, L, \dots$

respectively

Then clearly  $CB', DC', \dots$  are

parallel to one another and to  $AB$

Hence, by similar triangles,

$$BF : FA = B'F : FK$$

$$= CG : GK$$

$$= CG : GL$$

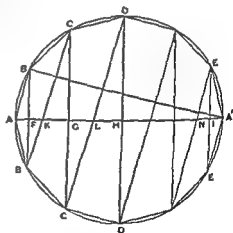
$$= EI : IA',$$

and, summing the antecedents and

consequents respectively, we have

$$(BB' + CC' + \dots) AA' = BF : FA$$

$$= A'B : BA$$

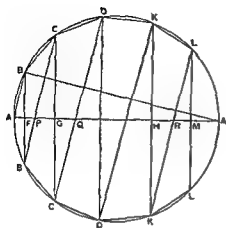


### PROPOSITION 22

If a polygon be inscribed in a segment of a circle  $LAL'$  so that all its sides excluding the base are equal and their number even as  $LK \dots A \dots K'L'$ ,  $A$  being the middle point of the segment, and if the lines  $BB', CC', \dots$  parallel to the base  $LL'$  and joining pairs of angular points be drawn, then

$$(BB' + CC' + \dots + LM) AM = A'B \cdot BA,$$

where  $M$  is the middle point of  $LL'$  and  $AA'$  is the diameter through  $M$ .



Joining  $CB'$ ,  $DC'$ ,  $LK'$ , as in the last proposition, and supposing that they meet  $AM$  in  $P$ ,  $Q$ ,  $R$ , while  $BB'$ ,  $CC'$ ,  $KK'$  meet  $AM$  in  $F$ ,  $G$ ,  $H$ , we have, by similar triangles,

$$\begin{aligned} BF : FA &= B'F : FP \\ &= CG : PG \\ &= C'G : GQ \end{aligned}$$

$$= LM : RM,$$

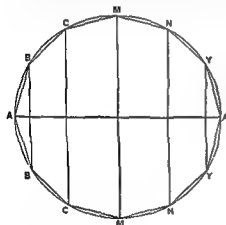
and, summing the antecedents and consequents we obtain

$$\begin{aligned} (BB' + CC' + \dots + LM) : AM \\ &= BF : FA \\ &= A'B : BA \end{aligned}$$

### PROPOSITION 23

Take a great circle  $ABC$  of a sphere, and inscribe in it a regular polygon whose sides are a multiple of four in number. Let  $AA'$ ,  $MM'$  be diameters at right angles and joining opposite angular points of the polygon

Then, if the polygon and great circle revolve together about the



sphere at right angles to the diameter  $AA'$ . Also the sides of the polygon will describe portions of conical surfaces  $\square g$   $BC$  will describe a surface forming part of a cone whose base is a circle about  $CC'$  as diameter and whose apex  $\square$  the point in which  $CB$   $C'B'$  produced meet each other and the diameter  $AA'$

Corollary 1. The surface of the hemisphere is greater than that of the inscribed figure, and the same is true of the other halves of the figures

Therefore [Assumptions 4] the surface of the hemisphere is greater than that of the inscribed figure, and the same is true of the other halves of the figures

Hence the surface of the sphere is greater than the surface described by the revolution of the polygon inscribed in the great circle about the diameter of the great circle



## PROPOSITION 24

If a regular polygon  $AB \dots A'$   $B'A$ , the number of whose sides is a multiple of four, be inscribed in a great circle of a sphere, and if  $BB'$  subtending two sides be joined, and all the other lines parallel to  $BB'$  and joining pairs of angular points be drawn, then the surface of the figure inscribed in the sphere by the revolution of the polygon about the diameter  $AA'$  is equal to a circle the square of whose radius is equal to the rectangle

$$BA(BB' + CC' + \dots)$$

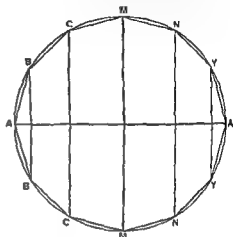
The surface of the figure is made up of the surfaces of parts of different cones

Now the surface of the cone  $ABB'$  is equal to a circle whose radius is  $\sqrt{BA \cdot \frac{1}{2}BB'}$  [Prop 14]

The surface of the frustum  $BB'C'C$  is equal to a circle of radius  $\sqrt{BC \cdot \frac{1}{2}(BB' + CC')}$ , [Prop 16] and so on

It follows, since  $BA = BC = \dots$ , that the whole surface is equal to a circle whose radius is equal to

$$\sqrt{BA(BB' + CC' + \dots + MM' + \dots + YY')}$$



## PROPOSITION 25

The surface of the figure inscribed in a sphere as in the last propositions, consisting of portions of conical surfaces, is less than four times the greatest circle in the sphere

Let  $AB \dots A'$   $B'A$  be a regular polygon inscribed in a great circle, the number of its sides being a multiple of four

As before, let  $BB'$  be drawn subtending two sides, and  $CC'$ ,  $YY'$  parallel to  $BB'$

Let  $R$  be a circle such that the square of its radius is equal to

$$AB(BB' + CC' + \dots + YY')$$

so that the surface of the figure inscribed in the sphere is equal to  $R$

[Prop 24]

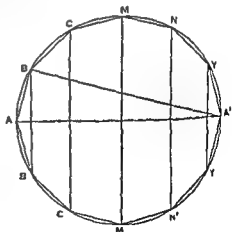
Now

$$(BB' + CC' + \dots + YY') \cdot AA' = A'B \cdot AB, \text{ [Prop 21]}$$

$$\text{whence } AB(BB' + CC' + \dots + YY') = AA' \cdot A'B$$

$$\text{Hence } (\text{radius of } R)^2 = AA' \cdot A'B < AA'^2$$

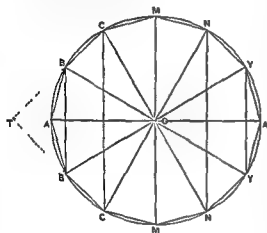
Therefore the surface of the inscribed figure, or the circle  $R$ , is less than four times the circle  $1/4 AA'M'$



## PROPOSITION 26

*The figure inscribed as above in a sphere is equal [in volume] to a cone whose base is a circle equal to the surface of the figure inscribed in the sphere and whose height is equal to the perpendicular drawn from the centre of the sphere to one side of the polygon*

Suppose, as before, that  $AB A' B'A$  is the regular polygon inscribed in a great circle, and let  $BB', CC',$  be joined



With apex  $O$  construct cones whose bases are the circles on  $BB', CC',$  as diameters in planes perpendicular to  $AA'$

Then  $OBAB'$  is a solid rhombus, and its volume is equal to a cone whose base is equal to the surface of the cone  $ABB'$  and whose height is equal to the perpendicular from  $O$  on  $AB$  [Prop 18] Let the length of the perpendicular be  $p$

Again, if  $CB, C'B'$  produced meet in  $T$ , the portion of the solid figure which is described by the revolution of the tri-

angle  $BOC$  about  $AA'$  is equal to the difference between the rhombi  $OCTC'$  and  $OBTB'$ , i.e. to a cone whose base is equal to the surface of the frustum  $BB'CC'$  and whose height is  $p$  [Prop 20]

Proceeding in this manner, and adding, we prove that, since cones of equal height are to one another as their bases the volume of the solid of revolution is equal to a cone with height  $p$  and base equal to the sum of the surfaces of the cone  $BAB'$ , the frustum  $BB'CC'$ , etc., i.e. a cone with height  $p$  and base equal to the surface of the solid

## PROPOSITION 27

*The figure inscribed in the sphere as before is less than four times the cone whose base is equal to a great circle of the sphere and whose height is equal to the radius of the sphere*

By Prop 26 the volume of the solid figure is equal to a cone whose base is equal to the surface of the solid and whose height is  $p$ , the perpendicular from  $O$  on any side of the polygon Let  $R$  be such a cone

Take also a cone  $S$  with base equal to the great circle, and height equal to the radius, of the sphere

Now, since the surface of the inscribed solid is less than four times the great circle [Prop 25], the base of the cone  $R$  is less than four times the base of the cone  $S$

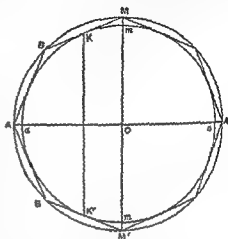
Also the height ( $p$ ) of  $R$  is less than the height of  $S$

Therefore the volume of  $R$  is less than four times that of  $S$ , and the proposition is proved

## PROPOSITION 28

Let a regular polygon, whose sides are a multiple of four in number, be circumscribed about a great circle of a given sphere, as  $AB A' B'A$ , and about the polygon describe another circle, which will therefore have the same centre as the great circle of the sphere. Let  $AA'$  bisect the polygon and cut the sphere in  $\alpha, \alpha'$

If the great circle and the circumscribed polygon revolve together about  $AA'$ , the great circle will describe the surface of a sphere, the angular points of the polygon except  $A, A'$  will move round the surface of a larger sphere, the points of contact of the sides of the polygon with the great circle of the inner sphere will describe circles on that sphere in planes perpendicular to  $AA'$ , and the sides of the polygon themselves will describe portions of conical surfaces. The circumscribed figure will thus be greater than the sphere itself



Let any side, as  $BM$ , touch the inner circle in  $K$ , and let  $K'$  be the point of contact of the circle with  $B'M'$

Then the circle described by the revolution of  $KK'$  about  $AA'$  is the boundary in one plane of two surfaces

- (1) the surface formed by the revolution of the circular segment  $KaK'$ , and
- (2) the surface formed by the revolution of the part  $KB A B'K'$  of the polygon

Now the second surface entirely includes the first, and they are both concave in the same direction,

therefore [Assumptions, 4] the second surface is greater than the first

The same is true of the portion of the surface on the opposite side of the circle on  $AA'$  as diameter

Hence, adding we see that the surface of the figure circumscribed to the given sphere is greater than that of the sphere itself

## PROPOSITION 29

In a figure circumscribed to a sphere in the manner shown in the previous proposition the surface is equal to a circle the square on whose radius is equal to

$$AB(BB' + CC' + \dots)$$

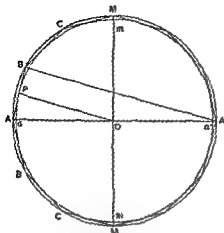
For the figure circumscribed to the sphere is inscribed in a larger sphere, and the proof of Prop. 21 applies

## PROPOSITION 30

The surface of a figure circumscribed as before about a sphere is greater than four times the great circle of the sphere

Let  $AB A' B'A$  be the regular polygon of  $4n$  sides which by its revolu-

tion about  $AA'$  describes the figure circumscribing the sphere of which  $ama'm'$



face of the circumscribed solid

**Now**

$$(BB' + CC' + \dots) AA' = A'B \quad BA, \quad \text{[as in Prop. 21]}$$

so that

$$AB(BB' + CC + \dots) = AA' A'B$$

Hence (radius of  $R$ ) =  $\sqrt{AA' \cdot A'B}$   
[Prop 29]

$\geq A'B$

But  $A'B = 2OP$ , where  $P$  is the point in which  $AB$  touches the circle  $amam'$ .

Therefore (radius of  $R$ )  $>$  (diameter of circle  $gma'm'$ ).

hence  $R$ , and therefore the surface of the circumscribed solid, is greater than four times the great circle of the given sphere

### PROPOSITION 31

The solid of revolution circumscribed as before about a sphere is equal to a cone whose base is equal to the surface of the solid and whose height is equal to the radius of the sphere

The solid  $\square$  as before a solid inscribed in a larger sphere, and, since the perpendicular on any side of the revolving polygon is equal to the radius of the inner sphere, the proposition is identical with Prop. 26.

**COR.** The solid circumscribed about the smaller sphere is greater than four times the cone whose base is a great circle of the sphere and whose height is equal to the radius of the sphere.

For, since the surface of the solid is greater than four times the great circle of the inner sphere [Prop 30] the cone whose base is equal to the surface of the solid and whose height is the radius of the sphere is greater than four times the cone of the same height which has the great circle for base [Lemma 1]

Hence by the proposition, the volume of the solid is greater than four times the latter cone.

### PROPOSITION 32

If a regular polygon with  $4n$  sides be inscribed in a great circle of a sphere, as  $ab$  a  $ba$  and a similar polygon  $AB$   $A'$   $BA$  be described about the great circle, and if the polygons radiate with the great circle about the diameters  $aa'$ ,  $AA'$  respectively, so that they describe the surfaces of solid figures inscribed in and circumscribed to the sphere respectively, then

(1) the surfaces of the circumscribed and inscribed figures are to one another in the duplicate ratio of their sides and

(2) the figures themselves [i.e. their volumes] are in the triplicate ratio of their sides

(1) Let  $AA'$ ,  $aa'$  be in the same straight line, and let  $MmOm'M'$  be a diameter at right angles to them

Join  $BB'$ ,  $CC'$ , and  $bb'$ ,  $cc'$ , which will all be parallel to one another and  $MM'$ .

Suppose  $R$ ,  $S$  to be circles such that

$R$  = (surface of circumscribed solid),

$S$  = (surface of inscribed solid)

Then (radius of  $R$ )<sup>2</sup> =  $AB(BB' + CC' + \dots)$  [Prop 23]

(radius of  $S$ )<sup>2</sup> =  $ab(bb' + cc' + \dots)$  [Prop 24]

And, since the polygons are similar, the rectangles in these two equations are similar, and are therefore in the ratio of

$$AB^2 : ab^2$$

Hence

(surface of circumscribed solid) (surface of inscribed solid) =  $AB^2 : ab^2$

(2) Take a cone  $V$  whose base is the circle  $R$  and whose height is equal to  $Oa$ , and a cone  $W$  whose base is the circle  $S$  and whose height is equal to the perpendicular from  $O$  on  $ab$ , which we will call  $p$

Then  $V$ ,  $W$  are respectively equal to the volumes of the circumscribed and inscribed figures

Now, since the polygons are similar,

$$AB : ab = Oa : p$$

$$= (\text{height of cone } V) : (\text{height of cone } W);$$

and as shown above, the bases of the cones (the circles  $R$ ,  $S$ ) are in the ratio of  $AB^2$  to  $ab^2$

Therefore

$$V : W = AB^2 : ab^2$$

### PROPOSITION 33

*The surface of any sphere is equal to four times the greatest circle in it.*

Let  $C$  be a circle equal to four times the great circle

Then if  $C$  is not equal to the surface of the sphere, it must either be less or greater

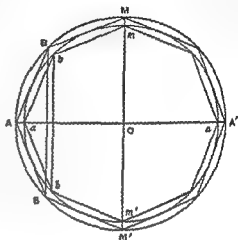
I Suppose  $C$  less than the surface of the sphere

It is then possible to find two lines  $\beta$ ,  $\gamma$ , of which  $\beta$  is the greater, such that  $\beta : \gamma < (\text{surface of sphere}) : C$  [Prop 2]

Take such lines and let  $\delta$  be a mean proportional between them

Suppose similar regular polygons with  $4n$  sides circumscribed about and inscribed in a great circle such that the ratio of their sides is less than the ratio  $\beta : \delta$  [Prop 3]

Let the polygons with the circle revolve together about a diameter common to all, describing solids of revolution as before

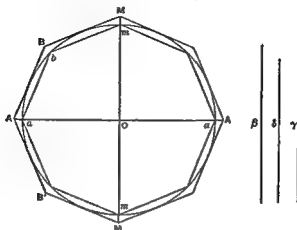


[Props 31, 26]

Then

(surface of outer solid) (surface of inner solid)  
 $= (\text{side of outer})^2 (\text{side of inner})^2$   
 $< \beta^2 \delta^2$ , or  $\beta \gamma$   
 $< (\text{surface of sphere}) \quad C$ , *a fortiori*

[Prop 32]



D A 1

Therefore  $C$  is not less than the surface of the sphere

H C

C 1 1 1

Then, in this case,

(surface of circumscribed solid) (surface of inscribed solid)  
 $< C$  (surface of sphere)

But this is impossible, because the surface of the circumscribed solid is greater than  $C$  [Prop 30] while the surface of the inscribed solid is less than that of the sphere [Prop 23]

Thus  $C$  is not greater than the surface of the sphere

Therefore, since it is neither greater nor less  $C$  is equal to the surface of the sphere

## PROPOSITION 34

Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere

Let the sphere be that of which  $ama'm'$  is a great circle

If now the sphere is not equal to four times the cone described it is either greater or less

I If possible, let the sphere be greater than four times the cone

Suppose  $V$  to be a cone whose base is equal to four times the great circle and whose height is equal to the radius of the sphere

Then, by hypothesis the sphere is greater than  $V$ , and two lines  $\beta, \gamma$  can be found (of which  $\beta$  is the greater) such that

$$\beta \cdot \gamma < (\text{volume of sphere}) \quad V$$

Between  $\beta$  and  $\gamma$  place two arithmetic means  $\delta, \epsilon$

As before, let similar regular polygons with sides  $4n$  in number be circumscribed about and inscribed in the great circle, such that their sides are in a ratio less than  $\beta : \delta$

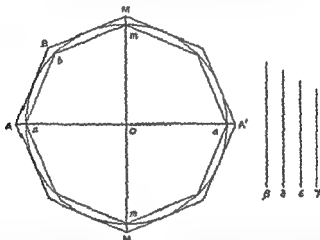
Imagine the diameter  $aa'$  of the circle to be in the same straight line with a diameter of both polygons and imagine the latter to revolve with the circle about  $aa'$  describing the surfaces of two solids of revolution. The volumes of these solids are therefore in the triplicate ratio of their sides [Prop 32]

Thus (vol of outer solid) (vol of inscribed solid)

$$< \beta^3 : \delta^3, \text{ by hypothesis,}$$

$$< \beta : \gamma, \text{ a fortiori (since } \beta : \gamma > \beta^3 : \delta^3),$$

$$< (\text{volume of sphere}) \quad V, \text{ a fortiori}$$



But this is impossible, since the volume of the circumscribed solid is greater than that of the sphere [Prop 28], while the volume of the inscribed solid is less than  $V$  [Prop 27]

Hence the sphere is not greater than  $V$ , or four times the cone described in the enunciation

II If possible let the sphere be less than  $V$

In this case we take  $\beta : \gamma$  ( $\beta$  being the greater) such that

$$\beta \cdot \gamma < V \quad (\text{volume of sphere})$$

The rest of the construction and proof proceeding as before, we have finally

$$(\text{volume of outer solid}) \quad (\text{volume of inscribed solid})$$

$$< V \quad (\text{volume of sphere})$$

But this is impossible because the volume of the outer solid is greater than  $V$  [Prop 31 Cor], and the volume of the inscribed solid is less than the volume of the sphere

Hence the sphere is not less than  $V$

Since then the sphere is neither less nor greater than  $V$ , it is equal to  $V$ , or to four times the cone described in the enunciation

# ON THE SPHERE AND CYLINDER I

COR From what has been  
the greatest circle in a sphere  
is  $\frac{2}{3}$  of the sphere and  
sphere

For the cylinder is three times the cone  
XII 10] i.e. six times the cone with the  
radius of the sphere

But the sphere is four times the latter cone [Prop 34] Therefore the cylinder  
is  $\frac{3}{4}$  of the sphere

Again the surface of a sphere is  
with

circle  
the  
]   
great  
A

area of sphere)

## PROPOSITION 35

If in a segment of a circle  $LAL$  (where  $A$  is the middle point of the arc) a polygon  
 $LKAHL$  be inscribed of which  $LL$  is one side while the other sides are equal  
in number and all equal and if the polygon be cut  
diameter  $AM$   
surface of the  
to the rectangle

$$AB \left( BB + CC + \dots + AA + \frac{LL}{2} \right)$$

The surface of the inscribed frustum  
is

is equal to  
a circle whose radius is  
 $\sqrt{AB \cdot \frac{BB}{2}}$  [Prop 14]

The surface of the frustum of a cone  
 $BCCB$  is equal to a circle whose radius  
is

$$\sqrt{AB \cdot \frac{BB + CC}{2}}, \text{ [Prop 16]}$$

and so on

Proceeding in this way and adding  
we find since circles are to one another  
as the squares of their radii that the

surface of the

$$+ AA + \frac{LL}{2}$$





the frustum  $BCC'B'$  and whose height is  $p$

[Prop 20]

Similarly for the part of the solid described by the triangle  $COD$  as the polygon revolves, and so on

Hence, by addition, the solid figure inscribed in the segment together with the cone  $OLL'$  is equal to a cone whose base is the surface of the inscribed solid and whose height is  $p$

**COR** The cone whose base is a circle with radius equal to  $AL$  and whose height is equal to the radius of the sphere is greater than the sum of the inscribed solid and the cone  $OLL'$

For, by the proposition, the inscribed solid together with the cone  $OLL'$  is equal to a cone with base equal to the surface of the solid and with height  $p$

This latter cone is less than a cone with height equal to  $OA$  and with base equal to the circle whose radius is  $AL$ , because the height  $p$  is less than  $OA$ , while the surface of the solid is less than a circle with radius  $AL$ . [Prop 37]

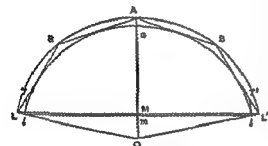
### PROPOSITION 39

Let  $lal'$  be a segment of a great circle of a sphere, being less than a semicircle. Let  $O$  be the centre of the sphere, and join  $Ol, Ol'$ . Suppose a polygon circumscribed about the sector  $Olal'$  such that its sides, excluding the two radii, are  $2n$  in number and all equal, as  $LK, BA, AB', K'L'$ , and let  $OA$  be that radius of the great circle which bisects the segment  $lal'$

The circle circumscribing the polygon will then have the same centre  $O$  as the given great circle

Now suppose the polygon and the two circles to revolve together about  $OA$ . The two circles will describe spheres, the angular points except  $A$  will describe

circles on the outer sphere, with diameters  $BB'$  etc, the points of contact of the sides with the inner segment will describe circles on the inner sphere, the sides themselves will describe the surfaces of cones or frusta of cones, and the whole figure circumscribed to the segment of the inner sphere by the revolution of the equal sides of the polygon will have for its base the circle on  $LL'$  as diameter



The surface of the solid figure so circumscribed about the sector of the sphere [excluding its base] will be greater than that of the segment of the sphere whose base is the circle on  $LL'$  as diameter

I or draw the tangents  $lT, l'T'$  to the inner segment at  $l, l'$ . These with the sides of the polygon will describe by their revolution a solid whose surface is greater than that of the segment [Assumptions, 4]

But the surface described by the revolution of  $lT$  is less than that described by the revolution of  $LT$ , since the angle  $TlL$  is a right angle, and therefore  $LT > lT$

Hence a fortiori the surface described by  $LK, AB, K'L'$  is greater than that of the segment

**Cor** The surface of the figure so described about the sector of the sphere is equal to a circle the square on whose radius is equal to the rectangle

$$AB(BB' + CC' + \dots + KK' + \frac{1}{2}LL')$$

For the circumscribed figure is inscribed in the outer sphere, and the proof of Prop 35 therefore applies

#### PROPOSITION 40

The surface of the figure circumscribed to the sector as before is greater than a circle whose radius is equal to  $al$

Let the diameter  $AaO$  meet the great circle and the circle circumscribing the revolving polygon again in  $a'$ ,  $A'$ . Join  $A'B$ , and let  $ON$  be drawn to  $N$ , the point of contact of  $AB$  with the inner circle

Now, by Prop 39 Cor, the surface of the solid figure circumscribed to the sector  $OLA'$  is equal to a circle the square on whose radius is equal to the rectangle

$$AB(BB' + CC' + \dots + KK' + \frac{LL'}{2})$$

But this rectangle is equal to  $A'B \cdot AM$  [as in Prop 22]

Next since  $AE', a'l'$  are parallel, the triangles  $AML', aml'$  are similar. And  $AL' > a'l'$ , therefore  $AM > am$

Also  $A'B = 2ON = aa'$

Therefore  $A'B \cdot AM > aa' \cdot am$   
 $> a'l'^2$

Hence the surface of the solid figure circumscribed to the sector is greater than a circle whose radius is equal to  $a'l'$ , or  $al$

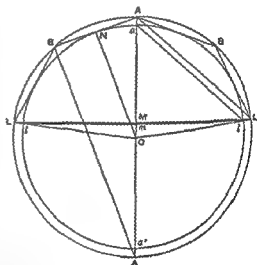
**Cor 1** The volume of the figure circumscribed about the sector together with the cone whose apex is  $O$  and base the circle on  $LL'$  as diameter, is equal to the volume of a cone whose base is equal to the surface of the circumscribed figure and whose height is  $ON$

For the figure is inscribed in the outer sphere which has the same centre as the inner. Hence the proof of Prop 38 applies

**Cor 2** The volume of the circumscribed figure with the cone  $OLL'$  is greater than the cone whose base is a circle with radius equal to  $al$  and whose height is equal to the radius ( $Oa$ ) of the inner sphere

For the volume of the figure with the cone  $OLL'$  is equal to a cone whose base is equal to the surface of the figure and whose height is equal to  $OV$

And the surface of the figure is greater than a circle with radius equal to  $al$  [Prop 40], while the heights  $Oa \cdot OV$  are equal



## PROPOSITION 41

Let  $lal'$  be a segment of a great circle of a sphere which is less than a semicircle

Suppose a polygon inscribed in the sector  $Oal'$  such that the sides  $la$ ,  $ba$ ,  $ab'$ ,  $l'l'$  are  $2n$  in number and all equal. Let a similar polygon be circumscribed about the sector so that its sides are parallel to those of the first polygon, and draw the circle circumscribing the outer polygon.

Now let the polygons and circles revolve together about  $OaA$ , the radius bisecting the segment  $lal'$ .

Then (1) the surfaces of the outer and inner solids of revolution so described are in the ratio of  $AB^2$  to  $ab^2$ , and (2) their volumes together with the corresponding cones with the same base and with apex  $O$  in each case are as  $AB^3$  to  $ab^3$ .

(1) For the surfaces are equal to circles the squares on whose radii are equal respectively to

$$AB \left( BB' + CC' + \dots + KK' + \frac{LL'}{2} \right), \quad [\text{Prop 39, Cor 1}]$$

$$\text{and} \quad ab \left( bb' + cc' + \dots + kk' + \frac{ll'}{2} \right) \quad [\text{Prop 35}]$$

But these rectangles are in the ratio of  $AB^2$  to  $ab^2$ . Therefore so are the surfaces.

(2) Let  $OnN$  be drawn perpendicular to  $ab$  and  $AB$ , and suppose the circles which are equal to the surfaces of the outer and inner solids of revolution to be denoted by  $S$ ,  $s$  respectively.

Now the volume of the circumscribed solid together with the cone  $OLL'$  is equal to a

cone whose base is  $S$  and whose height is  $ON$  [Prop 40, Cor 1]

And the volume of the inscribed figure with the cone  $Oll'$  is equal to a cone with base  $s$  and height  $On$  [Prop 38]

$$\text{But} \quad S = AB^2 \cdot ab^2,$$

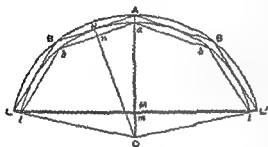
$$\text{and} \quad ON \cdot On = AB \cdot ab$$

Therefore the volume of the circumscribed solid together with the cone  $OLL'$  is to the volume of the inscribed solid together with the cone  $Oll'$  as  $AB^3$  is to  $ab^3$  [Lemma 5]

## PROPOSITION 42

If  $lal'$  be a segment of a sphere less than a hemisphere and  $Oa$  the radius perpendicular to the base of the segment, the surface of the segment is equal to a circle whose radius is equal to  $al$ .

Let  $R$  be a circle whose radius is equal to  $al$ . Then the surface of the segment, which we will call  $S$ , must, if it be not equal to  $R$ , be either greater or less than  $R$ .



I Suppose, if possible,  $S > R$

Let  $lal'$  be a segment of a great circle which is less than a semicircle. Join  $Ol$ ,  $Ol'$ , and let similar polygons with  $2n$  equal sides be circumscribed and inscribed to the sector, as in the previous propositions, but such that (circumscribed polygon) (inscribed polygon)  $< S$   $R$

[Prop 6]

Let the polygons now revolve with the segment about  $OaA$ , generating solids of revolution circumscribed and inscribed to the segment of the sphere

Then

(surface of outer solid) (surface of inner solid)

$$= AB^2 \quad ab^2$$

[Prop 41]

= (circumscribed polygon) (inscribed polygon)

$< S$   $R$ , by hypothesis

But the surface of the outer solid is greater than  $S$  [Prop 39]

Therefore the surface of the inner solid is greater than  $R$ , which is impossible, by Prop 37

II Suppose, if possible,  $S < R$

In this case we circumscribe and inscribe polygons such that their ratio is less than  $R$   $S$ , and we arrive at the result that

(surface of outer solid) (surface of inner solid)

$$< R \quad S$$

But the surface of the outer solid is greater than  $R$  [Prop 40] Therefore the surface of the inner solid is greater than  $S$  which is impossible [Prop 36]

Hence since  $S$  is neither greater nor less than  $R$ ,

$$S = R$$

### PROPOSITION 43

Even if the segment of the sphere is greater than a hemisphere, its surface is still equal to a circle whose radius is equal to  $al$

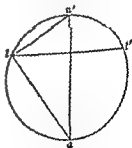
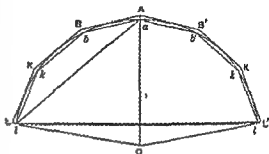
For let  $lal'$  be a great circle of the sphere  $aa'$  being the diameter perpendicular to  $ll'$ , and let  $la'l'$  be a segment less than a semicircle

Then, by Prop 42, the surface of the segment  $la'l'$  of the sphere is equal to a circle with radius equal to  $al$

Also the surface of the whole sphere is equal to a circle with radius equal to  $aa'$  [Prop 33]

But  $aa'^2 - a'l'^2 = al^2$  and circles are to one another as the squares on their radii

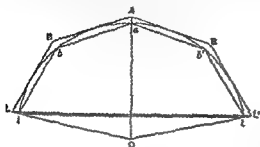
Therefore the surface of the segment  $la'l'$ , being the difference between the surfaces of the sphere and of  $la'l'$ , is equal to a circle with radius equal to  $al$



## PROPOSITION 44

The volume of any sector of a sphere is equal to a cone whose base is equal to the surface of the segment of the sphere included in the sector, and whose height is equal to the radius of the sphere

Let  $R$  be a cone whose base is equal to the surface of the segment  $lal'$  of a sphere and whose height is equal to the radius of the sphere, and let  $S$  be the volume of the sector  $Oal'$



Then, if  $S$  is not equal to  $R$ , it must be either greater or less

I Suppose, if possible, that  $S > R$

Find two straight lines  $\beta$ ,  $\gamma$ , of which  $\beta$  is the greater, such that

$\beta \gamma < S R$ ,  
and let  $\delta$ ,  $\epsilon$  be two arithmetic means between  $\beta$ ,  $\gamma$

Let  $lal'$  be a segment of a great circle of the sphere Join  $Oi$ ,  $Ol'$ , and let similar polygons with  $2n$  equal sides be circumscribed and inscribed to

the sector of the circle as before, but such that their sides are in a ratio less than  $\beta \delta$  [Prop 4]

Then let the two polygons revolve with the segment about  $OaA$ , generating two solids of revolution

Denoting the volumes of these solids by  $V$ ,  $v$  respectively, we have

$$\begin{aligned} (V + \text{cone } OLL') (v + \text{cone } Oll) &= AB^3 ab^3 & [\text{Prop 41}] \\ &< \beta^3 \delta^3 \\ &< \beta \gamma, \text{ a fortiori,} \\ &< S R, \text{ by hypothesis} \end{aligned}$$

Now

$$(V + \text{cone } OLL') > S$$

Therefore also

$$(v + \text{cone } Oll) > R$$

But this is impossible, by Prop 38 Cor combined with Props 42, 43

Hence

$$S > R$$

II Suppose, if possible that  $S < R$

In this case we take  $\beta$ ,  $\gamma$  such that

$$\beta \gamma < R S$$

and the rest of the construction proceeds as before

We thus obtain the relation

$$(V + \text{cone } OLL') (v + \text{cone } Oll) < R S$$

Now

$$(v + \text{cone } Oll) < S$$

Therefore

$$(V + \text{cone } OLL') < R,$$

which is impossible by Prop 40, Cor 2 combined with Props 42, 43

Since then  $S$  is neither greater nor less than  $R$ ,

$$S = R$$

# ON THE SPHERE AND CYLINDER

## BOOK TWO

ARCHIMEDES to Dositheus greeting

On a former occasion you asked me to write out the proofs of the problems the enunciations of which I had myself sent to Conon. In point of fact they depend for the most part on the theorems of which I have already sent you the demonstrations, namely (1) that the surface of any sphere is four times the greatest circle in the sphere, (2) that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of its base, (3) that the cylinder whose

which is equal to the surface of the segment of the sphere included in the sector and whose height is equal to the radius of the sphere. Such then of the theorems and problems as depend on these theorems I have written out in the book which I send herewith; those which are discovered by means of a different sort of investigation, those namely which relate to spirals and the conoids, I will endeavour to send you soon.

The first of the problems was as follows: *Given a sphere, to find a plane area equal to the surface of the sphere.*

The solution of this is obvious from the theorems aforesaid. For four times the greatest circle in the sphere is both a plane area and equal to the surface of the sphere.

The second problem was the following:

### PROPOSITION 1 (PROBLEM)

*Given a cone or a cylinder, to find a sphere equal to the cone or to the cylinder.*

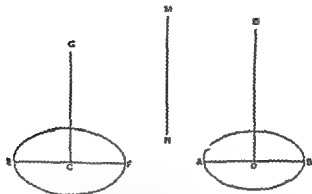
If  $\Gamma$  be the given cone or cylinder, we can make a cylinder equal to  $\Gamma$ . Let this cylinder be the cylinder whose base is the circle on  $AB$  as diameter and whose height is  $OD$ .

Its height is equal to the height (or to the diameter of the base) of the same cylinder.

to the cylinder  
height  $CG$

Then, since in equal cylinders the heights and bases are reciprocally proportional,

$$\begin{aligned} AB^2 \cdot EF^2 &= CG \cdot OD \\ &= EF \cdot OD \end{aligned} \quad (1)$$



Suppose  $MN$  to be such a line that

$$\begin{aligned} EF^2 &= AB \cdot MN \\ \text{Hence } AB \cdot EF &= EF \cdot MN, \end{aligned} \quad (2)$$

and, combining (1) and (2), we have

$$\begin{aligned} AB \cdot MN &= EF \cdot OD, \\ \text{or } AB \cdot EF &= MN \cdot OD \end{aligned}$$

Therefore

$$AB \cdot EF = EF \cdot MN = MN \cdot OD,$$

and  $EF, MN$  are two mean proportionals between  $AB, OD$

The synthesis of the problem is therefore as follows. Take two mean proportionals  $EF, MN$  between  $AB$  and  $OD$ , and describe a cylinder whose base is a circle on  $EF$  as diameter and whose height  $CG$  is equal to  $EF$

Then, since

$$AB \cdot EF = EF \cdot MN = MN \cdot OD,$$

and therefore

$$\begin{aligned} EF^2 &= AB \cdot MN, \\ AB^2 \cdot EF^2 &= AB \cdot MN \\ &= EF \cdot OD \\ &= CG \cdot OD, \end{aligned}$$

whence the bases of the two cylinders ( $OD$ ), ( $CG$ ) are reciprocally proportional to their heights

Therefore the cylinders are equal, and it follows that

$$\text{cylinder } (CG) = \frac{1}{2}V$$

The sphere on  $EF$  as diameter is therefore the sphere required, being equal to  $V$

#### PROPOSITION 2

If  $BAB'$  be a segment of a sphere,  $BB'$  a diameter of the base of the segment, and  $O$  the centre of the sphere and if  $AA'$  be the diameter of the sphere bisecting  $BB'$  in  $M$ , then the volume of the segment is equal to that of a cone whose base is the same as that of the segment and whose height is  $h$ , where

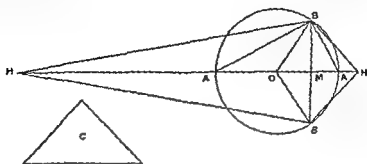
$$h \cdot AM = OA' + A'M \cdot A'M$$

Measure  $MH$  along  $MA$  equal to  $h$  and  $MH'$  along  $MA'$  equal to  $h'$ , where

$$h' \cdot A'M = OA + AM \cdot AM$$



Suppose the three cones constructed which have  $O, H, H'$  for their apices and the base  $(BB')$  of the segment for their common base. Join  $AB, A'B$



Let  $C$  be a cone whose base is equal to the surface of the segment  $BAB'$  of the sphere, i.e. to a circle with radius equal to  $AB$  [I 42], and whose height is equal to  $OA$

Then the cone  $C$  is equal to the solid sector  $OBAB'$  [I 44]

Now, since  $HM \cdot NA = OA' + A'M \cdot A'M$ ,  
 dividendo,  $HA \cdot AM = OA \cdot A'M$ ,  
 and, alternately,  $HA \cdot AO = AM \cdot MA'$ ,  
 so that

$$HO \cdot OA = AA' \cdot A'M$$

$$= AB^2 \cdot BM^2$$

$=$  (base of cone  $C$ ) (circle on  $BB'$  as diameter)

But  $OA$  is equal to the height of the cone  $C$ , therefore, since cones are equal if their bases and heights are reciprocally proportional, it follows that the cone  $C$  (or the solid sector  $OBAB'$ ) is equal to a cone whose base is the circle on  $BB'$  as diameter and whose height is equal to  $OH$

And this latter cone is equal to the sum of two others having the same base and with heights  $OM, MH$ , i.e. to the solid rhombus  $OBHB'$

Hence the sector  $OBAB'$  is equal to the rhombus  $OBHB'$

Taking away the common part, the cone  $OBB'$ ,

the segment  $BAB' =$  the cone  $HBB'$

Similarly, by the same method, we can prove that

the segment  $BA'B' =$  the cone  $H'BB'$

*Alternative proof of the latter property*

Suppose  $D$  to be a cone whose base is equal to the surface of the whole sphere and whose height is equal to  $OA$

Thus  $D$  is equal to the volume of the sphere

[I 33 31]

Now, since  $OA' + A'M \cdot AM = HM \cdot MA$ ,

dividendo and alternando, as before,

$$OA \cdot AH = A'M \cdot MA$$

Again, since

$$HM \cdot MA' = OA + AM \cdot AM,$$

$$H'A' \cdot OA = A'M \cdot MA$$

$$= OA \cdot AH, \text{ from above}$$

Componendo,

$$H'O \cdot OA = OH \cdot HA$$

Alternately,

$$H'O \cdot OH = OA \cdot AH$$

(1)

(2)

and, *componendo*,

$$\begin{aligned} HH' \cdot HO &= OH \cdot HA, \\ &= H'O \cdot OA, \text{ from (1),} \end{aligned}$$

whence

$$HH' \cdot OA = H'O \cdot OH \quad (3)$$

Next, since

$$\begin{aligned} H'O \cdot OH &= OA \cdot AH, \text{ by (2),} \\ &= A'M \cdot MA, \end{aligned}$$

$$(H'O + OH)^2 \cdot H'O \cdot OH = (A'M + MA)^2 \cdot A'M \cdot MA,$$

whence, by means of (3),

$$HH'^2 \cdot HH' \cdot OA = AA'^2 \cdot A'M \cdot MA,$$

or

$$HH' \cdot OA = AA'^2 \cdot \frac{A'M \cdot MA}{HH'}$$

Now the cone  $D$ , which is equal to the sphere, has for its base a circle whose radius is equal to  $AA'$ , and for its height a line equal to  $OA$

Hence this cone  $D$  is equal to a cone whose base is the circle on  $BB'$  as diameter and whose height is equal to  $HH'$ ,

therefore

the cone  $D$  = the rhombus  $HBH'B'$ ,

or

the rhombus  $HBH'B'$  = the sphere

But

the segment  $BAB'$  = the cone  $HBB'$ ,

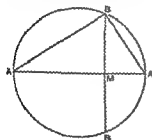
therefore the remaining segment  $BA'B'$  = the cone  $H'B'B'$

Cor. The segment  $BAB'$  is to a cone with the same base and equal height in the ratio of  $OA' + A'M$  to  $A'M$

### PROPOSITION 3 (PROBLEM)

To cut a given sphere by a plane so that the surfaces of the segments may have to one another a given ratio

Suppose the problem solved. Let  $AA'$  be a diameter of a great circle of the sphere, and suppose that a plane perpendicular to  $AA'$  cuts the plane of the great circle in the straight line  $BB'$ , and  $AA'$  in  $M$ , and that it divides the sphere so that the surface of the segment  $BAB'$  has to the surface of the segment  $BA'B'$  the given ratio



Now these surfaces are respectively equal to circles with radii equal to  $AB$ ,  $A'B'$  [I 42, 43]

Hence the ratio  $AB^2 : A'B'^2$  is equal to the given ratio, i.e.  $AM$  is to  $MA'$  in the given ratio

Accordingly the synthesis proceeds as follows

If  $H : K$  be the given ratio, divide  $AA'$  in  $M$  so that

$$AM : MA' = H : K$$

Then  $AM : MA' = AB^2 : A'B'^2$

= (circle with radius  $AB$ ) : (circle with radius  $A'B'$ )

= (surface of segment  $BAB'$ ) : (surface of segment  $BA'B'$ )

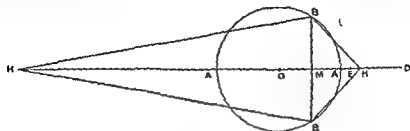
Thus the ratio of the surfaces of the segments is equal to the ratio  $H : K$ .

### PROPOSITION 4 (PROBLEM)

To cut a given sphere by a plane so that the volumes of the segments are to one another in a given ratio

Suppose the problem solved, and let the required plane cut the great circle

$ABA'$  at right angles in the line  $BB'$ . Let  $AA'$  be that diameter of the great circle which bisects  $BB'$  at right angles (in  $M$ ), and let  $O$  be the centre of the sphere.



Take  $H$  on  $OA$  produced, and  $H'$  on  $OA'$  produced, such that

$$OA' + A'M = A'M + HM = MA. \quad (1)$$

and  $OA + AM \quad AM = H'M \quad MA'$  (2)

$$OA+AM \quad AM=H'M \quad MA' \quad (2)$$

Join  $BH, B'H, BH', B'H'$

Then the cones  $HBB'$ ,  $H'B'B'$  are respectively equal to the segments  $BAB'$ ,  $BA'B'$  of the sphere [Prop. 2]

Hence the ratio of the cones, and therefore of their altitudes, is given, i.e.

$$HM/H'M = \text{the given ratio} \quad (3)$$

We have now three equations (1) (2), (3), in which there appear three as yet undetermined points  $M$ ,  $H$ ,  $H'$ , and it is first necessary to find, by means of them, another equation in which only one of these points ( $M$ ) appears, so that we have, so to speak, to eliminate  $H$ ,  $H'$ .

No. 10, and Archi-  
medes of the ratios

$$A'H' \quad "$$

secondly, to equate the ratio compounded of these two ratios to the known value of the ratio  $HH' : H'M$

(a) To find such a value for  $A'H' - H'M$

It is at once clear from equation (2) above that

$$A'H' \perp H'M \Rightarrow OA \perp AM \quad (4)$$

(b) To find such a value for  $HH' = A'H'$

From (1) we derive

$$\begin{aligned} AM \quad MA &= OA' + A'M \quad HM \\ &= OA' \quad AH \end{aligned} \quad (5)$$

and, from (2),  $A'M \cdot MA = H'M \cdot OA + AM^2$   
 $= A'H' \cdot OA$  (6)

Thus  $HA \cong AO \cong OA' \cong A'H'$ .

whence  $OH \cdot OA' = OH' \cdot A'H'$ .

$$\text{or } OH \quad OH' \equiv OA' \quad A'H'.$$

It follows that

Или  $OH' = OH'$   $A'B'$ .

$$III \quad H'A' \approx OH''$$

Therefore  $\begin{matrix} H & H' \\ H & A' \end{matrix} = \begin{matrix} OH^2 & H'A'^2 \\ AA'^2 & A'M^2 \end{matrix}$  by means of (6)

(c) To express the ratios  $A'H'/H'M$  and  $H'H'/H'M$  more simply we make

the following construction Produce  $OA$  to  $D$  so that  $OA = AD$  ( $D$  will be beyond  $H$ , for  $A'M > MA$ , and therefore, by (5),  $OA > AH$ )

$$\begin{aligned} \text{Then} \quad A'H' \cdot H'M &= OA \cdot OA + AM \\ &= AD \cdot DM \end{aligned} \quad (7)$$

Now divide  $AD$  at  $E$  so that

$$HH' \cdot H'M = AD \cdot DE \quad (8)$$

Thus, using equations (8), (7) and the value of  $HH' \cdot H'A'$  above found, we have

$$\begin{aligned} AD \cdot DE &= HH' \cdot H'M \\ &= (HH' \cdot H'A') (A'H' \cdot H'M) \\ &= (AA'^2 \cdot A'M^2) (AD \cdot DM) \end{aligned}$$

$$\text{But} \quad AD \cdot DE = (DM \cdot DE) (AD \cdot DM) \quad (9)$$

Therefore  $MD \cdot DE = AA'^2 \cdot A'M^2$

And  $D$  is given, since  $AD = OA$ . Also  $AD \cdot DE$  (being equal to  $HH' \cdot H'M$ ) is a given ratio. Therefore  $DE$  is given.

Hence the problem reduces itself to the problem of dividing  $A'D$  into two parts at  $M$  so that

$$MD \text{ (a given length)} = \text{(a given area)} \cdot A'M^2$$

Archimedes adds "If the problem is propounded in this general form, it requires a *diopisμus* [i.e. it is necessary to investigate the limits of possibility], but, if there be added the conditions subsisting in the present case, it does not require a *diopisμus*."

In the present case the problem is

Given a straight line  $A'A$  produced to  $D$  so that  $AA' = 2AD$ , and given a point  $E$  on  $AD$ , to cut  $AA'$  in a point  $M$  so that

$$AA'^2 \cdot A'M^2 = MD \cdot DE$$

"And the analysis and synthesis of both problems will be given at the end."

The synthesis of the main problem will be as follows. Let  $R/S$  be the given ratio,  $R$  being less than  $S$ .  $AA'$  being a diameter of a great circle, and  $O$  the centre, produce  $OA$  to  $D$  so that  $OA = AD$ , and divide  $AD$  in  $E$  so that

$$AE \cdot ED = R \cdot S$$

Then cut  $AA'$  in  $M$  so that

$$MD \cdot DE = AA'^2 \cdot A'M^2$$

Through  $M$  erect a plane perpendicular to  $AA'$ , this plane will then divide the sphere into segments which will be to one another as  $R$  to  $S$ .

Take  $H$  on  $A'A$  produced, and  $H'$  on  $AA'$  produced, so that

$$OA + A'M \cdot A'M = HM \cdot MA, \quad (1)$$

$$OA + AM \cdot AM = H'M \cdot MA' \quad (2)$$

We have then to show that

$$HM \cdot MH' = R \cdot S, \text{ or } AE \cdot ED$$

(a) We first find the value of  $HH' \cdot M'A'$  as follows

As was shown in the analysis (b)

$$HH' \cdot H'A' = OH^2,$$

$$\text{or} \quad HH' \cdot H'A' = OH^2 \cdot H'A'^2$$

$$= AA'^2 \cdot A'M^2$$

$$= MD \cdot DE, \text{ by construction}$$

<sup>1</sup>As Archimedes' commentator, Eutocius notes "we do not find the promise kept in any of the copies. Sir Thomas Heath's translation of Eutocius' note on the matter along with the solutions of Dionysodorus and Diocles is omitted from this edition.—Eo

(β) Next we have

$$\begin{aligned} H'A' \cdot H'M &= OA \cdot OA + AM \\ &= AD \cdot DM \end{aligned}$$

$$\begin{aligned} \text{Therefore } HH' \cdot H'M &= (HH' \cdot H'A') (H'A' \cdot H'M) \\ &= (MD \cdot DE) (AD \cdot DM) \\ &= AD \cdot DE, \end{aligned}$$

$$\begin{aligned} \text{whence } HM \cdot MH' &= AE \cdot ED \\ &= R \cdot S \end{aligned}$$

Q.E.D.

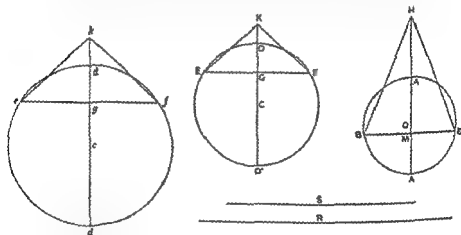
### PROPOSITION 5 (PROBLEM)

To construct a segment of a sphere similar to one segment and equal in volume to another

Let  $ABB'$  be one segment whose vertex is  $A$  and whose base is the circle on  $BB'$  as diameter, and let  $DEF$  be another segment whose vertex is  $D$  and whose base is the circle on  $EF$  as diameter. Let  $AA'$ ,  $DD'$  be diameters of the great circles passing through  $BB'$ ,  $EF$  respectively, and let  $O$ ,  $C$  be the respective centres of the spheres.

Suppose it required to draw a segment similar to  $DEF$  and equal in volume to  $ABB'$ .

*Analysis.* Suppose the problem solved, and let  $def$  be the required segment,  $d$  being the vertex and  $ef$  the diameter of the base. Let  $dd'$  be the diameter of the sphere which bisects  $ef$  at right angles,  $c$  the centre of the sphere.



Let  $M$ ,  $G$ ,  $g$  be the points where  $BB'$ ,  $EF$ ,  $ef$  are bisected at right angles by  $AA'$ ,  $DD'$ ,  $dd'$  respectively and produce  $OA$ ,  $CD$ ,  $cd$  respectively to  $H$ ,  $K$ ,  $I$ , so that

$$\left. \begin{aligned} OA + A'M \cdot A'M &= HM \cdot MA \\ CD + D'G \cdot D'G &= KG \cdot GD \\ cd + d'g \cdot d'g &= kg \cdot gd \end{aligned} \right\},$$

and suppose cones formed with vertices  $H$ ,  $K$ ,  $I$  and with the same bases as the respective segments. The cones will then be equal to the segments respectively [Prop 2]

Therefore, by hypothesis,

the cone  $HBB'$  = the cone  $\lambda ef$

Hence

(circle on diameter  $BB'$ ) (circle on diameter  $ef$ ) =  $kg$   $HM$ ,

so that  $BB'^2 ef^2 = kg$   $HM$  (1)

But, since the segments  $DEF$ ,  $def$  are similar, so are the cones  $\lambda EF$ ,  $\lambda ef$

Therefore  $\lambda G$   $EF = kg$   $ef$

And the ratio  $KG$   $EF$  is given Therefore the ratio  $kg$   $ef$  is given

Suppose a length  $R$  taken such that

$$kg$$
  $ef = HM$   $R$  (2)

Thus  $R$  is given

Again, since  $kg$   $HM = BB'^2$   $ef^2 = ef$   $R$ , by (1) and (2), suppose a length  $S$  taken such that

$$\begin{aligned} ef^2 &= BB' S, \\ BB'^2 ef^2 &= BB' S \\ BB' ef &= ef S = S R, \end{aligned}$$

Let  $R$  be a straight line such that

Let  $R$  be a straight line such that

$$\lambda G$$
  $EF = HM$   $R$ ,

Let  $R$  be a straight line such that

through  $d$ , and  $c$  the centre. Conceive a sphere constructed of which  $def$  is a great circle, and through  $ef$  draw a plane at right angles to  $dd'$

Then shall  $def$  be the required segment of a sphere

For the segments  $DEF$ ,  $def$  of the spheres are similar, like the circular segments  $DEF$ ,  $def$

Produce  $cd$  to  $\lambda$  so that

$$cd' + d'g$$
  $dg = kg$   $gd$

The cones  $\lambda EF$ ,  $\lambda ef$  are then similar

Therefore  $kg$   $ef = KG$   $EF = HM$   $R$ ,

whence  $kg$   $HM = ef$   $R$

But since  $BB'$ ,  $ef$ ,  $S$ ,  $R$  are in continued proportion,

$$\begin{aligned} BB'^2 ef^2 &= BB' S \\ &= ef R \\ &= kg HM \end{aligned}$$

Thus the bases of the cones  $HBB$ ,  $\lambda ef$  are reciprocally proportional to their heights. The cones are therefore equal, and  $def$  is the segment required, being equal in volume to the cone  $\lambda ef$  [Prop 2]

# PROPOSITION II (PROBLEM)

Given two segments of spheres to find a third segment of a sphere similar to one of the given segments and having its surface equal to that of the other

Let  $ABB$  be the segment to whose surface the surface of the required segment is to be equal,  $ABA$   $B$  the great circle whose plane cuts the plane of the

base of the segment  $ABB'$  at right angles in  $BB'$ . Let  $AA'$  be the diameter which bisects  $BB'$  at right angles.

Let  $DEF$  be the segment to which the required segment is to be similar,  $DED'F$  the great circle cutting the base of the segment at right angles in  $EF$ . Let  $DD'$  be the diameter bisecting  $EF$  at right angles in  $G$ .

Suppose the problem solved,  $def$  being a segment similar to  $DEF$  and having its surface equal to that of  $ABB'$ , and complete the figure for  $def$  as for  $DEF$ , corresponding points being denoted by small and capital letters respectively.

Join  $AB$ ,  $DF$ ,  $df$ .

Now, since the surfaces of the segments  $def$ ,  $ABB'$  are equal, so are the circles on  $df$ ,  $AB$  as diameters,

[1 42, 43]

that is  $df = AB$ .

From the similarity of the segments  $DEF$ ,  $def$  we obtain

$$d'd : dg = D'D : DG,$$

and  $dg : df = DG : DF$ ,

whence  $d'd : df = D'D : DF$ ,

or  $d'd : AB = D'D : DF$ .

But  $AB$ ,  $D'D$ ,  $DF$  are all given,

therefore  $d'd$  is given.

Accordingly the synthesis is as follows.

Take  $dd$  such that

$$d'd : AB = D'D : DF \quad (1)$$

Describe a circle on  $dd$  as diameter, and conceive a sphere constructed of which this circle is a great circle.

Divide  $dd$  at  $g$  so that

$$dg : gd = D'G : GD,$$

and draw through  $g$  a plane perpendicular to  $d'd$  cutting off the segment  $def$  of the sphere and intersecting the plane of the great circle in  $ef$ . The segments  $def$ ,  $DEF$  are thus similar and

$$dg : df = DG : DF$$

But from above *componendo*,

$$d'd : dg = D'D : DG$$

Therefore *ex aequali*,  $d'd : df = D'D : DF$

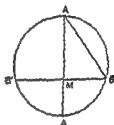
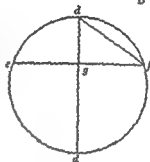
whence by (1),  $df = AB$ .

Therefore the segment  $def$  has its surface equal to the surface of the segment  $ABB'$  [1 42 43], while it is also similar to the segment  $DEF$ .

#### PROPOSITION 7 (PROBLEM)

From a given sphere to cut off a segment by a plane so that the segment may have a given ratio to the cone which has the same base as the segment and equal height.

Let  $AA'$  be the diameter of a great circle of the sphere. It is required to draw a plane at right angles to  $AA'$  cutting off a segment, as  $ABB'$ , such that the segment  $ABB'$  has to the cone  $ABB'$  a given ratio.



## Analysis

Suppose the problem solved and let the plane of section cut the plane of the great circle in  $BB'$ , and the diameter  $AA'$  in  $M$ . Let  $O$  be the centre of the sphere.

Produce  $OA'$  to  $H$  so that

$$OA' + A'M = HM = MA \quad (1)$$

Thus the cone  $HBB'$  is equal to the segment  $ABB'$

[Prop 2]

Therefore the given ratio must be equal to the ratio of the cone  $HBB'$  to the cone  $ABB'$ , i.e. to the ratio  $HM : MA$ .

Hence the ratio  $OA' + A'M : A'M$  is given, and therefore  $A'M$  is given.

*διόριστος*

Now  
so that

$$\begin{aligned} OA' + A'M &> OA' : A'A \\ OA' + A'M : A'M &> OA' + A'A : A'A \\ &> 3 : 2 \end{aligned}$$

Thus in order that a solution may be possible it is a necessary condition that the given ratio must be greater than  $3 : 2$ .

The synthesis proceeds thus

Let  $AA'$  be a diameter of a great circle of the sphere  $O$  the centre.

Take a line  $DE$  and a point  $F$  on it, such that  $DE : EF$  is equal to the given ratio, being greater than  $3 : 2$ .

Now, since

$$\begin{aligned} OA' + A'A : A'A &= 3 : 2 \\ DE : EF &> OA' + A'A : A'A \\ DF : FE &> OA' : A'A \end{aligned}$$

so that

Hence a point  $M$  can be found on  $AA'$  such that

$$DF : FE = OA' : A'M \quad (2)$$

Through  $M$  draw a plane at right angles to  $AA'$  intersecting the plane of the great circle in  $BB'$  and cutting off from the sphere the segment  $ABB'$ .

As before take  $H$  on  $OA'$  produced such that

$$OA' + A'M = HM = MA$$

Therefore  $HM : MA = DE : EF$  by means of (2).

It follows that the cone  $HBB'$  or the segment  $ABB'$  is to the cone  $ABB'$  in the given ratio  $DE : EF$ .

## PROPOSITION 8

If a sphere be cut by a plane passing through the centre into two segments  $ABB'$   $A'B'B$  of which  $ABB'$  is the greater then the ratio (segmt  $ABB'$ ) (segmt  $A'B'B$ )

$$< (\text{surface of } A'B'B)^2 : (\text{surface of } ABB')$$

$$\text{but} > (\text{surface of } A'B'B)^3 : (\text{surface of } ABB')^3$$

Let the plane of section cut a great circle  $ABAB'$  at right angles in  $BB'$ , and let  $AA'$  be the diameter bisecting  $BB'$  at right angles in  $M$ .

Let  $O$  be the centre of the sphere



$R$  will fall between  $O$  and  $M$

Also, since  $AB^2 = DE^2$ ,  $AR = CD$

Produce  $OA'$  to  $K$  so that  $OA' = A'K$ , and produce  $A'A$  to  $H$  so that

$$A'K : A'M = HA : AM,$$

or, *componendo*,  $A'K + A'M : A'M = HM : MA$  (1)

Thus the cone  $HBB'$  is equal to the segment  $ABB'$  [Prop 2]

Again, produce  $CD$  to  $F$  so that  $CD = DF$ , and the cone  $FEE'$  will be equal to the hemisphere  $DEE'$  [Prop 2]

Now  $AR : RA' > AM : MA'$ ,

and  $AR^2 = \frac{1}{2}AB^2 = \frac{1}{2}AM \cdot AA' = AM \cdot A'K$

Hence

$$AR : RA' + RA^2 > AM : MA' + AM \cdot A'K,$$

or  $AA' : AR > AM : MK$

$$> HM : A'M, \text{ by (1)}$$

Therefore  $AA' : A'M > HM : AR$ ,

or  $AB^2 : BM^2 > HM : AR$ ,

i.e.  $AR^2 : BM^2 > HM : 2AR$ , since  $AB^2 = 2AR^2$ ,

$$> HM : CF$$

Thus since  $AR = CD$ , or  $CE$ ,

$$(\text{circle on diam } EE') (\text{circle on diam } BB') > HM : CF$$

It follows that

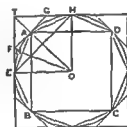
$$(\text{the cone } FEE') > (\text{the cone } HBB'),$$

and therefore the hemisphere  $DEE'$  is greater in volume than the segment  $ABB'$

# MEASUREMENT OF A CIRCLE

## PROPOSITION 1

*The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle*



Let  $ABCD$  be the given circle,  $K$  the triangle described

Then, if the circle is not equal to  $K$ , it must be either greater or less

I If possible, let the circle be greater than  $K$

Inscribe a square  $ABCD$ , bisect the arcs  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ , then bisect (if

the sides about the right angle in  $K$ . Also the perimeter of the polygon is less than the circumference of the circle, i.e. less than the other side about the right angle in  $K$ .

Therefore the area of the polygon is less than  $K$ , which is inconsistent with the hypothesis

Thus the area of the circle is not greater than  $K$

II If possible, let the circle be less than  $K$

tangents at the points of bisection. Let  $A$  be the middle point of the arc  $EH$ , and  $FAG$  the tangent at  $A$ .

Then the angle  $TAG$  is a right angle.

Therefore  $TG > GA$   
 $> GH$

It is greater than half the area  $TEAH$ .

be drawn, so that the area of the polygon is more than one-half the area of the circle.

Thus, by continuing the process, we may eventually arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of  $K$  over the area of the circle.

Thus the area of the polygon will be less than  $K$ .

Now, since the perpendicular from  $O$  on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle  $K$ , which is impossible.

Therefore the area of the circle is not less than  $K$ .

Since then the area of the circle is neither greater nor less than  $A$ , it is equal to it.

#### PROPOSITION 2

*The area of a circle is to the square on its diameter as 11 to 14.*

#### PROPOSITION 3

*The ratio of the circumference of any circle to its diameter is less than  $3\frac{1}{4}$  but greater than  $3\frac{1}{7}$ .*

Let  $AB$  be the diameter of any circle  $O$  its centre,  $AC$  the tangent at  $A$ , and let the angle  $AOC$  be one-third of a right angle.

Then  $OA : AC = \sqrt{3} : 1 > 265 : 153$  (1)

and  $OC : CA = 2 : 1 = 306 : 153$  (2)

First draw  $OD$  bisecting the angle  $AOC$  and meeting  $AC$  in  $D$ .

Now  $CO : OD = CD : DA$  [Eucl VI 3]

so that  $[CO + OD : OD = CA : DA, \text{ or}]$

$CO + OD : CA = OA : AD$

Therefore [by (1) and (2)]

$OA : AD > 571 : 153$  (3)

The text of this proposition is not satisfactory and Archimedes cannot have placed it before Proposition 3 as the approximation depends upon the result of that proposition.

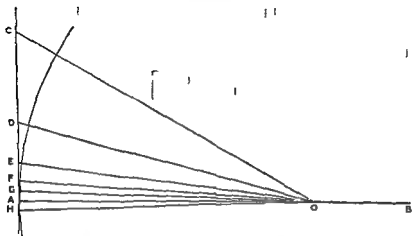
In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes it is necessary in reproducing it to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to  $\sqrt{3}$  (one being less and the other greater than the real value) without any explanation as to how he arrived at them, and in like manner approximations to the square roots of several large numbers which are not complete squares are merely stated.

Hence

$$\begin{aligned} OD^2 AD^2 &= (OA^2 + AD^2) AD^2 \\ &> (571^2 + 153^2) 153^2 \\ &> 349450 \quad 23409, \end{aligned}$$

so that

$$OD DA > 591\frac{1}{2} \quad 153 \quad (4)$$



Secondly, let  $OE$  bisect the angle  $AOD$ , meeting  $AD$  in  $E$

[Then  $DO OA = DE EA$ ,

so that  $DO + OA DA = OA AE$ ]

Therefore  $OA AE [> (591\frac{1}{2} + 571) 153, \text{ by (3) and (4)}]$

$$> 1162\frac{1}{2} \quad 153$$

(5)

[It follows that

$$\begin{aligned} OE^2 EA^2 &> \{(1162\frac{1}{2})^2 + 153^2\} 153^2 \\ &> (1350534\frac{1}{4} + 23409) \quad 23409 \\ &> 1373943\frac{1}{4} \quad 23409 \end{aligned}$$

Thus  $OE EA > 1172\frac{1}{2} \quad 153$

(6)

Thirdly, let  $OF$  bisect the angle  $AOE$  and meet  $AE$  in  $F$

We thus obtain the result [corresponding to (3) and (5) above] that

$$\begin{aligned} OA AF [> (1162\frac{1}{2} + 1172\frac{1}{2}) 153] \\ &> 2334\frac{1}{2} \quad 153 \end{aligned}$$

(7)

[Therefore  $OF^2 FA^2 > \{(2334\frac{1}{2})^2 + 153^2\} 153^2$

$$> 5472132\frac{1}{4} \quad 23409]$$

Thus  $OF FA > 2339\frac{1}{2} \quad 153$

(8)

Fourthly, let  $OG$  bisect the angle  $AOF$ , meeting  $AF$  in  $G$

We have then

$$\begin{aligned} OA AG [> (2334\frac{1}{2} + 2339\frac{1}{2}) 153, \text{ by means of (7) and (8)}] \\ &> 4673\frac{1}{2} \quad 153 \end{aligned}$$

Now the angle  $AOC$ , which is one-third of a right angle has been bisected four times and it follows that

$$\angle AOG = \frac{1}{16} \text{ (a right angle)}$$

Make the angle  $AOH$  on the other side of  $OA$  equal to the angle  $AOG$ , and let  $GA$  produced meet  $OH$  in  $H$

Then  $\angle GOH = \frac{1}{8} \text{ (a right angle)}$

Thus  $GH$  is one side of a regular polygon of 96 sides circumscribed to the given circle

And, since  
while  
it follows that

$$\begin{aligned} OA \ AG &> 4673\frac{1}{2} \quad 153, \\ AB &= 2OA, \ GH = 2AG, \end{aligned}$$

$$\begin{aligned} AB \text{ (perimeter of polygon of 96 sides)} &[> 4673\frac{1}{2} \quad 153 \times 96] \\ &> 4673\frac{1}{2} \quad 14688 \end{aligned}$$

But

$$\begin{aligned} \frac{14688}{4673\frac{1}{2}} &= 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} \\ &\left[ < 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}} \right] \\ &< 3\frac{1}{2} \end{aligned}$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than  $3\frac{1}{2}$  times the diameter  $AB$

If Next let  $AB$  be the diameter of a circle, and let  $AC$ , meeting the circle in  $C$ , make the angle  $CAB$  equal to one-third of a right angle Join  $BC$

Then  $AC \ CB [= \sqrt{3} \ 1] < 1351 \ 780$

First let  $AD$  bisect the angle  $BAC$  and meet  $BC$  in  $d$  and the circle in  $D$  Join  $BD$

$$\begin{aligned} \text{Then} \quad \angle BAD &= \angle DAC \\ &= \angle CBD, \end{aligned}$$

and the angles at  $D, C$  are both right angles

It follows that the triangles  $ADB \ ACd \ BDd$  are similar

$$\begin{aligned} \text{Therefore} \quad \frac{AD}{DB} &= \frac{BD}{Dd} \\ &= \frac{AC}{Cd} \\ &= \frac{AB}{Bd} \\ &= \frac{AB+AC}{BC} \end{aligned} \quad [\text{Eucl vi 3}]$$

$$\begin{aligned} \text{or} \quad BA+AC \ BC &= AD \ DB \\ \text{[But} \quad AC \ CB &< 1351 \ 780 \text{ from above,} \\ \text{while} \quad BA \ BC &= 2 \ 1 \\ &= 1560 \ 780 \end{aligned}$$

$$\text{Therefore} \quad AD \ DB < 2911 \ 780 \quad (1)$$

$$\begin{aligned} \text{[Hence} \quad AB^2 \ BD^2 &< (2911^2 + 780^2) \ 780^2 \\ &< 9082321 \ 608100 \end{aligned}$$

$$\text{Thus} \quad AB \ BD < 3013\frac{1}{2} \ 780 \quad (2)$$

Secondly let  $AE$  bisect the angle  $BAD$ , meeting the circle in  $E$ , and let  $BE$  be joined

Then we prove in the same way as before that

$$\begin{aligned} \frac{AE}{EB} &= \frac{BA+AD}{BD} \\ &< \frac{(3013\frac{1}{2} + 2911) \ 780}{\text{by (1) and (2)}} \\ &< 5921\frac{1}{2} \ 780 \\ &< 5921\frac{1}{2} \times \frac{1}{2} \ 780 \times \frac{1}{2} \\ &< 1823 \ 210 \end{aligned} \quad (3)$$

$$\begin{aligned} \text{[Hence} \quad AB^2 \ BE^2 &< (1823^2 + 210^2) \ 210^2 \\ &< 3380929 \ 57600 \end{aligned} \quad (4)$$

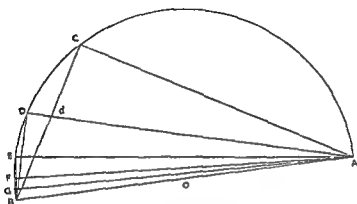
$$\text{Therefore} \quad AB \ BE < 1839\frac{1}{2} \ 210$$

Thirdly, let  $AF$  bisect the angle  $BAE$ , meeting the circle in  $F$

## MEASUREMENT OF A CIRCLE

$$\begin{aligned}
 AF \cdot FB & [= BA + AE \cdot BE] \\
 & < 3661 \frac{1}{11} \cdot 240, \text{ by (3) and (4)} \\
 & < 3661 \frac{1}{11} \times 1 \frac{1}{11} \cdot 240 \times 1 \frac{1}{11} \\
 & < 1007 \cdot 66
 \end{aligned}$$

(5)



[It follows that

$$\begin{aligned}
 AB^2 \cdot BF^2 & < (1007^2 + 66^2) \cdot 66^2 \\
 & < 1018405 \cdot 4356
 \end{aligned}$$

(6)

Therefore

Fourthly, let the angle BAF be bisected by AG meeting the circle in G

Then

$$\begin{aligned}
 AG \cdot GB & [= BA + AF \cdot BF] \\
 & < 2016 \frac{1}{4} \cdot 66, \text{ by (5) and (6)} \\
 AB^2 \cdot BG^2 & < \{ (2016 \frac{1}{4})^2 + 66^2 \} \cdot 66^2 \\
 & < 4069234 \frac{1}{4} \cdot 4356
 \end{aligned}$$

[And

$$\begin{aligned}
 AB \cdot BG & < 2017 \frac{1}{4} \cdot 66, \\
 BG \cdot AB & > 60 \cdot 2017 \frac{1}{4}
 \end{aligned}$$

Therefore  
whence

[Now the angle BAG which is the result of the fourth bisection of the angle BAC, or of one-third of a right angle, is equal to one-fortyeighth of a right angle]

Thus the angle subtended by BG at the centre is  $\frac{1}{48}$  (a right angle)

Therefore BG is a side of a regular inscribed polygon of 96 sides

It follows from (7) that

$$\begin{aligned}
 (\text{perimeter of polygon}) \cdot AB & > 96 \times 66 \cdot 2017 \frac{1}{4} \\
 & > 6336 \cdot 2017 \frac{1}{4}
 \end{aligned}$$

And

$$\frac{6336}{2017 \frac{1}{4}} > 3 \frac{1}{4}$$

Much more then is the circumference of the circle greater than  $3 \frac{1}{4}$  times the diameter

Thus the ratio of the circumference to the diameter  $< 3 \frac{1}{2}$  but  $> 3 \frac{1}{4}$

# ON CONOIDS AND SPHEROIDS

## INTRODUCTION<sup>1</sup>

<sup>1</sup> ARCHIMEDES to Dositheus greeting

' In this book I have set forth and send you the proofs of the remaining theorems not included in what I sent you before and also of some others discovered later which though I had often tried to investigate them previously I had failed to arrive at because I found their discovery attended with some difficulty And this is why even the propositions themselves were not published with the rest But afterwards when I had studied them with greater care I discovered what I had failed in before

Now the remainder of the earlier theorems were propositions concerning

I Concerning the right-angled conoid it was laid down that if a section of a right-angled cone [a parabola] be made to revolve about the diameter [axis] which remains fixed and return to the position from which it started the figure comprehended by the section of the right angled cone is called a *right-angled conoid* and the diameter which has remained fixed is called its *axis* while its *vertex* is the point in which the axis meets the surface of the conoid And if a

axis of the conoid

The questions propounded for consideration were

(1) why if a segment of the right angled conoid be cut off by a plane at right angles to the axis will the segment so cut off be half as large again as the

duplicate ratio of their axes

II Respecting the *obtuse-angled conoid* we lay down the following premises If there be in a plane a section of an obtuse-angled cone [a hyperbola] its

The whole of this introductory matter including the definitions is translated literally from the Greek faithfully represented by modern phraseology we come to the

the obtuse-angled cone [the asymptotes] will clearly comprehend an isosceles

*conoid* [hyperboloid of revolution], its *axis* is the diameter which has remained fixed and its *vertex* the point in which the axis meets the surface of the conoid. The cone comprehended by the nearest lines to the section of the obtuse-angled cone is called [the cone] *enveloping the conoid* and the straight line between the vertex of the conoid and the vertex of the cone enveloping the conoid is called [the line] *adjacent to the axis*. And if a plane touch the obtuse-angled conoid and another plane drawn parallel to the tangent plane cut off a

drawn through the vertex of the segment and the vertex of the cone enveloping the conoid, and the straight line between the said vertices is called *adjacent to the axis*.

"Right angled conoids are all similar, but of obtuse-angled conoids let those be called similar in which the cones enveloping the conoids are similar

the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis, and

(2) "why if a segment of the obtuse-angled conoid be cut off by a plane not at right angles to the axis, the segment so cut off will bear to the figure which has the same base as the segment and the same axis being a segment of a cone, the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis

III Concerning spheroidal figures we lay down the following premisses. If a section of an acute-angled cone [ellipse] be made to revolve about the greater diameter [major axis] which remains fixed and return to the position from

to the position from which it started, the figure comprehended by the section

middle point of the axis, and the diameter as the line drawn through the centre at right angles to the axis. And, if parallel planes touch, without cutting, either



their *vertices* as the points in which the parallel planes touch the spheroid, and their *axes* as the portions cut off within the segments from the straight line joining their vertices. And that the planes touching the spheroid meet its surface at one point only, and that the straight line joining the points of contact passes through the centre of the spheroid, we shall prove. Those spheroidal figures are called *similar* in which the axes have the same ratio to the 'diameters'. And let segments of spheroidal figures and conoids be called *similar* if they are cut off from similar figures and have their bases similar, while their axes, being either at right angles to the planes of the bases or making equal angles with the corresponding diameters [axes] of the bases, have the same ratio to one another as the corresponding diameters [axes] of the bases.

"The following questions about spheroids are propounded for consideration,"

(1) "why, if one of the spheroidal figures be cut by a plane through the centre at right angles to the axis, each of the resulting segments will be double

the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the lesser segment bears to the axis of the lesser segment, and (b) the lesser segment bears to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the greater segment bears to the axis of the greater segment",

(2) "why, if one of the spheroids be cut by a plane passing through the centre but not at right angles to the axis, each of the resulting segments will be double of the figure having the same base as the segment and the same axis and consisting of a segment of a cone

(3) "But if the plane cutting the spheroid be neither through the centre nor at right angles to the axis (a) the greater of the resulting segments will have to the figure which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the seg-

line joining the vertices of the segments and the axis of the greater segment bears to the axis of the greater segment. And the figure referred to is in these cases also a segment of a cone

"When the aforesaid theorems are proved, there are discovered by means of them many theorems and problems

"Such, for example, are the theorems"

(1) "that similar spheroids and similar segments both of spheroidal figures and conoids have to one another the triplicate ratio of their axes, and"

(2) "that in equal spheroidal figures the squares on the 'diameters' are reciprocally proportional to the axes, and, if in spheroidal figures the squares on

the 'diameters' are reciprocally proportional to the axes, the spheroids are equal

'Such also is the problem, From a given spheroidal figure or conoid to cut off a segment by a plane drawn parallel to a given plane so that the segment cut off is equal to a given cone or cylinder or to a given sphere

"After prefixing therefore the theorems and directions which are necessary for the proof of them, I will then proceed to expound the propositions themselves to you Farewell"

### DEFINITIONS

"If a cone be cut by a plane meeting all the sides [generators] of the cone, the section will be either a circle or a section of an acute-angled cone [an ellipse] If then the section be a circle, it is clear that the segment cut off from the cone towards the same parts as the vertex of the cone will be a cone But, if the section be a section of an acute-angled cone [an ellipse] let the figure cut off from the cone towards the same parts as the vertex of the cone be called a

of the cone to the centre of the section of the acute-angled cone

"And if a cylinder be cut by two parallel planes meeting all the sides [generators] of the cylinder, the sections will be either circles or sections of acute-

[ellipses] let the figure cut off from the cylinder between the parallel planes be called a *frustum of a cylinder* And let the bases of the frustum be defined as the

the cylinder "

### LEMMA

If in an ascending arithmetical progression consisting of the magnitudes  $A_1, A_2, \dots, A_n$  the common difference be equal to the least term  $A_1$ , then

$$n A_n < 2(A_1 + A_2 + \dots + A_n),$$

and

$$> 2(A_1 + A_2 + \dots + A_{n-1})$$

[The proof of this is given incidentally in the treatise *On Spirals* Prop 11 By placing lines side by side to represent the terms of the progression and then producing each so as to make it equal to the greatest term, Archimedes gives the equivalent of the following proof

$$\begin{aligned} \text{If} \quad S_n &= A_1 + A_2 + \dots + A_{n-1} + A_n, \\ \text{we have also} \quad S_n &= A_n + A_{n-1} + A_{n-2} + \dots + A_1 \\ \text{And} \quad A_1 + A_{n-1} &= A_2 + A_{n-2} = \dots = A_n \end{aligned}$$

$$\begin{aligned} \text{Therefore} \quad 2S_n &= (n+1)A_n, \\ \text{whence} \quad n A_n &< 2S_n, \\ \text{and} \quad n A_n &> 2S_{n-1} \end{aligned}$$

Thus, if the progression is a  $2a, na,$

$$S_n = \frac{n(n+1)}{2}a,$$

## ARCHIMEDES

$$n^2a < 2S_n \\ > 2S_{n-1}]$$

## PROPOSITION 1

If  $A_1, B_1, C_1, K_1$  and  $A_2, B_2, C_2, K_2$  be two series of magnitudes such that  
 $A_1, B_1 = A_2, B_2$   
 $B_1, C_1 = B_2, C_2$  and so on } (a)

and if  $A_2, B_2, C_2, K_2$  and  $A_3, B_3, C_3, K_3$  be two other series such that  
 $A_2, B_2 = A_3, B_3$  and so on } (b)

then

$$(A_1 + B_1 + C_1 + \dots + K_1) (A_2 + B_2 + C_2 + \dots + K_2) \\ = (A_2 + B_2 + C_2 + \dots + K_2) (A_3 + B_3 + C_3 + \dots + K_3)$$

The proof is as follows

Since

and

while

we have *ex aequali*,

Similarly

Again it follows from equations (a) that

$$A_1, A_2 = A_3, A_4, A_5, \\ A_1, B_1 = A_2, B_2, \\ B_1, B_2 = B_3, B_4, \\ A_2, B_2 = A_3, B_3, \\ B_2, C_2 = B_3, C_3, \text{ and so on } \}$$

(7)

Therefore

$$A_1, A_2 = (A_1 + B_1 + C_1 + \dots + K_1) (A_2 + B_2 + \dots + K_2), \\ (1 + B_1 + C_1 + \dots + K_1) A_1 = (A_2 + B_2 + C_2 + \dots + K_2) A_2,$$

or

and

while from equations (7) it follows in like manner that

$$A_2, (A_2 + B_2 + C_2 + \dots + K_2) = A_3, (A_3 + B_3 + C_3 + \dots + K_3) \\ B_1, (1 + B_1 + C_1 + \dots + K_1) (A_2 + B_2 + C_2 + \dots + K_2) \\ = (A_2 + B_2 + C_2 + \dots + K_2) (A_3 + B_3 + C_3 + \dots + K_3)$$

Con If any terms in the third and fourth series corresponding to terms in the first and second be left out the result is the same. For example, if the last terms  $A_1, A_2$  are absent

$$(A_1 + B_1 + C_1 + \dots + K_1) (A_2 + B_2 + C_2 + \dots + K_2) \\ = (A_2 + B_2 + C_2 + \dots + K_2) (A_3 + B_3 + C_3 + \dots + K_3),$$

where  $I$  immediately precedes  $A_1$  in each series

### LEMMA TO PROPOSITION 2

[On Spirals Prop 10]

If  $1, 1_1, A_1, A_n$  be  $n$  lines forming an ascending arithmetical progression in which the common difference is equal to the least term  $A_1$  then

$$(n+1) 1_1^2 + 1_1 (1 + A_1 + 1_2 + \dots + A_n) = 3(A_1^2 + A_2^2 + A_3^2 + \dots + A_n^2)$$

Let the lines  $1, 1_1, 1_2, \dots, 1_{n-1}, A_1$  be placed in a row from left to right. Produce  $1, 1_1, 1_2, \dots, 1_{n-1}$  until they are each equal to  $A_n$ , so that the parts produced are respectively equal to  $A_1, 1_2, \dots, A_{n-1}$ .

Taking each line successively we have

$$2 1_1^2 = 2 A_1^2 \\ (1 + 1_1 + 1_2)^2 = 1_1^2 + A_1^2 + 2 A_1 A_{n-1} \\ (1 + 1_1 + 1_2 + 1_3)^2 = 1_2^2 + A_2^2 + 2 A_2 A_{n-2}$$

$$(A_{n-1} + A_1)^2 = A_{n-1}^2 + A_1^2 + 2A_{n-1}A_1$$

And, by addition,

$$(n+1)A_n^2 = 2(A_1^2 + A_2^2 + \dots + A_n^2) + 2A_1A_{n-1} + 2A_2A_{n-2} + \dots + 2A_{n-1}A_1$$

Therefore, in order to obtain the required result, we have to prove that

$$2(A_1A_{n-1} + A_2A_{n-2} + \dots + A_{n-1}A_1) + A_1(A_1 + A_2 + A_3 + \dots + A_n) = A_1^2 + A_2^2 + \dots + A_n^2 \quad (\alpha)$$

Now

$$2A_2A_{n-2} = A_1^2 + A_{n-2}^2 \text{ because } A_2 = 2A_1$$

$$2A_3A_{n-3} = A_1^2 + A_{n-3}^2 \text{ because } A_3 = 3A_1$$

$$2A_{n-1}A_1 = A_1^2 + 2(n-1)A_1^2$$

It follows that

$$2(A_2A_{n-2} + A_3A_{n-3} + \dots + A_{n-1}A_1) + A_1(A_1 + A_2 + A_3 + \dots + A_n) = A_1\{A_n + 3A_{n-1} + 5A_{n-2} + \dots + (2n-1)A_1\}$$

And this last expression can be proved to be equal to

$$A_1^2 + A_2^2 + \dots + A_n^2$$

For

$$\begin{aligned} A_n^2 &= A_1\{nA_n\} \\ &= A_1\{A_n + (n-1)A_n\} \\ &= A_1\{A_n + 2(A_{n-1} + A_{n-2} + \dots + A_1)\}, \\ &\text{because } (n-1)A_n = A_{n-1} + A_1 \\ &\quad + A_{n-2} + A_2 \\ &\quad + \\ &\quad + A_1 + A_{n-1} \end{aligned}$$

$$\text{Similarly } A_{n-1}^2 = A_1\{A_{n-1} + 2(A_{n-2} + A_{n-3} + \dots + A_1)\},$$

$$A_2^2 = A_1(A_2 + 2A_1),$$

$$A_3^2 = A_1(A_3 + 4A_1),$$

whence by addition,

$$A_1^2 + A_2^2 + A_3^2 + \dots + A_n^2 = A_1\{A_n + 3A_{n-1} + 5A_{n-2} + \dots + (2n-1)A_1\}$$

Thus the equation marked ( $\alpha$ ) above is true, and it follows that

$$(n+1)A_n^2 + A_1(A_1 + A_2 + A_3 + \dots + A_n) = 3(A_1^2 + A_2^2 + \dots + A_n^2)$$

Con 1 From this it is evident that

$$nA_n^2 < 3(A_1^2 + A_2^2 + \dots + A_n^2) \quad (1)$$

Also  $A_n^2 = A_1\{A_n + 2(A_{n-1} + A_{n-2} + \dots + A_1)\}$ , as above,

so that  $A_n^2 > A_1(A_n + A_{n-1} + \dots + A_1)$ ,

and therefore

$$A_n^2 + A_1(A_1 + A_2 + \dots + A_n) < 2A_n^2$$

It follows from the proposition that

$$nA_n^2 > 3(A_1^2 + A_2^2 + \dots + A_n^2) \quad (2)$$

Con 2 All these results will hold if we substitute similar figures for squares on all the lines, for similar figures are in the duplicate ratio

## PROPOSITION 2

If  $A_1, A_2, \dots, A_n$  be any number of areas such that

$$A_1 = ax + x^2,$$

$$A_2 = a \cdot 2x + (2x)^2,$$

$$A_3 = a \cdot 3x + (3x)^2,$$

$$A_n = a \cdot nx + (nx)^2,$$

then  $n A_n (A_1 + A_2 + \dots + A_n) < (a + nx) \left( \frac{a}{2} + \frac{nx}{3} \right),$

and  $n A_n (A_1 + A_2 + \dots + A_{n-1}) > (a + nx) \left( \frac{a}{2} + \frac{nx}{3} \right)$

For, by the Lemma immediately preceding Prop 1,

$$n \cdot anx < (ax + a \cdot 2x + \dots + a \cdot nx),$$

and  $> 2(ax + a \cdot 2x + \dots + a \cdot (n-1)x)$

Also, by the Lemma preceding this proposition,

$$n (nx)^2 < 3\{x^2 + (2x)^2 + (3x)^2 + \dots + (nx)^2\}$$

and  $> 3\{x^2 + (2x)^2 + \dots + (n-1)x^2\}$

Hence

$$\frac{an^2x}{2} + \frac{n(nx)^2}{3} < [(ax + x^2) + \{a \cdot 2x + (2x)^2\} + \dots + \{a \cdot nx + (nx)^2\}],$$

and

$$> [(ax + x^2) + \{a \cdot 2x + (2x)^2\} + \dots + \{a \cdot (n-1)x + (n-1)x^2\}],$$

or  $\frac{an^2x}{2} + \frac{n(nx)^2}{3} < A_1 + A_2 + \dots + A_n,$

and  $> A_1 + A_2 + \dots + A_{n-1}$

It follows that

$$n \cdot 1_n (A_1 + A_2 + \dots + A_n) < n[a \cdot nx + (nx)^2] \left\{ \frac{an^2x}{2} + \frac{n(nx)^2}{3} \right\},$$

or  $n A_n (A_1 + A_2 + \dots + A_n) < (a + nx) \left( \frac{a}{2} + \frac{nx}{3} \right),$

also  $n \cdot 1_n (A_1 + A_2 + \dots + A_{n-1}) > (a + nx) \left( \frac{a}{2} + \frac{nx}{3} \right)$

## PROPOSITION 3

(1) If  $TP, TP'$  be two tangents to any conic meeting in  $T$ , and if  $Qq, Q'q'$  be any two chords parallel respectively to  $TP, TP'$  and meeting in  $O$ , then

$$QO \cdot Oq \cdot QO \cdot Oq' = TP^2 \cdot TP'^2$$

"And this is proved in the elements of conics"<sup>1</sup>

(2) If  $QQ'$  be a chord of a parabola bisected in  $V$  by the diameter  $PV$ , and if  $PV$  be of constant length, then the areas of the triangle  $PQQ'$  and of the segment  $PQQ'$  are both constant whatever be the direction of  $QQ'$

Let  $ABB'$  be the particular segment of the parabola whose vertex is  $A$ , so that  $BB'$  is bisected perpendicularly by the axis at the point  $H$ , where  $AH = PV$ .

Draw  $QD$  perpendicular to  $PV$

<sup>1</sup>In the treatises on conics by Aristaeus and Eudoxus

Let  $p_1$  be the parameter of the principal ordinates, and let  $p$  be another line of such length that

$$QV^2 = QD^2 = p \cdot p_1,$$

it will then follow that  $p$  is equal to the parameter of the ordinates to the diameter  $PV$ , i.e. those which are parallel to  $QV$ .

'For this is proved in the conics'.

Thus  $QV^2 = p \cdot PV$

And  $BH^2 = p_1 \cdot AH$  while  $AH = PV$

Therefore  $QV^2 \cdot BH^2 = p \cdot p_1$

But  $QV^2 \cdot QD^2 = p \cdot p_1$ ,

hence  $BH = QD$

Thus  $BH \cdot AH = QD \cdot PV$ ,

and therefore  $\triangle ABB = \triangle PQQ$ ,  
that is, the area of the triangle  $PQQ$  is constant so long as  $PV$  is of constant length

Hence also the area of the segment  $PQQ'$  is constant under the same conditions, for the segment is equal to  $\frac{1}{2} \triangle PQQ$  [Quadrature of the Parabola, Prop 17 or 24]

#### PROPOSITION 4

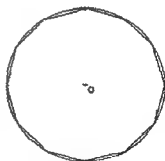
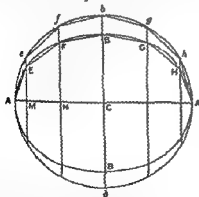
The area of any ellipse is to that of the auxiliary circle as the minor axis to the major

Let  $AA'$  be the major and  $BB'$  the minor axis of the ellipse, and let  $BB'$  meet the auxiliary circle in  $b, b'$

Suppose  $O$  to be such a circle that

$$(\text{circle } AbA'b') \quad O = CA \cdot CB$$

Then shall  $O$  be equal to the area of the ellipse



For if not  $O$  must be either greater or less than the ellipse

I. If possible let  $O$  be greater than the ellipse

We can then inscribe in the circle  $O$  an equilateral polygon of  $4n$  sides such that its area is greater than that of the ellipse [cf. *On the Sphere and Cylinder*, I 6]

The theorem which is here assumed by Archimedes as known is easily deduced from Apollonius I 49.

Ac  
on

Suppose that  $P$  denotes the area of the polygon inscribed in the auxiliary circle and  $P'$  that of the polygon inscribed in the ellipse

Then, since all the lines  $eM, fN, \dots$  are cut in the same proportions at  $E, F, \dots$ ,

$$eM : EM = fN : FN = \dots = bC : BC,$$

the pairs of triangles as  $eAM, EAM$ , and the pairs of trapeziums as  $eMNf, EMNI$ , are all in the same ratio to one another as  $bC$  to  $BC$ , or as  $CA$  to  $CB$

Therefore by addition

$$P' : P = CA : CB$$

Now  $P$  (polygon inscribed in  $O$ )

$$= (\text{circle } AbA'b') : O$$

$$= CA : CB, \text{ by hypothesis}$$

Therefore  $P$  is equal to the polygon inscribed in  $O$

But this is impossible, because the latter polygon is by hypothesis greater than the ellipse, and *a fortiori* greater than  $P'$

Hence  $O$  is not greater than the ellipse

II If possible let  $O$  be less than the ellipse

In this case we inscribe in the ellipse a polygon  $P$  with  $4n$  equal sides such that  $P > O$

Let the perpendiculars from the angular points on the axis  $AA'$  be produced to meet the auxiliary circle, and let the corresponding polygon ( $P'$ ) in the circle be formed

Inscribe in  $O$  a polygon similar to  $P$

Then  $P' : P = CA : CB$

$$= (\text{circle } AbA'b') : O \text{ by hypothesis,}$$

$$= P' : (\text{polygon inscribed in } O)$$

Therefore the polygon inscribed in  $O$  is equal to the polygon  $P$ , which is impossible because  $P > O$

Hence  $O$  being neither greater nor less than the ellipse is equal to it, and the required result follows

### PROPOSITION 5

If  $AA' : BB'$  be the major and minor axis of an ellipse respectively, and if  $d$  be the diameter of any circle then

$$(\text{area of ellipse}) : (\text{area of circle}) = AA' : BB' : d^2$$

For

$$\begin{aligned} (\text{area of ellipse}) : (\text{area of auxiliary circle}) &= BB' : AA' & [\text{Prop 4}] \\ &= AA' : BB' : AA'^2 \end{aligned}$$

And

$$(\text{area of aux. circle}) : (\text{area of circle with diam } d) = AA'^2 : d^2$$

Therefore the required result follows *ex aequali*

### PROPOSITION 6

The areas of ellipses are as the rectangles under their axes

This follows at once from Props 1-5

Cor. The areas of similar ellipses are as the squares of corresponding axes

## PROPOSITION 7

Given an ellipse with centre  $C$ , and a line  $CO$  drawn perpendicular to its plane, it is possible to find a circular cone with vertex  $O$  and such that the given ellipse is a section of it [or, in other words, to find the circular sections of the cone with vertex  $O$  passing through the circumference of the ellipse]

Conceive an ellipse with  $BB'$  as its minor axis and lying in a plane perpendicular to that of the paper. Let  $CO$  be drawn perpendicular to the plane of the ellipse, and let  $O$  be the vertex of the required cone. Produce  $OB, OC, OB'$  and in the same plane with them draw  $BED$  meeting  $OC, OB$  produced in  $E, D$  respectively and in such a direction that

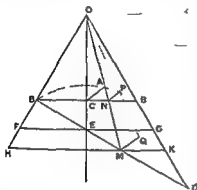
$$BE \cdot ED \cdot EO^2 = CA^2 \cdot CO^2,$$

where  $CA$  is half the major axis of the ellipse

"And this is possible, since

$$BE \cdot ED \cdot EO^2 > BC \cdot CB' \cdot CO^2"$$

[Both the construction and this proposition are assumed as known]



Now conceive a circle with  $BD$  as diameter lying in a plane at right angles to that of the paper, and describe a cone with this circle for its base and with vertex  $O$

We have therefore to prove that the given ellipse is a section of the cone or if  $P$  be any point on the ellipse, that  $P$  lies on the surface of the cone

Draw  $PN$  perpendicular to  $BB'$ . Join  $ON$  and produce it to meet  $BD$  in  $V$  and let  $UQ$  be drawn in the plane of the circle on  $BD$  as diameter perpendicular to  $BD$  and meeting the circle in  $Q$ . Also let  $FG, HK$  be drawn through  $E, M$  respectively parallel to  $BB'$

We have then

$$\begin{aligned} QM^2 \cdot HM \cdot MK &= BV \cdot MD \cdot HM \cdot MK \\ &= BE \cdot ED \cdot FE \cdot EG \\ &= (BE \cdot ED \cdot EO^2) (EO^2 \cdot FE \cdot EG) \\ &= (CA^2 \cdot CO^2) (CO^2 \cdot BC \cdot CB') \\ &= CA^2 \cdot CB'^2 \\ &= PN^2 \cdot BN \cdot NB' \end{aligned}$$

Therefore

$$QM^2 \cdot PN^2 = HM \cdot MK \cdot BN \cdot NB' = OM^2 \cdot ON^2,$$

whence, since  $PN, QV$  are parallel,  $OPQ$  is a straight line

But  $Q$  is on the circumference of the circle on  $BD$  as diameter, therefore  $OQ$  is a generator of the cone, and hence  $P$  lies on the cone

Thus the cone passes through all points on the ellipse

## PROPOSITION 8

Given an ellipse, a plane through one of its axes  $AA'$  and perpendicular to the plane of the ellipse, and a line  $CO$  drawn from  $C$ , the centre, in the given plane through  $AA'$  but not perpendicular to  $AA'$ , it is possible to find a cone with vertex



*O* such that the given ellipse is a section of it [or, in other words, to find the circular sections of the cone with vertex *O* whose surface passes through the circumference of the ellipse]

By hypothesis, *OA*, *OA'* are unequal. Produce *OA'* to *D* so that *OA* = *OD*. Join *AD*, and draw *FG* through *C* parallel to it.

The given ellipse is to be supposed to lie in a plane perpendicular to the plane of the paper. Let *BB'* be the other axis of the ellipse.

Conceive a plane through *AD* perpendicular to the plane of the paper, and in it describe either (a), if  $CB^2 = FC \cdot CG$ , a circle with diameter *AD*, or (b), if not, an ellipse on *AD* as axis such that, if *d* be the other axis,

$$d^2 \cdot AD^2 = CB^2 \cdot FC \cdot CG$$

Take a cone with vertex *O* whose surface passes through the circle or ellipse just drawn. This is possible even when the curve is an ellipse, because the line from *O* to the middle point of *AD* is perpendicular to the plane of the ellipse, and the construction is effected by means of Prop 7.

Let *P* be any point on the given ellipse, and we have only to prove that *P* lies on the surface of the cone so described.

Draw *PN* perpendicular to *AA'*. Join *ON*, and produce it to meet *AD* in *M*. Through *M* draw *HA* parallel to *A'A*.

Lastly, draw *MQ* perpendicular to the plane of the paper (and therefore perpendicular to both *HA* and *AD*) meeting the ellipse or circle about *AD* (and therefore the surface of the cone) in *Q*.

Then

$$\begin{aligned} QM^2 \cdot HM \cdot MK &= (QM^2 \cdot DM \cdot MA) \cdot (DM \cdot MA \cdot HM \cdot MK) \\ &= (d^2 \cdot AD^2) \cdot (FC \cdot CG \cdot A'C \cdot CA) \\ &= (CB^2 \cdot FC \cdot CG) \cdot (FC \cdot CG \cdot A'C \cdot CA) \\ &= CB^2 \cdot CA^2 \\ &= PN^2 \cdot A'N \cdot NA \end{aligned}$$

Therefore, alternately

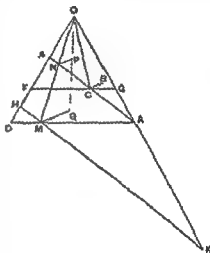
$$\begin{aligned} QM^2 \cdot PV^2 &= HM \cdot MK \cdot A'N \cdot NA \\ &= OM^2 \cdot ON^2 \end{aligned}$$

Thus since *PV*, *QM* are parallel *OPQ* is a straight line, and *Q* being on the surface of the cone, it follows that *P* is also on the surface of the cone.

Similarly all points on the ellipse are also on the cone, and the ellipse is therefore a section of the cone.

#### PROPOSITION 9

*Given an ellipse, a plane through one of its axes and perpendicular to that of the ellipse, and a straight line CO drawn from the centre C of the ellipse in the given plane through the axis but not perpendicular to that axis, it is possible to find a*



cylinder with axis  $OC$  such that the ellipse is a section of it [or, in other words, to find the circular sections of the cylinder with axis  $OC$  whose surface passes through the circumference of the given ellipse]

Let  $AA'$  be an axis of the ellipse, and suppose the plane of the ellipse to be perpendicular to that of the paper, so that  $OC$  lies in the plane of the paper

Draw  $AD, A'E$  parallel to  $CO$ , and let  $DE$  be the line through  $O$  perpendicular to both  $AD$  and  $A'E$

We have now three different cases according as the other axis  $BB'$  of the ellipse is (1) equal to, (2) greater than, or (3) less than,  $DE$

(1) Suppose  $BB' = DE$

Draw a plane through  $DE$  at right angles to  $OC$ , and in this plane describe a circle on  $DE$  as diameter. Through

this circle describe a cylinder with axis  $OC$

This cylinder shall be the cylinder required, or its surface shall pass through every point  $P$  of the ellipse

For, if  $P$  be any point on the ellipse, draw  $PN$  perpendicular to  $AA'$ , through  $N$  draw  $NM$  parallel to  $CO$  meeting  $DE$  in  $M$ , and through  $M$ , in the plane of the circle on  $DE$  as diameter, draw  $MQ$  perpendicular to  $DE$ , meeting the circle in  $Q$

Then, since

$$DE = BB', \quad PN^2 = AN \cdot NA' = DO^2 \cdot AC \cdot CA'$$

And

$$DM \cdot ME \cdot AN \cdot NA' = DO^2 \cdot AC^2,$$

since  $AD, NM, CO, A'E$  are parallel

Therefore

$$PN^2 = DM \cdot ME = QM^2,$$

by the property of the circle

Hence, since  $PN, QM$  are equal as well as parallel  $PQ$  is parallel to  $MIN$  and therefore to  $CO$ . It follows that  $PQ$  is a generator of the cylinder, whose surface accordingly passes through  $P$

(2) If  $BB' > DE$ , we take  $E$  on  $A'E$  such that  $DE = BB'$  and describe a circle on  $DE$  as diameter in a plane perpendicular to that of the paper, and the rest of the construction and proof is exactly similar to those given for case (1)

(3) Suppose  $BB' < DE$

Take a point  $K$  on  $CO$  produced such that

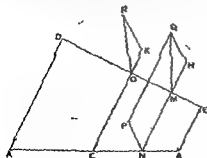
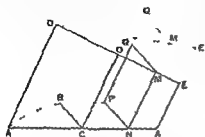
$$DO^2 - CB^2 = OK^2$$

From  $A$  draw  $AR$  perpendicular to the plane of the paper and equal to  $CB$

$$\text{Thus } OR^2 = OK^2 + CB^2 = OD^2$$

In the plane containing  $DE$ ,  $OR$  describe a circle on  $DE$  as diameter. Through this circle (which must pass through  $R$ ) draw a cylinder with axis  $OC$ .

We have then to prove that, if  $P$  be any point on the given ellipse  $P$  lies on the cylinder so described



Draw  $PN$  perpendicular to  $AA'$ , and through  $N$  draw  $NM$  parallel to  $CO$  meeting  $DE$  in  $M$ . In the plane of the circle on  $DE$  as diameter draw  $MQ$  perpendicular to  $DE$  and meeting the circle in  $Q$ .

Lastly, draw  $QH$  perpendicular to  $NM$  produced.  $QH$  will then be perpendicular to the plane containing  $AC$ ,  $DE$ , i.e. the plane of the paper.

Now  $QH^2 \cdot QM^2 = KR^2 : OR^2$ , by similar triangles.

And  $QM^2 \cdot AN \cdot NA' = DM \cdot ME \cdot AN \cdot NA'$   
 $= OD^2 : CA^2$

Hence, *ex aequali*, since  $OR = OD$ ,

$$\begin{aligned} QH^2 \cdot AN \cdot NA' &= KR^2 : CA^2 \\ &= CB^2 : CA^2 \\ &= PN^2 : AN \cdot NA' \end{aligned}$$

Thus  $QH = PN$ . And  $QH, PN$  are also parallel. Accordingly  $PQ$  is parallel to  $MN$ , and therefore to  $CO$ , so that  $PQ$  is a generator, and the cylinder passes through  $P$ .

### PROPOSITION 10

It was proved by the earlier geometers that any two cones have to one another the ratio compounded of the ratios of their bases and of their heights.<sup>1</sup> The same method of proof will show that any segments of cones have to one another the ratio compounded of the ratios of their bases and of their heights.

The proposition that any 'frustum' of a cylinder is triple of the manner as the proposition that the cylinder which has the same base as the frustum and equal height.<sup>2</sup>

### PROPOSITION 11

(1) If a paraboloid of revolution be cut by a plane through, or parallel to, the section will be a parabola equal to the original paraboloid. And the axis of the sec the cutting plane and the plane through the axis to the cutting plane.

If the paraboloid be cut by a plane at right angles to its axis, the section will be a circle whose centre is on the axis.

(2) If a hyperboloid of revolution be cut by a plane through the axis, parallel to the axis or through the centre, the section will be a hyperbola, (a) if the section be through the axis, equal (b) if parallel to the axis similar, (c) if through the centre, not similar to the original hyperbola which by its revolution generates the hyperboloid. And the axis of the section will be the intersection of the cutting plane and the plane through the axis of the hyperboloid at right angles to the cutting plane.

Any section of the hyperboloid by a plane at right angles to the axis will be a circle whose centre is on the axis.

(3) If any of the spheroidal figures be cut by a plane through the axis or parallel to the axis, the section will be an ellipse (a) if the section be through the axis, equal, (b) if parallel to the axis, similar, to the ellipse which by its revolution gen-

<sup>1</sup>This follows from LAUCHLIN 11 and 14 taken together. Cf. *On the Sphere and Cylinder* I, Lemma 1.

<sup>2</sup>This proposition was proved by Ludovius, as stated in the preface to *On the Sphere and Cylinder* I. Cf. FUCHS 10.

erates the figure. And the axis of the section will be the intersection of the cutting plane and the plane through the axis of the spheroid at right angles to the cutting plane.

If the section be by a plane at right angles to the axis of the spheroid, it will be a circle whose centre is on the axis.

(4) If any of the said figures be cut by a plane through the axis, and if a perpendicular be drawn to the plane of section from any point on the surface of the figure but not on the section, that perpendicular will fall within the section.

"And the proofs of all these propositions are evident."

### PROPOSITION 12

If a paraboloid of revolution be cut by a plane neither parallel nor perpendicular to the axis, and if the plane through the axis perpendicular to the cutting plane intersect it in a straight line of which the portion intercepted within the paraboloid is  $RR'$ , the section of the paraboloid will be an ellipse whose major axis is  $RR'$  and whose minor axis is equal to the perpendicular distance between the lines through  $R, R'$  parallel to the axis of the paraboloid.

Suppose the cutting plane to be perpendicular to the plane of the paper, and let the latter be the plane through the axis  $ANF$  of the paraboloid which intersects the cutting plane at right angles in  $RR'$ . Let  $RH$  be parallel to the axis of the paraboloid, and  $R'H$  perpendicular to  $RH$ .

Let  $Q$  be any point on the section made by the cutting plane, and from  $Q$  draw  $QV$  perpendicular to  $RR'$ .  $QM$  will therefore be perpendicular to the plane of the paper.

Through  $M$  draw  $DMFE$  perpendicular to the axis  $ANF$  meeting the parabolic section made by the plane of the paper in  $D, E$ . Then  $QV$  is perpendicular to  $DE$ , and, if a plane be drawn through  $DE, QM$ , it will be perpendicular to the axis and will cut the paraboloid in a circular section.

Since  $Q$  is on this circle,

$$QM^2 = DM \cdot ME$$

Again, if  $PT$  be that tangent to the parabolic section in the plane of the paper which is parallel to  $RR'$ , and if the tangent at  $A$  meet  $PT$  in  $O$ , then from the property of the parabola,

$$DM \cdot ME \cdot RM \cdot MR' = AO^2 \cdot OP^2 \quad [\text{Prop. II (1)}]$$

$$= AO^2 \cdot OT^2, \text{ since } AN = AT$$

$$\text{Therefore } QV^2 \cdot RV \cdot MR = AO^2 \cdot OT^2$$

$$= R'H^2 \cdot RR'^2,$$

by similar triangles

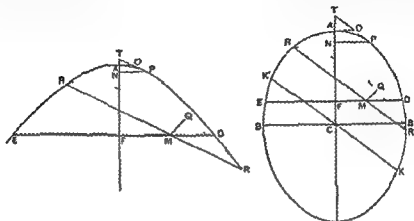
Hence  $Q$  lies on an ellipse whose major axis is  $RR'$  and whose minor axis is equal to  $R'H$ .

### PROPOSITIONS 13, 14

If a hyperboloid of revolution be cut by a plane meeting all the generators of the enveloping cone, or if an 'oblong' spheroid be cut by a plane not perpendicular to

the axis,<sup>1</sup> and if a plane through the axis intersect the cutting plane at right angles in a straight line on which the hyperboloid or spheroid intercepts a length  $RR'$ , then the section by the cutting plane will be an ellipse whose major axis is  $RR'$ .

Suppose the cutting plane to be at right angles to the plane of the paper, and suppose the latter plane to be that through the axis  $ANF$  which intersects the



cutting plane at right angles in  $RR'$ . The section of the hyperboloid or spheroid by the plane of the paper is thus a hyperbola or ellipse having  $ANF$  for its transverse or major axis.

Take any point on the section made by the cutting plane, as  $Q$ , and draw  $QM$  perpendicular to  $RR'$ .  $QM$  will then be perpendicular to the plane of the paper.

Through  $M$  draw  $DFE$  at right angles to the axis  $ANF$  meeting the hyperbola or ellipse in  $D$ ,  $E$ , and through  $QM$ ,  $DE$  let a plane be described. This plane will accordingly be perpendicular to the axis and will cut the hyperboloid or spheroid in a circular section.

Thus  $QM^2 = DM \cdot ME$ .

Let  $PT$  be that tangent to the hyperbola or ellipse which is parallel to  $RR'$ , and let the tangent at  $A$  meet  $PT$  in  $O$ .

Then by the property of the hyperbola or ellipse,

$$DM \cdot ME \cdot RM \cdot MR' = OA^2 \cdot OP^2,$$

or

$$QM^2 \cdot RM \cdot MR' = OA^2 \cdot OP^2.$$

Now (1) in the hyperbola  $OA < OP$ , because  $AT < AN$ , and accordingly

$$OT < OP, \text{ while } OA < OT.$$

(2) in the ellipse if  $AA'$  be the diameter parallel to  $RR'$ , and  $BB'$  the minor axis

$$BC \cdot CB' \cdot KC \cdot CK' = OA^2 \cdot OP^2,$$

$$\text{and } BC \cdot CB' < AC \cdot CA', \text{ so that } OA < OP.$$

Hence in both cases the locus of  $Q$  is an ellipse whose major axis is  $RR'$ .

COR. I. If the spheroid be a flat spheroid the section will be an ellipse, and everything will proceed as before except that  $RR'$  will in this case be the minor axis.

<sup>1</sup>Archimedes begins Prop. 14 for the spheroid with the remark that, when the cutting plane passes through or is parallel to the axis the case is clear. Cf. Prop. 11 (3).

COR 2 In all conoids or spheroids parallel sections will be similar since the ratio  $OA^2 : OP^2$  is the same for all the parallel sections

## PROPOSITION 15

*which is in the same direction as the convexity of the surface will fall without it and the part which is in the other direction within it*

For if a plane be drawn in the case of the paraboloid through the axis and the point and in the case of the hyperboloid through the given point and through the given straight line drawn through the vertex of the enveloping cone, the section by the plane will be (a) in the paraboloid a parabola whose axis is the axis of the paraboloid (b) in the hyperboloid a hyperbola in which the given line through the vertex of the enveloping cone is a diameter<sup>1</sup>

[Prop 11]

Hence the property follows from the plane properties of the conics

(2) *If a plane touch a conoid without cutting it it will touch it at one point only and the plane drawn through the point of contact and the axis of the conoid will be at right angles to the plane which touches it*

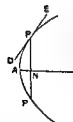
For if possible let the plane touch at two points Draw through each point a parallel to the axis The plane passing through both parallels will therefore either pass through or be parallel to the axis Hence the section of the conoid made by this plane will be a conic [Prop 11 (1) (2)] the two points will lie on this conic and the line joining them will lie within the conic and therefore within the conoid But this line will be in the tangent plane since the two points are in it Therefore some portion of the tangent plane will be within the conoid which is impossible since the plane does not cut it

Therefore the tangent plane touches in one point only

That the plane through the point of contact and the axis is perpendicular to the tangent plane is evident in the particular case where the point of contact

1

through the axis



If the point of contact  $P$  is not the vertex draw the plane passing through the axis  $AN$  and the point  $P$  It will cut the conoid in a conic whose axis is  $AN$  and the tangent plane in a line  $DPE$  touching the conic at  $P$  Draw  $PNP$  perpendicular to the axis and draw a plane through it also perpendicular to the axis This plane will make a circular section and meet the tangent

<sup>1</sup>There seems to be some error in the text here which says that the diameter (i.e. axis) of the hyperbola is the straight line drawn in the conoid from the vertex of the cone But this straight line is not in general the axis of the section.

## PROPOSITION 16

(1) *If a plane touch any of the spheroidal figures without cutting it, it will touch at one point only, and the plane through the point of contact and the axis will be at right angles to the tangent plane*

This is proved by the same method as the last proposition

(2) *If any conoid or spheroid be cut by a plane through the axis, and if through any tangent to the resulting conic a plane be erected at right angles to the plane of section, the plane so erected will touch the conoid or spheroid in the same point as that in which the line touches the conic*

For it cannot meet the surface at any other point. If it did, the perpendicular from the second point on the cutting plane would be perpendicular also to the tangent to the conic and would therefore fall outside the surface. But it must fall within it. [Prop 11 (4)]

(3) *If two parallel planes touch any of the spheroidal figures, the line joining the points of contact will pass through the centre of the spheroid*

If the planes are at right angles to the axis, the proposition is obvious. If not, the plane through the axis and one point of contact is at right angles to the tangent plane at that point. It is therefore at right angles to the parallel tangent plane and therefore passes through the second point of contact. Hence both points of contact lie on one plane through the axis, and the proposition is reduced to a plane one.

## PROPOSITION 17

*If two parallel planes touch any of the spheroidal figures, and another plane be drawn parallel to the tangent planes and passing through the centre, the line drawn through any point of the circumference of the resulting section parallel to the chord of contact of the tangent planes will fall outside the spheroid*

This is proved at once by reduction to a plane proposition

Archimedes adds that it is evident that, if the plane parallel to the tangent planes does not pass through the centre, a straight line drawn in the manner described will fall without the spheroid in the direction of the smaller segment but within it in the other direction.

## PROPOSITION 18

*Any spheroidal figure which is cut by a plane through the centre is divided, both as regards its surface and its volume into two equal parts by that plane*

To prove this Archimedes takes another equal and similar spheroid, divides it similarly by a plane through the centre, and then uses the method of application.

## PROPOSITIONS 19, 20

*Given a segment cut off by a plane from a paraboloid or hyperboloid of revolution, or a segment of a spheroid less than half the spheroid also cut off by a plane, it is possible to inscribe in the segment one solid figure and to circumscribe about it another solid figure, each made up of cylinders or "frusta" of cylinders of equal height, and such that the circumscribed figure exceeds the inscribed figure by a volume less than that of any given solid*

Let the plane base of the segment be perpendicular to the plane of the paper,

and let the plane of the paper be the plane through the axis of the conoid or spheroid which cuts the base of the segment at right angles in  $BC$ . The section in the plane of the paper is then a conic  $BAC$  [Prop 11]

spheroid at  $A$  [Prop 10]

(1) If the base of the segment is at right angles to the axis of the conoid or spheroid,  $A$  will be the vertex of the conoid or spheroid and its axis  $AD$  will bisect  $BC$  at right angles

(2) If the base of the segment is not at right angles to the axis of the conoid or spheroid, we draw  $AD$

(a) in the paraboloid, parallel to the axis,

(b) in the hyperboloid, through the centre (or the vertex of the enveloping cone),

(c) in the spheroid, through the centre

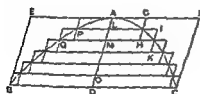
and in all the cases it will follow that  $AD$  bisects  $BC$  in  $D$

Then  $A$  will be the vertex of the segment, and  $AD$  will be its axis

Further, the base of the segment will be a circle or an ellipse with  $BC$  as diameter or as an axis respectively, and with centre  $D$ . We can therefore describe through this circle or ellipse a cylinder or a 'frustum' of a cylinder whose axis is  $AD$  [Prop 9]

Dividing this cylinder or frustum continually into equal parts by planes parallel to the base we shall at length arrive at a cylinder or frustum less in volume than any given solid

Let this cylinder or frustum be that whose axis is  $OD$ , and let  $AD$  be divided into parts equal to  $OD$ , at  $L, M,$



frusta of cylinders each with axis equal to  $OD$ , one of them lying in the direction of  $A$  and the other in the direction of  $D$  as shown in the figure

Then the cylinders or frusta of cylinders drawn in the direction of  $A$  make up a circumscribed figure, and those in the direction of  $D$  an inscribed figure, in relation to the segment

Also the cylinder or frustum  $PG$  in the circumscribed figure is equal to the cylinder or frustum  $PH$  in the inscribed figure,  $QI$  in the circumscribed figure is equal to  $QH$  in the inscribed figure, and so on

Therefore, by addition

(circumscribed fig) = (inscr fig) + (cylinder or frustum whose axis is  $OD$ )

But the cylinder or frustum whose axis is  $OD$  is less than the given solid figure whence the proposition follows

"Having set out these preliminary propositions, let us proceed to demonstrate the theorems propounded with reference to the figures"



## PROPOSITIONS 21, 22

Any segment of a paraboloid of revolution is half as large again as the cone or segment of a cone which has the same base and the same axis

Let the base of the segment be perpendicular to the plane of the paper and let the plane of the paper be the plane through the axis of the paraboloid which cuts the base of the segment at right angles in  $BC$  and makes the parabolic section  $BAC$

Let  $EF$  be that tangent to the parabola which is parallel to  $BC$ , and let  $A$  be the point of contact

Then (1) if the plane of the base of the segment is perpendicular to the axis of the paraboloid that axis is the line  $AD$  bisecting  $BC$  at right angles in  $D$

(2) If the plane of the base is not perpendicular to the axis of the paraboloid draw  $AD$  parallel to the axis of the paraboloid  $AD$  will then bisect  $BC$ , but not at right angles

the

$BC$  as major axis

Accordingly a cylinder or a frustum of a cylinder can be found passing through the circle or ellipse and having  $AD$  for its axis [Prop. 1], and likewise a cone or a segment of a cone can be drawn passing through the circle or ellipse and having  $A$  for vertex and  $AD$  for axis [Prop. 8]

Suppose  $\lambda$  to be a cone equal to  $\frac{1}{2}$  (cone or segment of cone  $ABC$ ) The cone  $\lambda$  is therefore equal to half the cylinder or frustum of a cylinder  $EC$

[Cf Prop. 10]

We shall prove that the volume of the segment of the paraboloid is equal to  $\lambda$

If not the segment must be either greater or less than  $\lambda$

I If possible let the segment be greater than  $\lambda$

We can then inscribe and circumscribe as in the last proposition figures made up of cylinders or frusta of cylinders with equal height and such that (circumscribed fig)  $-($ inscribed fig)  $< ($ segment)  $-\lambda$

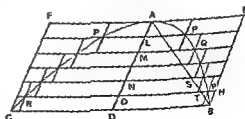
Let the greatest of the cylinders or frusta forming the circumscribed figure be that whose base is the circle or ellipse about  $BC$  and whose axis is  $OD$  and let the smallest of them be that whose base is the circle or ellipse about  $PP$  and whose axis is  $AL$

Let the greatest of the cylinders forming the inscribed figure be that whose base is the circle or ellipse about  $RR$  and whose axis is  $OD$  and let the smallest be that whose base is the circle or ellipse about  $PP$  and whose axis is  $LV$

Produce all the plane bases of the cylinders or frusta to meet the surface of the complete cylinder or frustum  $EC$

Now, since

$$(\text{circumscribed fig}) - (\text{inscr fig}) < (\text{segment}) - \lambda$$



it follows that (inscribed figure)  $> \lambda$  ( $\alpha$ )

Next comparing successively the cylinders or frusta with heights equal to  $OD$  and respectively forming parts of the complete cylinder or frustum  $EC$  and of the inscribed figure we have

$$\begin{aligned} & \text{(first cylinder or frustum in } EC) \text{ (first in inscr fig)} \\ & = BD \quad RO \\ & = AD \quad AO \\ & = BD \quad TO \text{ where } AB \text{ meets } OR \text{ in } T \end{aligned}$$

And (second cylinder or frustum in  $EC$ ) (second in inscr fig)  
 $= HO \quad SN$  in like manner,

and so on

ical

progression

But [Lemma preceding Prop 1]

$$BD + HO + \dots > 2(TO + SN + \dots)$$

Therefore (cylinder or frustum  $EC$ )  $> 2$  (inscribed fig)  
 or  $X > \text{(inscribed fig)}$ ,

which is impossible, by ( $\alpha$ ) above

II If possible let the segment be less than  $X$

In this case we inscribe and circumscribe figures as before but such that  
 (circumscribed fig)  $-$  (inscr fig)  $< \lambda -$  (segment)

whence it follows that

$$\text{(circumscribed figure)} < X \quad (\beta)$$

1 2 3 4 5 6 7 8 9 cylinder or

$$= BD^2 \quad BD^2$$

$$= BD \quad BD$$

(second in  $CE$ ) (second in circumscribed fig)

$$= HO^2 \quad RO^2$$

$$= AD \quad AO$$

$$= HO \quad TO$$

and so on

Hence [Prop 1]

(cylinder or frustum  $CE$ ) (circumscribed fig)

$$= (BD + HO + \dots) (BD + TO + \dots)$$

$$< 2 \quad 1$$

[Lemma preceding Prop 1]

and it follows that

$$Y < \text{(circumscribed fig)}$$

which is impossible by ( $\beta$ )

Thus the segment being neither greater nor less than  $X$  is equal to it and therefore to  $\frac{1}{2}$  (cone or segment of cone  $ABC$ )

#### PROPOSITION 23



equal [Prop 23] and, since the segments  $APp$   $AP'p'$  are half as large again as the cones  $APp$   $AP'p'$  respectively we have only to show that the cones are in the ratio of  $AV$  to  $AV'$ .

But

$$\begin{aligned}(\text{cone } APp) (\text{cone } AP'p') &= (PV^2 \cdot P'V'^2) (AV \cdot AV') \\ &= (AV \cdot AV') (AV \cdot AV') \\ &= AV^2 \cdot AV'^2,\end{aligned}$$

thus the proposition is proved.

### PROPOSITION 25 26

In an hyperboloid of revolution if  $A$  be the vertex and  $AD$  the axis of any segment cut off by a plane and if  $CA$  be the semi-diameter of the hyperboloid through  $A$  ( $CA$  being of course in the same straight line with  $AD$ ) then

$$\begin{aligned}(\text{segment}) (\text{cone with same base and axis}) \\ = (1D + 3CA) (AD + 2CA)\end{aligned}$$

Let the plane cutting off the segment be perpendicular to the plane of the paper and let the latter plane be the plane through the axis of the hyperboloid which intersects the cutting plane at right angles in  $BB$ , and makes the hyperbolic segment  $BAB$ . Let  $C$  be the centre of the hyperboloid (or the vertex of the enveloping cone)

Let  $EF$  be that tangent to the hyperbolic section which is parallel to  $BB$ . Let  $EF$  touch at  $A$  and join  $CA$ . Then  $CA$  produced will bisect  $BB'$  at  $D$ .  $CA$  will be a semi-diameter of the hyperboloid.  $A$  will be the vertex of the segment and  $AD$  its axis. Produce  $AC$  to  $A'$  and  $H$ , so that  $AC = CA' = A'H$ .

Through  $EF$  draw a plane parallel to the base of the segment. This plane will touch the hyperboloid at  $A$ .

Then (1), if the base of hyperboloid  $A$  will be the axis of the segment and the diameter

(2) If the base of the segment is not perpendicular to the axis of the hyperboloid the base will be an ellipse on  $BB$  as major axis [Prop 13]

Then we can draw a cylinder or a frustum of a cylinder  $FBBI$  passing through the circle or ellipse about  $BB$  and having  $1D$  for its axis and we can describe a cone or a segment of a cone through the circle or ellipse and having  $A$  for its vertex.

We have to prove that

$$(\text{segment } ABB) (\text{cone or segment of cone } ABB) = HD \cdot 1D$$

Let  $1$  be a cone such that

$$1 (\text{cone or segment of cone } ABB) = HD \cdot AD \quad (\alpha)$$

and we have to prove that  $1$  is equal to the segment

Now

$$(\text{cylinder or frustum } EB) (\text{cone or segment of cone } ABB) = 3 \cdot 1$$

$$\text{Therefore by means of } (\alpha), (\text{cylinder or frustum } EB) \cdot 1 = 1D \cdot \frac{HD}{3} \quad (\beta)$$

If the segment is not equal to  $1$  it must either be greater or less

I. If possible let the segment be greater than  $1$

Inscribe and circumscribe to the segment figures made up of cylinders or frusta of cylinders with axes along  $AD$  and all equal to one another such that

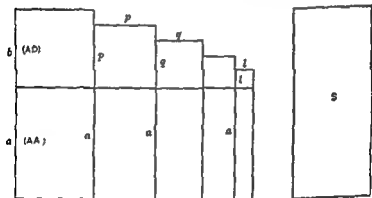
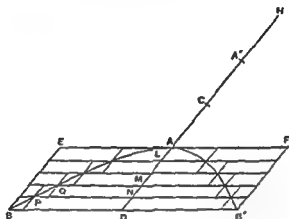
(circumscribed fig) - (inscr. fig) < (segmt.) -  $V$ ,

whence

(inscribed figure) >  $V$ .

(7)

Produce all the planes forming the bases of the cylinders or frusta of cylinders to meet the surface of the complete cylinder or frustum  $EB'$ .



Then, if  $ND$  be the axis of the greatest cylinder or frustum in the circumscribed figure, the complete cylinder will be divided into cylinders or frusta each equal to this greatest cylinder or frustum

Let there be a number of straight lines  $a$  equal to  $AA'$  and as many in number as the parts into which  $AD$  is divided by the bases of the cylinders or frusta. To each line  $a$  apply a rectangle which shall overlap it by a square, and let the greatest of the rectangles be equal to the rectangle  $AD A'D$  and the least equal to the rectangle  $AN A'N$ .

$b, p, q,$

respec

$(ap+p^2),$

$A'L$

$(al+l^2)$

$A'L$

the overlapping squares  
Thus  $b, p, q, l$  will be  
the rectangles  $(ab+b^2),$

$AD A'D, AN A'N, AL$

$A'L$

the largest

les

cylind

der or frustum  $EB$  and (2) in the unscripted figure, beginning from the base of the segment, we have

(first cylinder or frustum in  $EB$ ) (first in inser figure)

$$= BD^* \quad PN^2$$

$= AD \cdot A'D \cdot AN \cdot A'N$ , from the hyperbola,

$$= S \quad (ap \vdash p^*)$$

Again

(second cylinder or frustum in  $EB$ ) (second in inner fig)

$\cong BD^2 \quad QM^2$

$$= AD \quad A'D \quad AM \quad A'M$$

$$= S \quad (a q + q^2)$$

and so on

The last cylinder or frustum in the complete cylinder or frustum  $EB'$  has no cylinder or frustum corresponding to it in the inscribed figure

Combining the proportions, we have

[Prop 1]

(cylinder or frustum  $EB'$ ) (inscribed figure)

$$= (\text{sum of all the spaces } S) \quad (ap+p^2) + (aq+q^2) +$$

$$> (a+b) \left( \frac{a}{2} + \frac{b}{2} \right) \quad [\text{Prop 2}]$$

$$> AD \quad \frac{HD}{3} \text{ since } a=AA \quad b=AD,$$

$\geq (EB) \quad V$  by (8) above

 $(\text{unsubscribed figure}) < T'$ 

Hence

But this is impossible because by (7) above the inscribed figure is greater than  $V$ .

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

nt).

whence we derive

$$V > (\text{circumscribed figure}) \quad (d)$$

We now compare successive cylinders or frusta in the complete cylinder or frustum and in the circumscribed figure and we have

(first cylinder or frustum in  $EB'$ ) (first in circumscribed fig)

-2-

$$= S \quad (ab + b^2)$$

(second in  $EB$ ) (second in circumscribed fig)

$$= S \quad (ap + p^2)$$

and so on

Hence [Prop 1]

(cylinder or frustum  $EB$ ) (circumscribed fig)

$$= (\text{sum of all prices } S) \quad (ab + b^2) + (ap + p^2) +$$

$$\leq (a+b) \left( \frac{a}{2} + \frac{b}{3} \right) \quad [\text{Prop 2}]$$

$\angle A D \frac{H D}{3}$

$\langle EB' \rangle$  1 by (β) above

Hence the circumscribed figure is greater than  $V'$ ; which is impossible, by (6) above

Thus the segment is neither greater nor less than  $V'$ , and is therefore equal to it

Therefore, by (a),

$$\begin{aligned} &(\text{segment } ABB') \quad (\text{cone or segment of cone } ABB') \\ &= (AD + 3CA) \quad (AD + 2CA) \end{aligned}$$

#### PROPOSITIONS 27, 28, 29, 30

(1) In any spheroid whose centre is  $C$ , if a plane meeting the axis cut off a segment not greater than half the spheroid and having  $A$  for its vertex and  $AD$  for its axis, and if  $A'D$  be the axis of the remaining segment of the spheroid, then

$$\begin{aligned} &(\text{first segmt}) \quad (\text{cone or segmt of cone with same base and axis}) \\ &= CA + A'D \quad A'D \\ &= [3CA - AD] \quad [2CA - AD] \end{aligned}$$

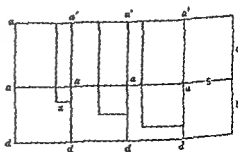
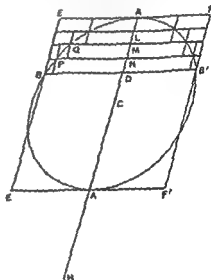
(2) As a particular case, if the plane passes through the centre, so that the segment is half the spheroid, half the spheroid is double of the cone or segment of a cone which has the same vertex and axis

Let the plane cutting off the segment be at right angles to the plane of the paper, and let the latter plane be the plane through the axis of the spheroid which intersects the cutting plane in  $BB'$  and makes the elliptic section  $AB'B'$

Let  $EF$ ,  $E'I'$  be the two tangents to the ellipse which are parallel to  $BB'$  let them touch it in  $A$ ,  $A'$ , and through the tangents draw planes parallel to the base of the segment These planes will touch the spheroid at  $A$ ,  $A'$ , which will be the vertices of the two segments into which it is divided Also  $AA'$  will pass through the centre  $C$  and bisect  $BB'$  in  $D$

Then (1) if the base of the segments be perpendicular to the axis of the spheroid,  $A$ ,  $A'$  will be the vertices of the spheroid as well as of the segments  $AA'$  will be the axis of the spheroid, and the base of the segments will be a circle on  $BB'$  as diameter,

(2) if the base of the segments be not perpendicular to the axis of the spheroid, the base of the segments will be an ellipse of which  $BB'$  is one axis, and  $AD$ ,  $A'D$  will be the axes of the segments respectively



We can now draw a cylinder or a frustum of a cylinder  $EBB'F$  through the circle or ellipse about  $BB'$  and having  $AD$  for its axis, and we can also draw a cone or a segment of a cone passing through the circle or ellipse about  $BB'$  and having  $A$  for its vertex.

We have then to show that, if  $CA'$  be produced to  $H$  so that  $CA' = A'H$ ,  
(segment  $ABB'$ ) (cone or segment of cone  $ABB'$ ) =  $HD \cdot A'D$

Let  $V$  be such a cone that

$$V \text{ (cone or segment of cone } ABB') = HD \cdot A'D, \quad (\alpha)$$

and we have to show that the segment  $ABB'$  is equal to  $V$ .

But, since

$$(\text{cylinder or frustum } EB') : (\text{cone or segment of cone } ABB') = 3 : 1,$$

we have, by the aid of  $(\alpha)$ ,

$$(\text{cylinder or frustum } EB') : V = A'D : \frac{HD}{3} \quad (\beta)$$

there greater or less  
ment consisting of  
d all equal to one

another, such that

$$(\text{circumscribed fig}) - (\text{inscribed fig}) < (\text{segment}) - V,$$

whence it follows that

$$(\text{inscribed fig}) > V \quad (\gamma)$$

Produce all the planes forming the bases of the cylinders or frusta to meet the surface of the complete cylinder or frustum  $EB'$ . Thus, if  $ND$  be the axis of the greatest cylinder or frustum of a cylinder in the circumscribed figure, the complete cylinder or frustum  $EB'$  will be divided into cylinders or frusta of cylinders each equal to the greatest of those in the circumscribed figure.

Take straight lines  $da'$  each equal to  $A'D$  and as many in number as the parts into which  $AD$  is divided by the bases of the cylinders or frusta, and measure  $da$  along  $da'$  equal to  $AD$ . It follows that  $aa' = 2CD$ .

Apply to each of the lines  $a'd$  rectangles with height equal to  $ad$  and draw the squares on each of the lines  $ad$  as in the figure. Let  $S$  denote the area of each complete rectangle.

From the first rectangle take away a gnomon with breadth equal to  $AN$  (i.e. with each end of a length equal to  $AN$ ), take away from the second rectangle a gnomon with breadth equal to  $AM$ , and so on, the last rectangle having no gnomon taken from it.

Then

$$\begin{aligned} \text{the first gnomon} &= A'D \cdot ID - ND \cdot (A'D - AN) \\ &= A'D \cdot AN + ND \cdot AN \\ &= AN \cdot I'N \end{aligned}$$

Similarly,

$$\text{the second gnomon} = A'U \cdot A'U,$$

and so on.

And the last gnomon (that in the last rectangle but one) is equal to  $AL \cdot A'L$ .

Also, after the gnomons are taken away from the rectangles, the



length  $aa'$  and "exceeding by squares" whose sides are respectively equal to  $DN$ ,  $DM$ ,  $DA$

For brevity, let  $DV$  be denoted by  $x$ , and  $aa'$  or  $2CD$  by  $c$ , so that

$$R_1 = cx + x^2, R_2 = c \cdot 2x + (2x)^2,$$

Then, comparing a  
complete cylinder or  
(first cylind

$$= BD^2 - PV^2$$

$$= AD \cdot A'D - AN \cdot A'N$$

$$= S \text{ (first gnomon),}$$

$$\text{(second cylinder or frustum in } EB') \text{ (second in inscribed fig.)}$$

$$= S \text{ (second gnomon),}$$

and so on

The last of the cylinders or frusta in the cylinder or frustum  $EB$  has none corresponding to it in the inscribed figure, and there is no corresponding gnomon

Combining the proportions, we have [by Prop. 1]

$$\text{(cylinder or frustum } EB) \text{ (inscribed fig.)}$$

$$= (\text{sum of all spaces } S) \text{ (sum of gnomons)}$$

Now the differences between  $S$  and the successive gnomons are  $R_1, R_2, \dots, R_n$ , while

$$R_1 = cx + x^2,$$

$$R_2 = c \cdot 2x + (2x)^2,$$

$$R_n = cb + b^2 = S,$$

where  $b = nx = AD$

Hence [Prop. 2]

$$(\text{sum of all spaces } S) \text{ (} R_1 + R_2 + \dots + R_n) < (c+b) \left( \frac{c}{2} + \frac{b}{3} \right)$$

It follows that

$$\begin{aligned} (\text{sum of all spaces } S) \text{ (sum of gnomons)} &> (c+b) \left( \frac{c}{2} + \frac{2b}{3} \right) \\ &> A'D \cdot \frac{HD}{3} \end{aligned}$$

Thus

$$\text{(cylinder or frustum } EB') \text{ (inscribed fig.)}$$

$$> A'D \cdot \frac{HD}{3}$$

$$> \text{(cylinder or frustum } EB') \cdot V,$$

from (8) above

$$\text{Therefore (inscribed fig.)} < V,$$

which is impossible by (7) above

whence

$$I > \text{(circumscribed fig.)}$$

(segment),

(8)

In this case we compare the cylinders or frusta in  $(EB)$  with those in the circumscribed figure

Thus

$$\begin{aligned} & \text{(first cylinder or frustum in } EB') \quad \text{(first in circumscribed fig)} \\ & \qquad \qquad \qquad = S \quad S, \\ & \text{(second in } EB') \quad \text{(second in circumscribed fig)} \\ & \qquad \qquad \qquad = S \quad \text{(first gnomon),} \end{aligned}$$

and so on

$$\begin{aligned} \text{Lastly} \quad & \text{(last in } EB') \quad \text{(last in circumscribed fig)} \\ & = S \quad \text{(last gnomon)} \end{aligned}$$

Now

$$\{S + (\text{all the gnomons})\} = nS - (R_1 + R_2 + \dots + R_{n-1})$$

And

$$nS - R_1 + R_2 + \dots + R_{n-1} > (c+b) \left(\frac{c}{2} + \frac{b}{3}\right), \quad [\text{Prop 2}]$$

so that

$$nS - \{S + (\text{all the gnomons})\} < (c+b) \left(\frac{c}{2} + \frac{2b}{3}\right)$$

It follows that, if we combine the above proportions as in Prop 1, we obtain  
 (cylinder or frustum  $EB'$ ) (circumscribed fig)

$$< (c+b) \left(\frac{c}{2} + \frac{2b}{3}\right)$$

$$< A'D \frac{HD}{3}$$

$$< (EB) \quad V, \text{ by } (\beta) \text{ above}$$

Hence the circumscribed figure is greater than  $V$ , which is impossible, by  $(\delta)$  above

Thus, since the segment  $ABB'$  is neither greater nor less than  $V$ , it is equal to it, and the proposition is proved

(2) The particular case [Props 27, 28] where the segment is half the spheroid differs from the above in that the distance  $CD$  or  $c/2$  vanishes, and the rectangles  $cb + b^2$  are simply squares ( $b^2$ ), so that the gnomons are simply the differences between  $b^2$  and  $x^2$ ,  $b^2$  and  $(2x)^2$ , and so on

Instead therefore of Prop 2 we use the *Lemma to Prop 2, Cor 1*, given above [On Spirals, Prop 10], and instead of the ratio  $(c+b) \left(\frac{c}{2} + \frac{2b}{3}\right)$  we obtain the ratio  $\frac{2}{3}$ , whence (segment  $ABB'$ ) (cone or segment of cone  $ABE$ ) = 2 : 1

## PROPOSITIONS 31, 32

If a plane divide a spheroid into two unequal segments, and if  $AN$ ,  $A'N$  be the axes of the lesser and greater segments respectively, while  $C$  is the centre of the spheroid, then

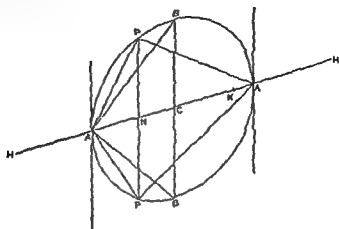
(greater segmt) (cone or segmt of cone with same base and axis)

$$= CA + AN \quad AN$$

the the  
 the optic  
 section  $PAP'A'$

Draw the tangents to the ellipse which are parallel to  $PP'$ , let them touch the ellipse at  $A$ ,  $A'$ , and through the tangents draw planes parallel to the base of the segments. These planes will touch the spheroid at  $A$ ,  $A'$ , the line  $AA'$

will pass through the centre  $C$  and bisect  $PP'$  in  $N$ , while  $AN$ ,  $A'N$  will be the axes of the segments



Then (1) if the cutting plane be perpendicular to the axis of the spheroid  $AA'$  will be that axis and  $A, A'$  will be the vertices of the spheroid as well as of the segments. Also the sections of the spheroid by the cutting plane and all planes parallel to it will be circles.

(2) If the cutting plane be not perpendicular to the axis, the base of the segments will be an ellipse of which  $PP'$  is an axis and the sections of the spheroid by all planes parallel to the cutting plane will be similar ellipses.

Draw a plane through  $C$  parallel to the base of the segments and meeting the plane of the paper in  $BB'$ .

Construct three cones or segments of cones two having  $A$  for their common vertex and the circle  $BB'$  for their respective bases and  $A'$  for its vertex

$$AH = A'H' = CA$$

We have then to prove that

$$\begin{aligned} & (\text{segment } A PP') \quad (\text{cone or segment of cone } A' PP') \\ & \quad = CA + AN \quad AN \\ & \quad = NH \quad HN \end{aligned}$$

Now half the spheroid is double of the cone or segment of a cone  $ABB'$  [Props 27-28] Therefore

$$(\text{the spheroid}) = 4(\text{cone or segment of cone } ABB')$$

But

$$\begin{aligned} & (\text{cone or segmt. of cone } ABB') \quad (\text{cone or segmt. of cone } APP') \\ & \quad = (CA \cdot AN) (BC^2 \cdot PN^2) \\ & \quad = (CA \cdot AN) (CA \cdot CA \cdot AN \cdot A'N) \end{aligned} \quad (a)$$

If we measure  $AK$  along  $AA'$  so that

$$\begin{aligned} & \frac{AK}{AN} = \frac{AC}{AN} = \frac{CA}{AN}, \\ & \text{we have} \quad \frac{AK}{AN} = \frac{AC}{AN} = \frac{CA}{AN}, \\ & \text{and the compound ratio in (a) becomes} \end{aligned}$$

$$\frac{(AK \cdot AN \cdot CA \cdot AN)}{AK \cdot CA' \cdot AN \cdot A'N} \quad (CA \cdot CA' \cdot AN \cdot A'N),$$

Thus

$$\begin{aligned} & (\text{cone or segmt of cone } ABB') \quad (\text{cone or segmt of cone } APP') \\ & \quad = AK \ CA' \ AN \ A'N \end{aligned}$$

But

$$\begin{aligned} & (\text{cone or segment of cone } APP') \quad (\text{segment } APP') \\ & \quad = A'N \ NH' \quad [\text{Props } 29, 30] \\ & \quad = AN \ A'N \ AN \ NH' \end{aligned}$$

Therefore, *ex aequali*,

$$\begin{aligned} & (\text{cone or segment of cone } ABB') \quad (\text{segment } APP') \\ & \quad = AK \ CA' \ AN \ NH', \end{aligned}$$

so that

$$\begin{aligned} & (\text{spheroid}) \quad (\text{segment } APP') \\ & \quad = HH' \ AK \ AN \ NH', \end{aligned}$$

since

$$HH' = 4CA'$$

Hence

$$\begin{aligned} & (\text{segment } A'PP') \quad (\text{segment } APP') \\ & \quad = (HH' \ AK - AN \ NH') \ AN \ NH' \\ & \quad = (AK \ NH + NH' \ NK) \ AN \ NH' \end{aligned}$$

Further,

$$\begin{aligned} & (\text{segment } APP') \quad (\text{cone or segment of cone } APP') \\ & \quad = NH' \ A'N \\ & \quad = AN \ NH' \ AN \ A'N, \end{aligned}$$

and

$$\begin{aligned} & (\text{cone or segmt of cone } APP') \quad (\text{cone or segmt of cone } A'PP') \\ & \quad = AN \ A'N \\ & \quad = AN \ 4'N \ A'N^2 \end{aligned}$$

From the last three proportions we obtain, *ex aequali*,

$$\begin{aligned} & (\text{segment } A'PP') \quad (\text{cone or segment of cone } A'PP') \\ & \quad = (AK \ NH + NH' \ NK) \ A'N^2 \\ & \quad = (AK \ NH + NH' \ NK) \ (CA^2 + NH' \ CN) \\ & \quad = (AK \ NH + NH' \ NK) \ (AK \ AN + NH' \ CN) \quad (\beta) \end{aligned}$$

But

$$\begin{aligned} AK \ NH \ AK \ AN &= NH \ AN \\ &= CA + AN \ AN \\ &= AK + CA \ CA \ (\text{since } AK \ AC = AC \ AN) \\ &= HK \ CA \\ &= HK - NH \ CA - AN \\ &= NK \ CN \\ &= NH' \ NK \ NH' \ CN \end{aligned}$$

Hence the ratio in  $(\beta)$  is equal to the ratio

$$AK \ NH \ AK \ AN, \text{ or } NH \ AN$$

Therefore

$$\begin{aligned} & (\text{segment } A'PP') \quad (\text{cone or segment of cone } A'PP') \\ & \quad = NH \ AN \\ & \quad = CA + AN \ AN \end{aligned}$$

## ON SPIRALS

"ARCHIMEDES to Dositheus greeting

' Of most of the theorems which I sent to Conon and of which you ask me from time to time to send you the proofs the demonstrations are already before you in the books brought to you by Heracleides, and some more are also contained in that which I now send you. Do not be surprised at my taking <sup>1</sup> considerable time before publishing these proofs. This has been owing to my desire to communicate them first to persons engaged in mathematical studies and anxious to investigate them. In fact, how many theorems in geometry which have seemed at first impracticable are in time successfully worked out! Now Conon died before he had sufficient time to investigate the theorems

mathematics and that his industry was extraordinary. But, though many years have elapsed since Conon's death I do not find that any one of the problems has been stirred by a single person. I wish now to put them in review one by one, particularly as it happens that there are two included among them which are impossible of realisation [and which may serve as a warning] how

you have already received the proofs and those of which the proofs are contained in this book respectively, I think it proper to specify. The first of the problems was Given a sphere to find a plane area equal to the surface of the sphere, and this was first made manifest on the publication of the book concerning the sphere for when it is once proved that the surface of any sphere is four times the greatest circle in the sphere, it is clear that it is possible to find a plane area equal to the surface of the sphere. The second was Given a cone or a cylinder to find a sphere equal to the cone or cylinder, the third To cut a given sphere by a plane so that the segments of it have to one another an assigned ratio the fourth To cut a given sphere by a plane so that the segments of the surface have to one another an assigned ratio. The fifth To cut a sphere by a plane so that the segments of the surface have to one another an assigned ratio. 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shall be similar to one of the segments and have its surface equal to the surface of the other segment. The seventh was From a given sphere to cut off a segment by a plane so that the segment bears to the cone which has the same base

<sup>1</sup>Cf. *On the Sphere and Cylinder* II. 5

as the segment and equal height an assigned ratio greater than that of three to two. Of all the propositions just enumerated Heracleides brought you the proofs. The proposition stated next after these was wrong, viz that, if a sphere be cut by a plane into unequal parts, the greater segment will have to the less the duplicate ratio of that which the greater surface has to the less. That this is wrong is obvious by what I sent you before, for it included this proposition

of any sphere be cut so that the square on the greater segment is triple of the

theorems which I before sent you. For it was there proved that the hemisphere

remains fixed, be made to revolve so that the diameter [axis] is the axis [of revolution], let the figure described by the section of the right-angled cone be called a *conoid*. And if a plane touch the conoidal figure and another plane drawn parallel to the tangent plane cut off a segment of the conoid, let the base

circle, but it needs to be proved that the segment cut off will be half as large again as the cone which has the same base as the segment and equal height. And if two segments be cut off from the conoid by planes drawn in any manner, it is clear that the sections will be sections of acute-angled cones [ellipses] if the cutting planes be not at right angles to the axis, but it needs to be proved that the segments will bear to one another the ratio of the squares on the lines drawn from their vertices parallel to the axis to meet the cutting planes. The proofs of these propositions are not yet sent to you

follows. If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which

<sup>1</sup>See *On the Sphere and Cylinder* II 8

<sup>2</sup>This should be presumably "the conoid" not "the cone"

the one revolution and if a straight line touch the spiral at the extreme end  
 $\frac{1}{2} \frac{1}{2} = \frac{1}{2} \frac{1}{2} = \dots$  at each angle to the line

moving along it make several revolutions and return to the position from  
 the spiral in the  
 that in the fourth  
 areas added in the  
 second revolution

according to the successive numbers while the area bounded by the spiral in the first revolution is a sixth part of that added in the second revolution. Also if on the spiral described in one revolution two points be taken and straight lines be drawn joining them to the fixed extremity of the revolving line and if two circles be drawn with the fixed point as centre and radii the lines drawn to the fixed extremity of the straight line and the shorter of the two lines be produced I say that (1) the area bounded by the circumference of the greater circle in the direction of (the part of) the spiral included between the straight lines the spiral (itself) and the produced straight line will bear to (2) the area bounded by the circumference of the lesser circle the same (part of the) spiral and the straight line joining their extremities the ratio which (3) the radius of the lesser circle together with two thirds of the excess of the radius of the greater circle over the radius of the lesser bears to (4) the radius of the lesser circle together with one third of the said excess

The proofs then of these theorems and others relating to the spiral are given in the present book. Prefixed to them after the manner usual in other geometrical works are the propositions necessary to the proofs of them. And here too as in the books previously published I assume the following lemma, that if there be (two) unequal lines or (two) unequal arcs the excess by which the greater exceeds the less can by being [continually] added to itself be made to exceed any given magnitude among those which are comparable with [it and with] one another

#### PROPOSITION 1

*If a point move at a uniform rate along any line, and two lengths be taken on it they will be proportional to the times of describing them*

Two unequal lengths are taken on a straight line, and two lengths on another straight line representing the times and they are proved to be proportional by taking equimultiples of each length and the corresponding time after the manner of Eucl. 1. Def. 5

#### PROPOSITION 2

*If each of two points on different lines respectively move along them each at a uniform rate and if lengths be taken one on each line forming pairs such that each pair are described in equal times the lengths will be proportional*

This is proved at once by equating the ratio of the lengths taken on one line to that of the times of description which must also be equal to the ratio of the lengths taken on the other line

## PROPOSITION 3

Given any number of circles, it is possible to find a straight line greater than the sum of all their circumferences

For we have only to describe polygons about each and then take a straight line equal to the sum of the perimeters of the polygons

## PROPOSITION 4

Given two unequal lines, viz a straight line and the circumference of a circle, it is possible to find a straight line less than the greater of the two lines and greater than the less

For, by the Lemma, the excess can, by being added a sufficient number of times to itself, be made to exceed the lesser line

Thus e.g. if  $c > l$  (where  $c$  is the circumference of the circle and  $l$  the length of the straight line), we can find a number  $n$  such that

$$n(c-l) > l$$

Therefore

$$c-l > \frac{l}{n}$$

and

$$c > l + \frac{l}{n} > l$$

Hence we have only to divide  $l$  into  $n$  equal parts and add one of them to  $l$ . The resulting line will satisfy the condition

## PROPOSITION 5

Given a circle with centre  $O$ , and the tangent to it at a point  $A$  it is possible to draw from  $O$  a straight line  $OPF$ , meeting the circle in  $P$  and the tangent in  $F$ , such that, if  $c$  be the circumference of any given circle whatever,

$$FP \cdot OP < (\text{arc } AP) \cdot c$$

Take a straight line as  $D$ , greater than the circumference = [Prop 3]

Through  $O$  draw  $OH$  parallel to the given tangent, and draw through  $A$  a line  $APH$ , meeting the circle in  $P$  and  $OH$  in  $H$ , such that the portion  $PH$  intercepted between the circle and the line  $OH$  may be equal to  $D$ . Join  $OP$  and produce it to meet the tangent in  $F$ .

$$\begin{aligned} \text{Then } FP \cdot OP &= AP \cdot PH, \text{ by parallels} \\ &= AP \cdot D \\ &< (\text{arc } AP) \cdot c \end{aligned}$$

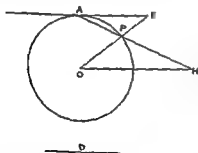
## PROPOSITION 6

Given a circle with centre  $O$ , a chord  $AB$  less than the diameter, and  $OM$  the perpendicular on  $AB$  from  $O$ , it is possible to draw a straight line  $OFP$ , meeting the chord  $AB$  in  $F$  and the circle in  $P$ , such that

$$FP \cdot PB = D \cdot E,$$

where  $D \cdot E$  is any given ratio less than  $BM \cdot MO$

Draw  $OH$  parallel to  $AB$ , and  $BT$  perpendicular to  $BO$  meeting  $OH$  in  $T$





Then the triangles  $BMO$ ,  $OBT$  are similar, and therefore

$$BM \cdot MO = OB \cdot BT,$$

whence

$$D \cdot E < OB \cdot BT$$

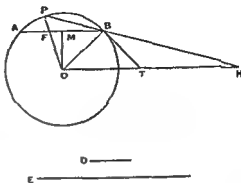
Suppose that  $\equiv$  line  $PH$  (greater than  $BT$ ) is taken such that

$$D \cdot E = OB \cdot PH,$$

and let  $PH$  be so placed that it passes through  $B$  and  $P$  lies on the circumference of the circle, while  $H$  is on the line  $OH$  ( $PH$  will fall outside  $BT$ , because  $PH > BT$ ). Join  $OP$  meeting  $AB$  in  $F$ .

We now have

$$\begin{aligned} FP \cdot PB &= OP \cdot PH \\ &= OB \cdot PH \\ &= D \cdot E \end{aligned}$$



#### PROPOSITION 7

Given a circle with centre  $O$ , a chord  $AB$  less than the diameter, and  $OM$  the perpendicular on it from  $O$ , it is possible to draw from  $O$  a straight line  $OPF$ , meeting the circle in  $P$  and  $AB$  produced in  $F$ , such that

$$FP \cdot PB = D \cdot E,$$

where  $D \cdot E$  is any given ratio greater than  $BM \cdot MO$ .

Draw  $OT$  parallel to  $AB$ , and  $BT$  perpendicular to  $BO$  meeting  $OT$  in  $T$ . In this case

$$D \cdot E > BM \cdot MO$$

$$> OB \cdot BT, \text{ by similar triangles}$$

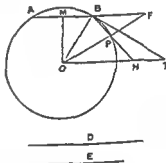
Take a line  $PH$  (less than  $BT$ ) such that

$$D \cdot E = OB \cdot PH,$$

and place  $PH$  so that  $P$ ,  $H$  are on the circle and on  $OT$  respectively, while  $HP$  produced passes through  $B$ .

Then

$$\begin{aligned} FP \cdot PB &= OP \cdot PH \\ &= D \cdot E \end{aligned}$$



#### PROPOSITION 8

Given a circle with centre  $O$ , a chord  $AB$  less than the diameter, the tangent at  $B$ , and the perpendicular  $OM$  from  $O$  on  $AB$ , it is possible to draw from  $O$  a straight line  $OPG$ , meeting the chord  $AB$  in  $F$ , the circle in  $P$  and the tangent in  $G$ , such that

$$FP \cdot BG = D \cdot E$$

where  $D \cdot E$  is any given ratio less than  $BM \cdot MO$ .

If  $OT$  be drawn parallel to  $AB$  meeting the tangent at  $B$  in  $T$ ,

$$BM \cdot MO = OB \cdot BT,$$

so that

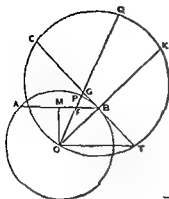
$$D \cdot E < OB \cdot BT$$

Take a point  $C$  on  $TB$  produced such that

$$D \cdot E = OB \cdot BC,$$

whence

$$BC > BT$$



Through the points  $O$ ,  $T$ ,  $C$  describe a circle and let  $OB$  be produced to meet this circle in  $K$ .

Then since  $BC > BT$ , and  $OB \perp CT$ , it is possible to draw from  $O$  a straight line  $OGQ$ , meeting  $CT$  in  $G$  and the circle about  $OTC$  in  $Q$ , such that  $GQ = BA$ .

Let  $OGQ$  meet  $AB$  in  $F$  and the original circle in  $P$ .

Now  $CG = GT = OG = GQ$ ,  
and  $OF = OG = BT = GT$ ,  
so that  $OF = GT = OG = BT$ .

It follows that

$$\begin{aligned} CG & GT & OF & GT = OG & GQ & OG & BT \\ CG & OF & < GQ & BT \\ & & < BA & BT, \text{ by construction,} \\ & & < BC & OB \\ & & < BC & OP \\ OP & OF & < BC & CG, \\ PF & OP & < BG & BC, \\ PF & BG & < OP & BC \\ & & < OB & BC \\ & & < D & E \end{aligned}$$

or

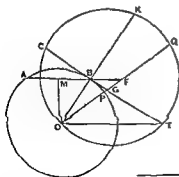
Hence  
and therefore  
or

#### PROPOSITION 9

Given a circle with centre  $O$ , a chord  $AB$  less than the diameter, the tangent at  $B$ , and the perpendicular  $OM$  from  $O$  on  $AB$ , it is possible to draw from  $O$  a straight line  $OPGF$ , meeting the circle in  $P$ , the tangent in  $G$ , and  $AB$  produced in  $F$ , such that

$$FP \cdot BG = D \cdot E,$$

where  $D \cdot E$  is any given ratio greater than  $BM \cdot MO$ .



Let  $OT$  be drawn parallel to  $AB$  meeting the tangent at  $B$  in  $T$ .

Then

$$D \cdot E > BM \cdot MO$$

$$> OB \cdot BT, \text{ by similar triangles}$$

Produce  $TB$  to  $C$  so that

$$D \cdot E = OB \cdot BC,$$

whence  $BC < BT$ .

Describe a circle through the points  $O$ ,  $T$ ,  $C$ , and produce  $OB$  to meet this circle in  $K$ .

Then since  $TB > BC$ , and  $OB$  is

in  $Q$ , such that  $GQ = BA$ . Let  $OQ$  meet the original circle in  $P$  and  $AB$  produced in  $F$ .

We now prove, exactly as in the 1st proposition, that

$$\begin{aligned} CO \quad OF &= BH \quad BT \\ &= BC \quad OP \end{aligned}$$

Thus as before,

$$\begin{aligned} OP \quad OF &= BC \quad CG, \\ OP \quad PF &= BC \quad BG, \\ PF \quad BG &= OP \quad HC \\ &= OB \quad BC \\ &= D \quad F \end{aligned}$$

and  
whence

### PROPOSITION 10

If  $A_1, A_2, A_3, \dots, A_n$  be  $n$  lines forming an ascending arithmetical progression in which the common difference is equal to 1, the least term then

$$(n+1)A_n^2 + A_1(A_1 + A_2 + \dots + A_n) = 3(1^2 + 2^2 + \dots + A_n^2)$$

[Archimedes' proof of this proposition is given above, pp 456-7, and it is there pointed out that the result is equivalent to

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}]$$

CON 1 It follows from this proposition that

$$n \cdot 1_n^2 < 3(1_1^2 + 1_2^2 + \dots + A_n^2),$$

and also that

$$n \cdot 1_n^2 > 3(A_1^2 + A_2^2 + \dots + A_{n-1}^2)$$

[For the proof of the latter inequality see p 457 above]

CON 2 All the results will equally hold if similar figures are substituted for squares

### PROPOSITION 11

If  $A_1, A_2, A_3, \dots, A_n$  be  $n$  lines forming an ascending arithmetical progression [in which the common difference is equal to the least term  $A_1$ ], then

$$(n-1)A_n^2 - (1_n^2 + 1_{n-1}^2 + \dots + A_2^2) < A_n^2 \{A_n - A_1 + \frac{1}{2}(A_n - A_1)^2\},$$

but

$$(n-1)A_n^2 - (A_{n-1}^2 + A_{n-2}^2 + \dots + A_1^2) > A_n^2 \{A_n - A_1 + \frac{1}{2}(A_n - A_1)^2\}$$

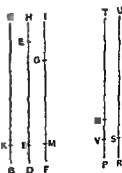
[Archimedes sets out the terms side by side in the manner shown in the figure where  $BC = A_n$ ,  $DE = A_{n-1}$ ,  $RS = A_1$  and produces  $DE$ ,  $FG$ ,  $RS$  until they are respectively equal to  $BC$  or  $A_n$  so that  $EH$ ,  $GI$ ,  $SU$  in the figure are respectively equal to  $1, 1_2, \dots, A_{n-1}$ . He further measures lengths  $BK, DL, FM, PV$  along  $BC, DE, FG, PQ$  respectively each equal to  $RS$ .

The figure makes the relations between the terms easier to see with the eye but the use of so large a number of letters makes the proof somewhat difficult to follow and it may be more clearly represented as follows]

It is evident that  $(A_n - A_1) = A_{n-1}$

The following proportion is therefore obviously true viz

$$(n-1)A_n^2 - (n-1)(A_n - A_1 + \frac{1}{2}A_{n-1}^2) = A_n^2 \{1 - A_1 + \frac{1}{2}(A_n - A_1)^2\}$$



In order therefore to prove the desired result, we have only to show that

$$(n-1)A_n A_1 + \frac{1}{2}(n-1)A_{n-1}^2 < (A_n^2 + A_{n-1}^2 + \dots + A_1^2) \\ > (A_{n-1}^2 + A_{n-2}^2 + \dots + A_1^2)$$

but

I To prove the first inequality, we have

$$(n-1)A_n A_1 + \frac{1}{2}(n-1)A_{n-1}^2 \\ = (n-1)A_1^2 + (n-1)A_1 A_{n-1} + \frac{1}{2}(n-1)A_{n-1}^2 \quad (1)$$

And

$$A_n^2 + A_{n-1}^2 + \dots + A_2^2 = (A_{n-1} + A_1)^2 + (A_{n-2} + A_1)^2 + \dots + (A_1 + A_1)^2 \\ = (A_{n-1}^2 + A_{n-2}^2 + \dots + A_1^2) \\ + (n-1)A_1^2 \\ + 2A_1(A_{n-1} + A_{n-2} + \dots + A_1) \\ = (A_{n-1}^2 + A_{n-2}^2 + \dots + A_1^2) \\ + (n-1)A_1^2 \\ + A_1\{A_{n-1} + A_{n-2} + A_{n-3} + \dots + A_1 \\ + A_1 + A_2 + \dots + A_{n-2} + A_{n-1}\} \\ = (A_{n-1}^2 + A_{n-2}^2 + \dots + A_1^2) \\ + (n-1)A_1^2 \\ + nA_1 A_{n-1} \quad (2)$$

Comparing the right-hand sides of (1) and (2), we see that  $(n-1)A_1^2$  is common to both sides, and

$$(n-1)A_1 A_{n-1} < nA_1 A_{n-1}$$

while, by Prop 10, Cor 1,

$$\frac{1}{2}(n-1)A_{n-1}^2 < A_{n-1}^2 + A_{n-2}^2 + \dots + A_1^2$$

It follows therefore that

$$(n-1)A_n A_1 + \frac{1}{2}(n-1)A_{n-1}^2 < (A_n^2 + A_{n-1}^2 + \dots + A_1^2),$$

and hence the first part of the proposition is proved

II We have now, in order to prove the second result, to show that

$$(n-1)A_n A_1 + \frac{1}{2}(n-1)A_{n-1}^2 > (A_{n-1}^2 + A_{n-2}^2 + \dots + A_1^2)$$

The right-hand side is equal to

$$(A_{n-2} + A_1)^2 + (A_{n-3} + A_1)^2 + \dots + (A_1 + A_1)^2 + A_1^2 \\ = A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2 \\ + (n-1)A_1^2 \\ + 2A_1(A_{n-2} + A_{n-3} + \dots + A_1) \\ = (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2) \\ + (n-1)A_1^2 \\ + A_1\{A_{n-2} + A_{n-3} + \dots + A_1 \\ + A_1 + A_2 + \dots + A_{n-2} + A_{n-1}\} \\ = (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2) \\ + (n-1)A_1^2 \\ + (n-2)A_1 A_{n-1} \quad (3)$$

Comparing this expression with the right-hand side of (1) above, we see that  $(n-1)A_1$  is common to both sides and

$$(n-1)A_1 A_{n-1} > (n-2)A_1 A_{n-1}$$

while by Prop 10 Cor 1,

$$\frac{1}{2}(n-1)A_{n-1}^2 > (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2)$$

Hence  $(n-1)A_n A_1 + \frac{1}{2}(n-1)A_{n-1}^2 > (A_{n-1}^2 + A_{n-2}^2 + \dots + A_1^2)$ ,

and the second required result follows

Cor The results in the above proposition are equally true if similar figures be substituted for squares on the several lines

DEFINITIONS

1 If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started and if at the same time as the line revolves a point move at a uniform rate along the straight line beginning from the extremity which remains fixed the point will describe a *spiral* (ἐλῆξ) in the plane

2 Let the extremity of the straight line which remains fixed while the straight line revolves be called the *origin* of the spiral

3 And let the position of the line from which the straight line began to revolve be called the *initial line* in the revolution

4 Let the length which the point that moves along the straight line describes in one revolution be called the *first distance* that which the same point describes in the second revolution the *second distance* and similarly let the distances described in further revolutions be called after the number of the particular revolution

5 Let the area bounded by the spiral described in the first revolution and the *first distance* be called the *first area* that bounded by the spiral described in the second revolution and the *second distance* the *second area* and similarly for the rest in order

6 If from the origin of the spiral any straight line be drawn let that side of it which is in the same direction as that of the revolution be called *forward* (πρὸς ἄγόμενα) and that which is in the other direction *backward* (εἰς ὀπίσθια)

7 Let the circle drawn with the *origin* as centre and the *first distance* as radius be called the *first circle* that drawn with the same centre and twice the radius the *second circle* and similarly for the succeeding circles

PROPOSITION 12

If an *n* number of straight lines drawn from the origin to meet the spiral make equal angles with one another the lines will be in arithmetical progression  
[The proof is obvious]

PROPOSITION 13

only

points *P Q* Join *OP OQ* and bisect the angle *POQ* by the straight line *OR* meeting the spiral in *R*

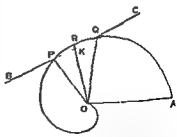
Then [Prop 12] *OR* is an arithmetic mean between *OP* and *OQ* or

$$OP + OQ = 2OR$$

But in any triangle *POQ* if the bisector of the angle *POQ* meets *PQ* in *K*

$$OP + OQ > 2OK$$

Therefore *OK* < *OR* and it follows that some point on *BC* between *P* and *Q* lies within the spiral Hence *BC* cuts the spiral which is contrary to the hypothesis



## PROPOSITION 14

If  $O$  be the origin, and  $P, Q$  two points on the first turn of the spiral, and if  $OP, OQ$  produced meet the "first circle"  $AKP'Q'$  in  $P', Q'$  respectively,  $OA$  being the initial line, then

$$OP \cdot OQ = (\text{arc } AKP') \cdot (\text{arc } AKQ')$$

For, while the revolving line  $OA$  moves about  $O$ , the point  $A$  on it moves uniformly along the circumference of the circle  $AKP'Q'$ , and at the same time the point describing the spiral moves uniformly along  $OA$ .

Thus, while  $A$  describes the arc  $AKP'$ , the moving point on  $OA$  describes the length  $OP$ , and, while  $A$  describes the arc  $AKQ'$ , the moving point on  $OA$  describes the distance  $OQ$ .

Hence

$$OP \cdot OQ = (\text{arc } AKP') \cdot (\text{arc } AKQ')$$

[Prop 2]

## PROPOSITION 15

If  $P, Q$  be points on the second turn of the spiral, and  $OP, OQ$  meet the "first circle"  $AKP'Q'$  in  $P', Q'$ , as in the last proposition, and if  $c$  be the circumference of the "first circle," then

$$OP \cdot OQ = c + (\text{arc } AKP') \cdot c + (\text{arc } AKQ')$$

arc  $AKQ'$

Con. Similarly, if  $P, Q$  are on the  $n$ th turn of the spiral,

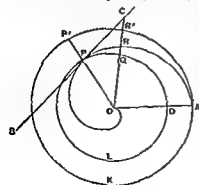
$$OP \cdot OQ = (n-1)c + (\text{arc } AKP') \cdot (n-1)c + (\text{arc } AKQ')$$

## PROPOSITIONS 16, 17

If  $BC$  be the tangent at  $P$ , any point on the spiral,  $PC$  being the "forward" part of  $BC$ , and if  $OP$  be joined, the angle  $OPC$  is obtuse while the angle  $OPB$  is acute.

I Suppose  $P$  to be on the first turn of the spiral.

Let  $OA$  be the initial line,  $AKP'$  the "first circle." Draw the circle  $DLP$  with centre  $O$  and radius  $OP$ , meeting  $OA$  in  $D$ . This circle must then, in the "forward" direction from  $P$ , fall within the spiral, and in the "backward" direction outside it, since the radii vectors of the spiral are on the "forward side" greater and on the "backward" side less than  $OP$ . Hence the angle  $OPC$  cannot be acute, since it cannot be less than the angle between  $OP$  and



the tangent to the circle at  $P$ , which is a right angle

It only remains therefore to prove that  $OPC$  is not a right angle

If possible, let it be a right angle  $BC$  will then touch the circle at  $P$

Therefore [Prop 5] it is possible to draw a line  $OQC$  meeting the circle through  $P$  in  $Q$  and  $BC$  in  $C$ , such that

$$CQ - OQ < (\text{arc } PQ) \quad (\text{arc } DLP) \quad (1)$$

Suppose that  $OC$  meets the spiral in  $R$  and the "first circle" in  $R'$ ; and produce  $OP$  to meet the "first circle" in  $P'$

From (1) it follows, *componendo*, that

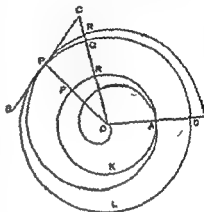
$$\begin{aligned} CO - OQ &< (\text{arc } DLQ) + (\text{arc } DLP) \\ &< (\text{arc } AKR') + (\text{arc } AKP') \\ &< OR - OP \quad [\text{Prop 14}] \end{aligned}$$

But this is impossible, because  $OQ = OP$ , and  $OR < OC$

Hence the angle  $OPC$  is not a right angle. It was also proved not to be acute

Therefore the angle  $OPC$  is obtuse, and the angle  $QPB$  consequently acute

II If  $P$  is on the second, or the  $n$ th turn, the proof is the same, except that in the proportion (1) above we have to substitute for the arc  $DLP$  an arc equal to  $(p + \text{arc } DLP)$  or  $(n-1)p + \text{arc } DLP$ , where  $p$  is the perimeter of the circle  $DLP$  through  $P$ . Similarly, in the later steps,  $p$  or  $(n-1)p$  will be added to each of the arcs  $DLQ$  and  $DLP$ , and  $c$  or  $(n-1)c$  to each of the arcs  $AKR'$ ,  $AKP'$ , where  $c$  is the circumference of the "first circle"  $AKP'$ .



### PROPOSITIONS 18, 19

I If  $OA$  be the initial line,  $A$  the end of the first turn of the spiral, the tangent to the spiral at  $A$  be drawn, the straight line  $OB$  perpendicular to  $OA$  will meet the said tangent in some point  $B$ , and  $OB$  will be equal to 2 (circumference of the "first circle")

II If  $A'$  be the end of the second turn, the perpendicular  $OB'$  will meet the tangent at  $A'$  in some point  $B'$ , and  $OB'$  will be equal to 2 (circumference of "second circle")

III Generally, if  $A_n$  be the end of the  $n$ th turn, and  $OB$  meet the tangent at  $A_n$  in  $B_n$ , then  $OB_n = nc_n$ , where  $c_n$  is the circumference of the " $n$ th circle"

I Let  $AA'C$  be the "first circle." Then since the "backward" angle between  $OA$  and the tangent at  $A$  is acute [Prop 16] the tangent will meet the "first circle" in a second point  $C$ . And the angles  $COA$ ,  $BOA$  are together less than two right angles, therefore  $OB$  will meet  $AC$  produced in some point  $B$

Then, if  $c$  be the circumference of the first circle, we have to prove that  $OB = c$

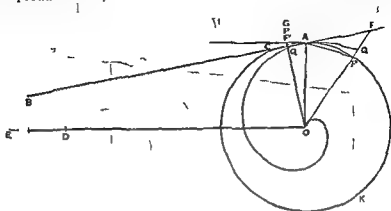
If not,  $OB$  must be either greater or less than  $c$

(1) If possible, suppose  $OB > c$

Measure along  $OB$  a length  $OD$  less than  $OB$  but greater than  $c$

We have then a circle  $AA'C$ , a chord  $AC$  in it less than the diameter, and a ratio  $AO : OD$  which is greater than the ratio  $AO : OB$  or (what is, by similar

triangles equal to it) the ratio of  $\frac{1}{2}AC$  to the perpendicular from  $O$  on  $AC$ . Therefore [Prop. 7] we can draw a straight line  $OPF$ , meeting the circle in  $P$  and  $CA$  produced in  $F$ , such that



Thus alternately, since

$FP$	$PA=AO$	$OD$
	$AO=PO$	
$FP$	$PO=PA$	$OD$
	$\angle(\text{arc } PA)$	$c.$

since  $(\text{arc } PA) > PA$ , and  $OD > c$

### Componendo

$$\angle POA < (\angle C + \angle PAO) < \angle COA$$

where  $OF$  meets the spiral in  $O$

[Prop 15]

Therefore since  $OA = OP$ ,  $FO < OQ$  which is impossible

Hence

ОВУС

(2) If possible, suppose

 $OB \leq c$ 

in  $G$ , such that

ГР АГ=АО ОЕ

Let  $OP'G$  cut the spiral in  $Q$

Then we have alternately

$$FP \cdot PO = AG \cdot OE$$

$$> (\text{arc } AP) \cdot c$$

because  $AG > (\text{arc } AP)$  and  $OE < c$

Therefore

$$\Gamma O \quad P O < (\text{arc } A K P) \quad c$$

[Prop 14]

But this is impossible since  $OA = OP$  and  $OQ < OF$

Hence

OB < c

Since therefore  $OB$  is neither greater nor less than  $c$ ,

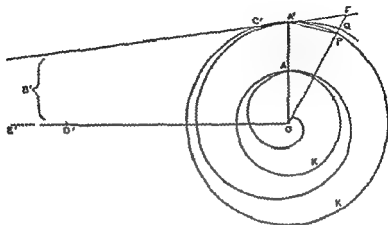
$$OB=c$$

II Let  $A'AC$  be the second circle  $AC$  being the tangent to the spiral



at  $A'$  (which will cut the second circle, since the "backward" angle  $O.A'C'$  is acute) Thus, as before, the perpendicular  $OB'$  to  $O.A'$  will meet  $A'C'$  produced in some point  $B'$

If then  $c'$  is the circumference of the "second circle," we have to prove that  $OB' = 2c'$



For, if not,  $OB'$  must be either greater or less than  $2c'$

(1) If possible, suppose  $OB' > 2c'$

Measure  $OD'$  along  $OB'$  so that  $OD'$  is less than  $OB'$  but greater than  $2c'$ .

Then, as in the case of the "first circle" above, we can draw a straight line  $OPF$  meeting the "second circle" in  $P$  and  $C'A'$  produced in  $F$ , such that

$$FP \cdot PA' = A'O \cdot OD'$$

Let  $OF$  meet the spiral in  $Q$

We now have, since  $A'O = PO$ ,

$$\begin{aligned} FP \cdot PO &= PA' \cdot OD' \\ &< (\text{arc } A'P) \cdot 2c', \end{aligned}$$

because  $(\text{arc } A'P) > A'P$  and  $OD' > 2c'$

$$\begin{aligned} \text{Therefore} \quad FO \cdot PO &< (2c' + \text{arc } A'P) \cdot 2c' \\ &< OQ \cdot OA' \end{aligned}$$

[Prop 15, Cor]

Hence  $FO < OQ$ , which is impossible

Thus  $OB' > 2c'$

Similarly as in the case of the "first circle" we can prove that

$$OB' < 2c'$$

Therefore  $OB' = 2c'$

III Proceeding, in like manner, to the "third and succeeding circles, we shall prove that

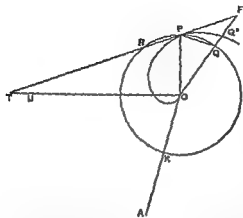
$$OB_n = 2c_n$$

#### PROPOSITION 20

I If  $P$  be any point on the first turn of the spiral and  $OT$  be drawn perpendicular to  $OP$ ,  $OT$  will meet the tangent at  $P$  to the spiral in some point  $T$ , and, if the circle drawn with centre  $O$  and radius  $OP$  meet the initial line in  $K$ , then  $OT$  is equal to the arc of this circle between  $K$  and  $P$  measured in the "forward" direction of the spiral

II Generally, if  $P$  be a point on the  $n$ th turn, and the notation be as before, while  $p$  represents the circumference of the circle with radius  $OP$ ,  
 $OT = (n-1)p + \text{arc } KP$  (measured "forward")

I Let  $P$  be a point on the first turn of the spiral,  $OA$  the initial line,  $PR$  the tangent at  $P$  taken in the "backward" direction



Then [Prop 16] the angle  $OPR$  is acute. Therefore  $PR$  meets the circle through  $P$  in some point  $R$ , and also  $OT$  will meet  $PR$  produced in some point  $T$ .

If now  $OT'$  is not equal to the arc  $KRP$ , it must be either greater or less.

(1) If possible, let  $OT$  be greater than the arc  $ARP$

Measure  $OU$  along  $OT'$  less than  $OT$  but greater than the arc  $KRP$

Then, since the ratio  $PO : OU$  is greater than the ratio  $PO : OT$ , or (what is, by similar triangles,

equal to it) the ratio of  $\frac{1}{2}PR$  to the perpendicular from  $O$  on  $PR$ , we can draw a line  $OQF$ , meeting the circle in  $Q$  and  $RP$  produced in  $F$ , such that

$$FQ \cdot PQ = PO \cdot OU \quad [\text{Prop. 7}]$$

Let  $OF$  meet the spiral in  $Q'$

We have then

FO 00-PO 0U

 $\angle(\text{arc } PQ) = \angle(\text{arc } KRP)$ , by hypothesis

**Componendo,**

$$FO \quad QO < (\text{arc } KRQ) \quad (\text{arc } KRP) \\ < OQ' \quad OP$$

[Prop 14]

But

$$OO=OP$$

Therefore  $FO < OQ'$ , which is impossible

Hence  $OT \nless (arc KRP)$

(2) The proof that  $OT \triangleleft (\text{arc } KRP)$  follows the method of Prop 18, I (2), exactly as the above follows that of Prop 18, I (1)

Since then  $OT \equiv$  neither greater nor less than the arc  $KRP$ , it is equal to it.

II If  $P$  be on the second turn, the same method shows that

$$OT' = p + (\text{arc } KRP).$$

and, similarly, we have, for a point  $P$  on the  $n$ th turn,

$$OT' = (n-1)p + (\text{arc } KRP)$$

### PROPOSITIONS 21, 22, 23

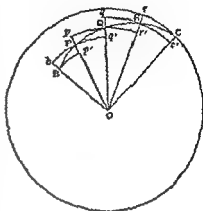
*Given an area bounded by any arc of a spiral and the lines joining the extremities of the arc to the origin, it is possible to circumscribe about the area one figure, and to inscribe in it another figure, each consisting of similar sectors of circles, and such that the circumscribed figure exceeds the inscribed by less than any assigned area.*

For let  $BC$  be any arc of the spiral,  $O$  the origin. Draw the circle with centre

$O$  and radius  $OC$ , where  $C$  is the "forward" end of the arc

Then, by bisecting the angle  $BOC$ , bisecting the resulting angles, and so on continually, we shall ultimately arrive at an angle  $COc$  cutting off a sector of the circle less than any assigned area. Let  $COc$  be this sector

Let the other lines dividing the angle  $BOC$  into equal parts meet the spiral in  $P, Q$ , and let  $Or$  meet it in  $R$ . With  $O$  as centre and radii  $OB, OP, OQ, Or$  respectively describe arcs of circles  $Bp', bBq', pQr', qRc'$ , each meeting the adjacent radii as shown in the figure. In each case the arc in the "forward" direction from each point will fall within, and the arc in the "backward" direction outside, the spiral



We have now a circumscribed figure and an inscribed figure each consisting of similar sectors of circles. To compare their areas, we take the successive sectors of each beginning from  $OC$ , and compare them.

The sector  $OCr$  in the circumscribed figure stands alone  
And

$$(\text{sector } ORq) = (\text{sector } ORc'),$$

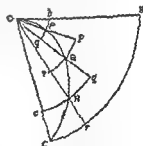
$$(\text{sector } OQp) = (\text{sector } OQr'),$$

$$(\text{sector } OPb) = (\text{sector } OPq'),$$

while the sector  $OBp'$  in the inscribed figure stands alone

Hence if the equal sectors be taken away, the difference between the circumscribed and inscribed figures is equal to the difference between the sectors  $OCr$  and  $OBp'$ , and this difference is less than the sector  $OCr$ , which is itself less than any assigned area.

The proof is exactly the same whatever be the number of angles into which the angle  $BOC$  is divided, the only difference being that, when the arc begins from the origin, the smallest sectors  $OPb, OPq'$  in each figure are equal, and there is therefore no inscribed sector standing by itself, so that the difference between the circumscribed and inscribed figures is equal to the sector  $OCr$  itself.



Thus the proposition is universally true

COR. Since the area bounded by the spiral is intermediate in magnitude between the circumscribed and inscribed figures, it follows that

(1) a figure can be circumscribed to the area such that it exceeds the area by less than any assigned space

(2) a figure can be inscribed such that the area exceeds it by less than any assigned space

#### PROPOSITION 24

The area bounded by the first turn of the spiral and the initial line is equal to one-third of the "first circle" [ $= \frac{1}{3}\pi(2ra)^2$ , where the spiral is  $r = a\theta$ ]

[The same proof shows equally that, if  $OP$  be any radius vector in the first turn of the spiral, the area of the portion of the spiral bounded thereby is equal to one third of that sector of the circle drawn with radius  $OP$  which is bounded by the initial line and  $OP$ , measured in the "forward" direction from the initial line]

Let  $O$  be the origin,  $OA$  the initial line,  $A$  the extremity of the first turn

Draw the "first circle," i.e. the circle with  $O$  as centre and  $OA$  as radius

Then, if  $C_1$  be the area of the first circle,  $E_1$  that of the first turn of the spiral bounded by  $OA$ , we have to prove that

$$R_1 = \frac{1}{3}C_1$$

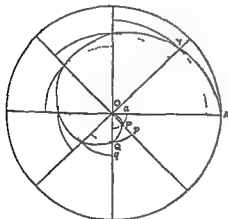
For, if not,  $R_1$  must be either greater or less than  $C_1$

I If possible, suppose  $R_1 < \frac{1}{3}C_1$

We can then circumscribe a figure about  $R_1$  made up of similar sectors of circles such that, if  $F$  be the area of this figure,

$$F - R_1 < \frac{1}{3}C_1 - R_1,$$

whence  $F < \frac{1}{3}C_1$



Let  $OP$ ,  $OQ$ , be the radii of the circular sectors, beginning from the smallest. The radius of the largest is of course  $OA$

The radii then form an ascending arithmetical progression in which the common difference is equal to the least term  $OP$ . If  $n$  be the number of the sectors, we have [by Prop 10, Cor 1]

$n \cdot OA^2 < 3(OP^2 + OQ^2 + \dots + OA^2)$ ,  
and, since the similar sectors are proportional to the squares on their radii, it follows that

$$\begin{aligned} C_1 &< 3F, \\ F &> \frac{1}{3}C_1 \end{aligned}$$

or

But this is impossible, since  $F$  was less than  $\frac{1}{3}C_1$

Therefore  $R_1 < \frac{1}{3}C_1$

II If possible, suppose  $R_1 > \frac{1}{3}C_1$

We can then inscribe a figure made up of similar sectors of circles such that, if  $f$  be its area

$$R_1 - f < R_1 - \frac{1}{3}C_1,$$

whence  $f > \frac{1}{3}C_1$

If there are  $(n-1)$  sectors, their radii, as  $OP$ ,  $OQ$ , , form an ascending arithmetical progression in which the least term is equal to the common difference, and the greatest term, as  $OY$ , is equal to  $(n-1)OP$

Thus [Prop 10, Cor 1]

$$n \cdot OA^2 > 3(OP^2 + OQ^2 + \dots + OY^2),$$

whence

$$C_1 > 3f,$$

or

$$f < \frac{1}{3}C_1,$$

which is impossible, since

$$f > \frac{1}{3}C_1$$

Therefore

$$R_1 = \frac{1}{3}C_1$$

Since then  $R_1$  is neither greater nor less than  $\frac{1}{3}C_1$ ,

$$R_1 = \frac{1}{3}C_1$$

## PROPOSITIONS 25-27

[Prop 25] If  $A_2$  be the area of the

second turn and  $OA$

the ratio of  $\{r_1 r_2 + \frac{1}{2}\}$

"second" circles resp

[Prop 26] If  $BC$  be any arc measured in the "forward" direction on any turn of a spiral, not being greater than the complete turn, and if a circle be drawn with  $O$  as centre and  $OC$  as radius meeting  $OB$  in  $B'$ , then

$$\{\text{area of spiral between } OB, OC\} : \{\text{sector } OB'C\}$$

$$= \{OC \cdot OB + \frac{1}{2}(OC - OB)^2\} : OC^2$$

[Prop 27] If  $R_1$  be the area of the first turn of the spiral bounded by the initial line,  $R_2$  the area of the ring added by the second complete turn,  $R_3$  that of the ring added by the third turn, and so on, then

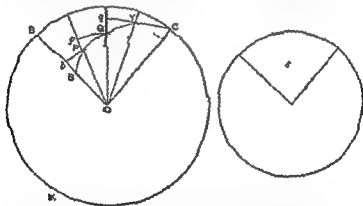
$$R_2 = 2R_1, R_3 = 3R_1, R_4 = 4R_1, \dots, R_n = (n-1)R_1$$

Also

$$R_2 = 6R_1$$

[Archimedes' proof of Prop 25 is, *mutatis mutandis*, the same as his proof of the more general Prop 26. The latter will accordingly be given here, and applied to Prop 25 as a particular case.]

Let  $BC$  be an arc measured in the "forward" direction on any turn of the spiral,  $CKB'$  the circle drawn with  $O$  as centre and  $OC$  as radius



Take a circle such that the square of its radius is equal to

$$OC \cdot OB + \frac{1}{2}(OC - OB)^2,$$

and let  $\sigma$  be a sector in it whose central angle is equal to the angle  $BOC$

Thus  $\sigma$  (sector  $OB'C$ ) =  $\{OC \cdot OB + \frac{1}{2}(OC - OB)^2\} : OC^2$ ,

and we have therefore to prove that

$$\{\text{area of spiral } OBC\} = \sigma$$

For, if not, the area of the spiral  $OBC$  (which we will call  $S$ ) must be either greater or less than  $\sigma$

I Suppose, if possible,  $S < \sigma$

Circumscribe to the area  $S$  a figure made up of similar sectors of circles, such that, if  $F$  be the area of the figure,

$$F - S < \sigma - S,$$

whence

$$F < \sigma$$

Let the radii of the successive sectors, starting from  $OB$ , be  $OP$ ,  $OQ$ ,  $OC$ .  
Produce  $OP$ ,  $OQ$ , to meet the circle  $CKB'$ ,

If then the lines  $OB$ ,  $OP$ ,  $OQ$ ,  $OC$  be  $n$  in number, the number of sectors in the circumscribed figure will be  $(n-1)$ , and the sector  $OB'C$  will also be divided into  $(n-1)$  equal sectors. Also  $OB$ ,  $OP$ ,  $OQ$ ,  $OC$  will form an ascending arithmetical progression of  $n$  terms

Therefore [see Prop 11 and Cor]

$$(n-1)OC^2 (OP^2 + OQ^2 + \dots + OC^2) < OC^2 \{OC \cdot OB + \frac{1}{2}(OC - OB)^2\} \\ < (\text{sector } OB'C) \sigma, \text{ by hypothesis}$$

Hence, since similar sectors are as the squares of their radii,

$$(\text{sector } OB'C) F < (\text{sector } OB'C) \sigma,$$

so that

$$F > \sigma$$

But this is impossible, because

$$F < \sigma$$

Therefore

$$S \neq \sigma$$

II Suppose, if possible,  $S > \sigma$

Inscribe in the area  $S$  a figure made up of similar sectors of circles such that, if  $f$  be its area,

$$S - f < S - \sigma,$$

whence

$$f > \sigma$$

Suppose  $OB$ ,  $OP$ ,  $OY$  to be the radii of the successive sectors making up the figure  $f$ , being  $(n-1)$  in number

We shall have in this case [see Prop 11 and Cor]

$$(n-1)OC^2 (OB^2 + OP^2 + \dots + OY^2) > OC^2 \{OC \cdot OB + \frac{1}{2}(OC - OB)^2\},$$

whence

$$(\text{sector } OB'C) f > (\text{sector } OB'C) \sigma,$$

so that

$$f < \sigma$$

But this is impossible, because

$$f > \sigma$$

Therefore

$$S > \sigma$$

Since then  $S$  is neither greater nor less than  $\sigma$ , it follows that

$$S = \sigma$$

In the particular case where  $B$  coincides with  $A_1$ , the end of the first turn of the spiral, and the complete

Thus

$$= \{r_2 r_1 + \frac{1}{2}(r_2 - r_1)^2\} r_2^2 \\ = (2 + \frac{1}{2}) 4 \quad (\text{since } r_2 = 2r_1) \\ = 7 \frac{1}{2}$$

Again, the area of the spiral bounded by  $OA_2$  is equal to  $R_1 + R_2$  (i.e. the area bounded by the first turn and  $OA_1$  together with the ring added by the second turn). Also the "second circle" is four times the "first circle," and therefore equal to  $12 R_1$

Hence

$$(R_1 + R_2) 12 R_1 = 7 \frac{1}{2} 12,$$

or

$$R_1 + R_2 = 7 R_1$$

Thus

$$R_2 = 6 R_1$$

(1)

Next, for the third turn, we have

$$(R_1 + R_2 + R_3) \text{ ("third circle")} = \{r_3 r_2 + \frac{1}{2}(r_3 - r_2)^2\} r_3^2 \\ = (3 + 2 + \frac{1}{2}) 3^2 \\ = 19 \frac{1}{2}$$

and

$$\text{("third circle")} = 9 \text{ ("first circle")} \\ = 27 R_1,$$

therefore

$$R_1 + R_2 + R_3 = 19R_1,$$

and, by (1) above, it follows that

$$\begin{aligned} R_2 &= 12R_1 \\ &= 2R_3, \end{aligned} \quad (2)$$

and so on

Generally, we have

$$\begin{aligned} (R_1 + R_2 + \dots + R_n) \text{ (nth circle)} &= \{r_n r_{n-1} + \frac{1}{2}(r_n - r_{n-1})^2\} r_n^2, \\ (R_1 + R_2 + \dots + R_{n-1}) \text{ (n-1th circle)} &= \{r_{n-1} r_{n-2} + \frac{1}{2}(r_{n-1} - r_{n-2})^2\} r_{n-1}^2, \\ \text{and} \quad & \text{(nth circle) (n-1th circle)} = r_n^2 - r_{n-1}^2 \end{aligned}$$

Therefore

$$\begin{aligned} (R_1 + R_2 + \dots + R_n) (R_1 + R_2 + \dots + R_{n-1}) \\ &= \{n(n-1) + \frac{1}{2}\} \{(n-1)(n-2) + \frac{1}{2}\} \\ &= \{3n(n-1) + 1\} \{3(n-1)(n-2) + 1\} \end{aligned}$$

Dirimendo,

$$R_n (R_1 + R_2 + \dots + R_{n-1}) = 6(n-1) \{3(n-1)(n-2) + 1\} \quad (\alpha)$$

Similarly

$$R_{n-1} (R_1 + R_2 + \dots + R_{n-2}) = 6(n-2) \{3(n-2)(n-3) + 1\},$$

from which we derive

$$\begin{aligned} R_{n-1} (R_1 + R_2 + \dots + R_{n-1}) \\ &= 6(n-2) \{6(n-2) + 3(n-2)(n-3) + 1\} \\ &= 6(n-2) \{3(n-1)(n-2) + 1\} \end{aligned} \quad (\beta)$$

Combining ( $\alpha$ ) and ( $\beta$ ), we obtain

$$R_n R_{n-1} = (n-1) (n-2)$$

Thus

$$R_2, R_3, R_4, \dots, R_n \text{ are in the ratio of the successive numbers } 1, 2, 3, \dots, (n-1)$$

### PROPOSITION 28

If  $O$  be the origin and  $BC$  any arc measured in the "forward" direction on any turn of the spiral, let two circles be drawn (1) with centre  $O$ , and radius  $OB$ , meeting  $OC$  in  $C'$ , and (2) with centre  $O$  and radius  $OC$ , meeting  $OB$  produced in  $B'$ . Then, if  $E$  denote the area bounded by the larger circular arc  $B'C$ , the line  $BB'$  and the spiral  $BC$ , while  $F$  denotes the area bounded by the smaller arc  $BC'$ , the line  $CC'$  and the spiral  $BC$

$$E - F = \{OB + \frac{1}{2}(OC - OB)\} \{OB + \frac{1}{2}(OC - OB)\}$$

Let  $\sigma$  denote the area of the lesser sector  $OBC'$ ,

then the larger sector  $OBC$  is equal to  $\sigma + F + E$

Thus [Prop 26]

$$\begin{aligned} (\sigma + F) (\sigma + F + E) &= \\ &= \{OC \cdot OB + \frac{1}{2}(OC - OB)^2\} OC^2 \quad (1) \end{aligned}$$

whence

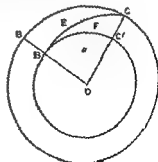
$$\begin{aligned} E (\sigma + F) &= \{OC(OC - OB) - \frac{1}{2}(OC - OB)^2\} \\ &= \{OC \cdot OB + \frac{1}{2}(OC - OB)^2\} \\ &= \{OB(OC - OB) + \frac{1}{2}(OC - OB)^2\} \\ &= \{OC \cdot OB + \frac{1}{2}(OC - OB)^2\} \quad (2) \end{aligned}$$

Again

$$(\sigma + F + E) \cdot \sigma = OC^2 \cdot OB^2$$

Therefore, by the first proportion above *ex aequali*,

$$(\sigma + F) \cdot \sigma = \{OC \cdot OB + \frac{1}{2}(OC - OB)^2\} OB^2,$$



whence

$$(e+F) F = \{OC \ OB + \frac{1}{2}(OC-OB)^2\} \\ \{OB(OC-OB) + \frac{1}{2}(OC-OB)^2\}$$

Combining this with (2) above we obtain

$$E F = \{OB(OC-OB) + \frac{1}{2}(OC-OB)^2\} \{OB(OC-OB) + \frac{1}{2}(OC-OB)^2\} \\ = \{OB + \frac{1}{2}(OC-OB)\} \{OB + \frac{1}{2}(OC-OB)\}$$



# ON THE EQUILIBRIUM OF PLANES OR THE CENTRES OF GRAVITY OF PLANES

## BOOK ONE

"I POSTULATE the following"

1 "Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance"

their centres of gravity similarly coincide"

5 "In figures which are unequal but similar, the centres of gravity will be similarly situated By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles, they

6 equa agnitudes

7 "In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure"

### PROPOSITION 1

*Weights which balance at equal distances are equal*

For, if they are unequal, take away from the greater the difference between the two The remainders will then not balance [Post 3], which is absurd

Therefore the weights cannot be unequal

### PROPOSITION 2

*Unequal weights at equal distances will not balance but will incline towards the greater weight*

For take away from the greater the difference between the two The equal remainders will therefore balance [Post 1] Hence, if we add the difference again, the weights will not balance but incline towards the greater [Post 2]

### PROPOSITION 3

*Unequal weights will balance at unequal distances, the greater weight being at the lesser distance*

Let  $A, B$  be two unequal weights (of which  $A$  is the greater) balancing about  $C$  at distances  $AC, BC$  respectively



Hence  $AC < CB$

Conversely, if the weights balance, and  $AC < CB$ , then  $A > B$

Then shall  $AC$  be less than  $BC$   
For, if not, take away from  $A$  the weight  $(A - B)$ . The remainders will then incline towards  $B$  [Post 3] But this is impossible, for (1) if  $AC = CB$ , the equal remainders will balance, or (2) if  $AC > CB$ , they will incline towards  $A$  at the greater distance [Post 1]

#### PROPOSITION 4

If two equal weights have not the same centre of gravity, the centre of gravity of both taken together is at the middle point of the line joining their centres of gravity  
[Proved from Prop 3 by *reductio ad absurdum*]

#### PROPOSITION 5

If three equal magnitudes have their centres of gravity on a straight line at equal distances, the centre of gravity of the system will coincide with that of the middle magnitude

[This follows immediately from Prop 4]

COR 1 The same is true of any odd number of magnitudes if those which are at equal distances from the middle one are equal, while the distances between their centres of gravity are equal

COR 2 If there be an even number of magnitudes with their centres of gravity situated at equal distances on one straight line, and if the two middle ones be equal, while those which are equidistant from them (on each side) are equal respectively, the centre of gravity of the system is the middle point of the line joining the centres of gravity of the two middle ones

#### PROPOSITIONS 6, 7

Two magnitudes, whether commensurable [Prop 6] or incommensurable [Prop 7],



We have then to prove that, if  $A$  be placed at  $E$  and  $B$  at  $D$ ,  $C$  is the centre of gravity of the two taken together

Since  $A, B$  are commensurable, so are  $DC, CE$ . Let  $N$  be a common measure of  $DC, CE$ . Make  $DH, DK$  each equal to  $CE$ , and  $EL$  (on  $CE$  produced) equal to  $CD$ . Then  $EH = CD$ , since  $DH = CE$ . Therefore  $LH$

is bisected at  $E$ , as  $HK$  is bisected at  $D$

Thus  $LH, HA$  must each contain  $N$  an even number of times

Take a magnitude  $O$  such that  $O$  is contained as many times in  $A$  as  $\Lambda$  is contained in  $LH$ , whence

$$\begin{aligned} A &= LH \cdot N \\ B &= CE \cdot DC \\ &= HK \cdot LH \end{aligned}$$

But

Hence *ex aequali*  $B = HK \cdot N$ , or  $O$  is contained in  $B$  as many times as  $\Lambda$  is contained in  $HK$ .

Thus  $O$  is a common measure of  $A$  &  $B$ .

Divide  $LH$  &  $HK$  into parts each equal to  $\Lambda$  and  $A$  &  $B$  into parts each equal to  $O$ . The parts of  $A$  will therefore be equal in number to those of  $LH$ , and the parts of  $B$  equal in number to those of  $HK$ . Place one of the parts of  $A$  at the middle point of each of the parts  $\Lambda$  of  $LH$  and one of the parts of  $B$  at the middle point of each of the parts  $N$  of  $HK$ .

Then the centre of gravity of the parts of  $A$  placed at equal distances on  $LH$  will be at  $E$ , the middle point of  $LH$  [Prop 5 Cor 2] and the centre of gravity of the parts of  $B$  placed at equal distances along  $HK$  will be at  $D$ , the middle point of  $HK$ .

Thus we may suppose  $A$  itself applied at  $E$  and  $B$  itself applied at  $D$ .

But the system formed by the parts  $O$  of  $A$  and  $B$  together is a system of equal magnitudes even in number and placed at equal distances along  $LH$ . And since  $LE = CD$  and  $EC = DK$ ,  $LC = CK$  so that  $C$  is the middle point of  $LK$ . Therefore  $C$  is the centre of gravity of the system ranged along  $LK$ .

Therefore  $A$  acting at  $E$  and  $B$  acting at  $D$  balance about the point  $C$ .

II Suppose the magnitudes to be incommensurable and let them be  $(A+a)$  and  $B$  respectively. Let  $DE$  be a line divided at  $C$  so that

$$(A+a) \cdot B = DC \cdot CE$$

Then if  $(A+a)$  placed at  $E$  and  $B$  placed at  $D$  do not balance about  $C$  ( $A+a$ ) is either too great to balance  $B$  or not great enough.

Suppose if possible that  $(A+a)$  is too great to balance  $B$ . Take from  $(A+a)$  a magnitude  $a$  smaller than the deduction which would make the remainder balance  $B$  but such that the remainder  $i$  and the magnitude  $B$  are commensurable.

Then since  $A$  &  $B$  are commensurable and

$$A \cdot B < DC \cdot CE$$

$A$  and  $B$  will not balance [Prop 6] but  $D$  will be depressed.

But this is impossible since the deduction  $a$  was an insufficient deduction from  $(A+a)$  to produce equilibrium so that  $E$  was still depressed.

Therefore  $(A+a)$  is not too great to balance  $B$ , and similarly it may be proved that  $B$  is not too great to balance  $(A+a)$ .

Hence  $(A+a)$  &  $B$  taken together have their centre of gravity at  $C$ .

#### PROPOSITION 8

If  $AB$  be a magnitude whose centre of gravity is  $C$  and  $AD$  a part of it whose centre of gravity is  $F$  then the centre of gravity of the remaining part will be a point  $G$  on  $FC$  produced such that



$$GC \cdot CF = (AD) \cdot (DE)$$

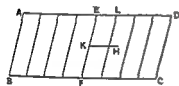
For, if the centre of gravity of the remainder (DE) be not G, let it be a point H. Then an absurdity follows at once from Props 6, 7

## PROPOSITION 9

*The centre of gravity of any parallelogram lies on the straight line joining the middle points of opposite sides*

Let ABCD be a parallelogram, and let EF join the middle points of the opposite sides AD, BC

If the centre of gravity does not lie on EF, suppose it to be H, and draw HK parallel to AD or BC meeting EF in K



Then it is possible, by bisecting ED, then bisecting the halves and so on continually, to arrive at a length EL less than KH. Divide both AE and ED into parts each equal to EL, and through the points of division draw parallels to AB or CD

We have then a number of equal and similar parallelograms, and, if any one be applied to any other, their centres of gravity coincide [Post 4]. Thus we have an

ograms [Prop 5, Cor 2]

But this is impossible for H is outside the middle parallelograms

Therefore the centre of gravity cannot but lie on EF

## PROPOSITION 10

*The centre of gravity of a parallelogram is the point of intersection of its diagonals*  
in each of the lines  
their intersection,

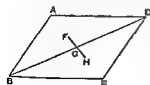
*Alternative proof*

Let ABCD be the given parallelogram, and BD a diagonal. Then the triangles ABD, CDB are equal and similar, so that [Post 4] if one be applied to the other, their centres of gravity will fall one upon the other

Suppose F to be the centre of gravity of the triangle ABD. Let G be the middle point of BD. Join FG and produce it to H so that  $FG = GH$

If we then apply the triangle ABD to the triangle CDB so that AD falls on CB and AB on CD, the point F will fall on H

But [by Post 4] F will fall on the centre of gravity of CDB. Therefore H is the centre of gravity of CDB



## PROPOSITION 11

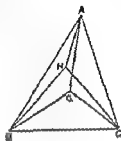
If  $abc$ ,  $ABC$  be two similar triangles, and  $g$ ,  $G$  two points in them similarly situated with respect to them respectively, then, if  $g$  be the centre of gravity of the triangle  $abc$ ,  $G$  must be the centre of gravity of the triangle  $ABC$

Suppose

$$ab \text{ be } ca = AB \text{ } BC \text{ } CA$$

The proposition is proved by an obvious *reductio ad absurdum*. For, if  $G$  be not the centre of gravity of the triangle  $ABC$ , suppose  $H$  to be its centre of gravity.

Post 5 requires that  $g$ ,  $H$  shall be similarly situated with respect to the triangles respectively, and this leads at once to the absurdity that the angles  $HAB$ ,  $GAB$  are equal



## PROPOSITION 12

Given two similar triangles  $abc$ ,  $ABC$ , and  $d$ ,  $D$  the middle points of  $bc$ ,  $BC$  respectively, then, if the centre of gravity of  $abc$  lie on  $ad$ , that of  $ABC$  will lie on  $AD$

Let  $g$  be the point on  $ad$  which is the centre of gravity of  $abc$

Take  $G$  on  $AD$  such that

$$ad \text{ } ag = AD \text{ } AG,$$

and join  $gb$ ,  $gc$ ,  $GB$ ,  $GC$

Then, since the triangles are similar, and  $bd$ ,  $BD$  are the halves of  $bc$ ,  $BC$  respectively,

$$ab \text{ } bd = AB \text{ } BD,$$

and the angles  $abd$ ,  $ABD$  are equal

Therefore the triangles  $abd$ ,  $ABD$  are similar, and

$$\angle bad = \angle BAD$$

Also

$$ba \text{ } ad = BA \text{ } AD,$$

while from above,

$$ad \text{ } ag = AD \text{ } AG$$

Therefore  $ba \text{ } ag = BA \text{ } AG$ , while the angles  $bag$ ,  $BAG$  are equal

Hence the triangles  $bag$ ,  $BAG$  are similar, and

$$\angle abg = \angle ABG$$

And since the angles  $abd$ ,  $ABD$  are equal it follows that

$$\angle gbd = \angle GBD$$

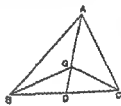
In exactly the same manner we prove that

$$\angle gac = \angle GAC,$$

$$\angle acg = \angle ACG,$$

$$\angle gcd = \angle GCD$$

Therefore  $g$ ,  $G$  are similarly situated with respect to the triangles respectively, whence [Prop 11]  $G$  is the centre of gravity of  $ABC$

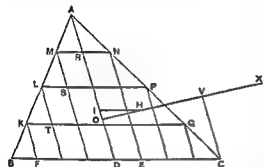


## PROPOSITION 13

*In any triangle the centre of gravity lies on the straight line joining any angle to the middle point of the opposite side*

Let  $ABC$  be a triangle and  $D$  the middle point of  $BC$ . Join  $AD$ . Then shall the centre of gravity lie on  $AD$ .

For, if possible, let this not be the case, and let  $H$  be the centre of gravity. Draw  $HI$  parallel to  $CB$  meeting  $AD$  in  $I$ .



Then, if we bisect  $DC$ , then bisect the halves, and so on, we shall at length arrive at a length, as  $DE$ , less than  $HI$ . Divide both  $BD$  and  $DC$  into lengths each equal to  $DE$ , and through the points of division draw lines each parallel to  $DA$  meeting  $BA$  and  $AC$  in points as  $K, L, M$  and  $N, P, Q$  respectively.

Join  $MN, LP, KQ$ , which lines will then be each parallel to  $BC$ .

We have now a series of parallelograms as  $FQ, TP, SN$ , and  $AD$  bisects opposite sides in each. Thus the centre of gravity of each parallelogram lies on  $AD$  [Prop 9], and therefore the centre of gravity of the figure made up of them all lies on  $AD$ .

Let the centre of gravity of all the parallelograms taken together be  $O$ . Join  $OH$  and produce it, also draw  $CV$  parallel to  $DA$  meeting  $OH$  produced in  $V$ .

Now, if  $n$  be the number of parts into which  $AC$  is divided,

$$\begin{aligned}\triangle ADC \text{ (sum of triangles on } AN, NP, \dots) &= AC^2 (AN^2 + NP^2 + \dots) \\ &= n^3 \cdot n \\ &= n \cdot 1 \\ &= AC \cdot AN\end{aligned}$$

Similarly

$$\triangle ABD \text{ (sum of triangles on } AM, ML, \dots) = AB \cdot AM$$

And

$$AC \cdot AN = AB \cdot AM$$

It follows that

$$\begin{aligned}\triangle ABC \text{ (sum of all the small } \triangle s) &= CA \cdot AN \\ &> VO \cdot OH, \text{ by parallels}\end{aligned}$$

Suppose  $OV$  produced to  $X$  so that

$$\triangle ABC \text{ (sum of small } \triangle s) = XO \cdot OH,$$

whence, dividendo,

Prop 8 that the centre of gravity of the remaining portion consisting of all the small triangles taken together is at  $X$ .

But this is impossible, since all the triangles are on one side of the line through  $X$  parallel to  $AD$ .

Therefore the centre of gravity of the triangle cannot but lie on  $AD$ .

*Alternative proof*

Suppose, if possible, that  $H$ , not lying on  $AD$ , is the centre of gravity of the triangle  $ABC$ . Join  $AH, BH, CH$ . Let  $E, F$  be the middle points of  $CA, AB$  respectively, and join  $DE, EF, FD$ . Let  $EF$  meet  $AD$  in  $M$ .

Draw  $FK, EL$  parallel to  $AH$  meeting  $BH, CH$  in  $K, L$  respectively. Join  $KD, HD, LD, AL$ . Let  $KL$  meet  $DH$  in  $N$ , and join  $MN$ .

Since  $DE$  is parallel to  $AB$ , the triangles  $ABC, EDC$  are similar.

And, since  $CE = EA$ , and  $EL$  is parallel to  $AH$ , it follows that  $CL = LH$ . And  $CD = DB$ . Therefore  $BH$  is parallel to  $DL$ .

Thus in the similar and similarly situated triangles  $ABC, EDC$  the straight lines  $AH, BH$  are respectively parallel to  $EL, DL$ , and it follows that  $H, L$  are similarly situated with respect to the triangles respectively.

But  $H$  is, by hypothesis, the centre of gravity of  $ABC$ . Therefore  $L$  is the centre of gravity of  $EDC$ . [Prop 11]

Similarly the point  $K$  is the centre of gravity of the triangle  $FBD$ .

And the triangles  $FBD, EDC$  are equal so that the centre of gravity of both together is at the middle point of  $KL$ , i.e. at the point  $N$ .

The remainder of the triangle  $ABC$ , after the triangles  $FBD, EDC$  are deducted, is the parallelogram  $AFDE$ , and the centre of gravity of this parallelogram is at  $M$ , the intersection of its diagonals.

It follows that the centre of gravity of the whole triangle  $ABC$  must be on  $MN$ , that is,  $MN$  must pass through  $H$ , which is impossible (since  $MN$  is parallel to  $AH$ ).

Therefore the centre of gravity of the triangle  $ABC$  cannot but be on  $AD$ .

## PROPOSITION 14

It follows at once from the last proposition that the centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.

## PROPOSITION 15

If  $AD, BC$  be the two parallel sides of a trapezium  $ABCD$ ,  $AD$  being the smaller, and if  $AD, BC$  be bisected at  $E, F$  respectively, then the centre of gravity of the trapezium is at a point  $G$  on  $EF$  such that

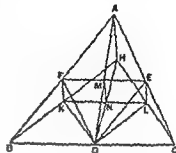
$$GE : GF = (2BC + AD) : (2AD + BC)$$

Produce  $BA, CD$  to meet at  $O$ . Then  $FE$  produced will also pass through  $O$ , since  $AE = ED$  and  $BF = FC$ .

Now the centre of gravity of the triangle  $OAD$  will lie on  $OE$ , and that of the triangle  $OBC$  will lie on  $OF$ . [Prop 13]

It follows that the centre of gravity of the remainder, the trapezium  $ABCD$ , will also lie on  $OF$ . [Prop 8]

Join  $BD$ , and divide it at  $L, M$  into three equal parts. Through  $L, M$  draw  $PQ, RS$  parallel to  $BC$  meeting  $BA$  in  $P, R$ ,  $FE$  in  $W, V$ , and  $CD$  in  $Q, S$  respectively.

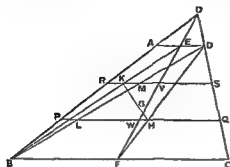


Join  $DF$ ,  $BE$  meeting  $PQ$  in  $H$  and  $RS$  in  $K$  respectively

Now, since

$$BL = \frac{1}{3}BD,$$

$$FH = \frac{1}{3}FD$$



of the triangle  $ADB$

Therefore the centre of gravity of the triangles  $DBC$ ,  $ADB$  together, i.e. of the trapezium, lies on the line  $HK$

But it also lies on  $OF$

Therefore, if  $OF$ ,  $HK$  meet in  $G$ ,  $G$  is the centre of gravity of the trapezium

Hence [Props 6, 7]

$$\begin{aligned} \triangle DBC \quad \triangle ABD &= KG \quad GH \\ &= VG : GW \end{aligned}$$

$$\begin{aligned} \triangle DBC \quad \triangle ABD &= BC \quad AD \\ BC \quad AD &= VG \cdot GW \end{aligned}$$

But

Therefore

It follows that

$$\begin{aligned} (2BC + AD) \quad (2AD + BC) &= (2VG + GW) \quad (2GW + VG) \\ &= EG \cdot GF \end{aligned}$$

Q.E.D.



# ON THE EQUILIBRIUM OF PLANES

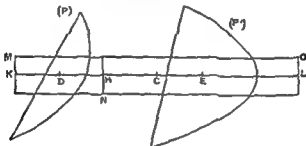
## BOOK TWO

### PROPOSITION 1

If  $P, P'$  be two parabolic segments and  $D, E$  their centres of gravity respectively, the centre of gravity of the two segments taken together will be at a point  $C$  on  $DE$  determined by the relation

$$P \cdot P' = CE \cdot CD$$

In the same straight line with  $DE$  measure  $EH, EL$  each equal to  $DC$ , and  $DK$  equal to  $DH$ , whence it follows at once that  $DK = CE$ , and also that  $KC = CL$



Apply a rectangle  $MN$  equal in area to the parabolic segment  $P$  to a base equal to  $KH$ , and place the rectangle so that  $KH$  bisects it, and is parallel to its base

Then  $D$  is the centre of gravity of  $MN$ , since  $KD = DH$

Produce the sides of the rectangle which are parallel to  $KH$ , and complete the rectangle  $NO$  whose base is equal to  $HL$ . Then  $E$  is the centre of gravity of the rectangle  $NO$

Now

$$\begin{aligned} (MN) \cdot (NO) &= KH \cdot HL \\ &= DH \cdot EH \\ &= CE \cdot CD \\ &= P \cdot P' \end{aligned}$$

But

$$(MN) = P$$

Therefore

$$(NO) = P'$$

Also, since  $C$  is the middle point of  $KL$ ,  $C$  is the centre of gravity of the whole parallelogram made up of the two parallelograms  $(MN), (NO)$ , which are equal to, and have the same centres of gravity as,  $P, P'$  respectively

Hence  $C$  is the centre of gravity of  $P, P'$  taken together

## DEFINITION AND LEMMAS PRELIMINARY TO PROPOSITION 2

ing segments triangles be inscribed in the same manner, let the resulting figure be said to be *inscribed in the recognised manner* in the segment

"And it is plain"

(1) "that the lines joining the two angles of the figure so inscribed which are nearest to the vertex of the segment, and the next pairs of angles in order, will be parallel to the base of the segment,"

(2) "that the said lines will be bisected by the diameter of the segment, and"

(3) "that they will cut the diameter in the proportions of the successive odd numbers, the number one having reference to [the length adjacent to] the vertex of the segment"

"And these properties will have to be proved in their proper places"

## PROPOSITION 2

If a figure be "inscribed in the recognised manner" in a parabolic segment, the centre of gravity of the figure so inscribed will lie on the diameter of the segment

For, in the figure of the foregoing lemmas, the centre of gravity of the trapezium  $BRrb$  must lie on  $XO$ , that of the trapezium  $RQqr$  on  $WX$ , and so on, while the centre of gravity of the triangle  $PAP$  lies on  $AV$

Hence the centre of gravity of the whole figure lies on  $AO$

## PROPOSITION 3

If  $BAB'$ ,  $bab'$  be two similar parabolic segments whose diameters are  $AO$ ,  $ao$  respectively, and if a figure be inscribed in each segment "in the recognised manner," the number of sides in each figure being equal, the centres of gravity of the inscribed figures will divide  $AO$ ,  $ao$  in the same ratio<sup>1</sup>

Suppose  $BRQPAP'Q'R'B'$ ,  $brqpap'q'r'b'$  to be the two figures inscribed "in the recognised manner" Join  $PP'$ ,  $QQ'$ ,  $RR'$  meeting  $AO$  in  $L$ ,  $M$ ,  $N$ , and  $pp'$ ,  $qq'$ ,  $rr'$  meeting  $ao$  in  $l$ ,  $m$ ,  $n$

Then [Lemma (3)]

$$\begin{array}{ccccccc} AL & LM & MN & NO & = & 1 & 3 & 5 & 7 \\ & & & & = & al & lm & mn & no, \end{array}$$

so that  $AO$ ,  $ao$  are divided in the same proportion

Also, by reversing the proof of Lemma (3), we see that

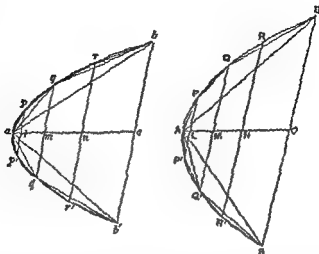
$$PP' : pp' = QQ' : qq' = RR' : rr' = BB' : bb'$$

Since then  $RR' : BB' = rr' : bb'$ , and these ratios respectively determine the proportion in which  $NO$ ,  $no$  are divided by the centres of gravity of the trapezia  $BRR'B'$ ,  $br'r'b'$  [15], it follows that the centres of gravity of the trapezia divide  $NO$ ,  $no$  in the same ratio

Similarly the centres of gravity of the trapezia  $RQQ'R'$ ,  $rqg'r'$  divide  $MN$ ,  $mn$  in the same ratio respectively, and so on

<sup>1</sup>Archimedes enunciates this proposition as true of similar segments but it is equally true of segments which are not similar as the course of the proof will show

Lastly, the centres of gravity of the triangles  $PAP'$ ,  $pap'$  divide  $AL$ ,  $al$  respectively in the same ratio



Moreover the corresponding trapezia and triangles are, each to each, in the same proportion (since their sides and heights are respectively proportional), while  $AO$   $ao$  are divided in the same proportion

Therefore the centres of gravity of the complete inscribed figures divide  $AO$   $ao$  in the same proportion

#### PROPOSITION 4

The centre of gravity of any parabolic segment cut off by a straight line lies on the diameter of the segment

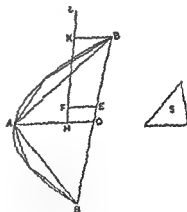
Let  $BAB$  be a parabolic segment  $A$  its vertex and  $AO$  its diameter

Then if the centre of gravity of the segment does not lie on  $AO$  suppose it to be, if possible the point  $F$  Draw  $FE$  parallel to  $AO$  meeting  $BB$  in  $E$

Inscribe in the segment the triangle  $ABB$  having the same vertex and height as the segment and take an area  $S$  such that

$$\triangle ABB \quad S = BE \quad EO$$

We can then inscribe in the segment "in the recognised manner" a figure such that the segments of the parabola left over are together less than  $S$ <sup>1</sup>



Let the inscribed figure be drawn accordingly, its centre of gravity then lies on  $AO$  [Prop 2] Let it be the point  $H$

Join  $HF$  and produce it to meet in  $K$  the line through  $B$  parallel to  $AO$

Then we have

$$\begin{aligned} (\text{inscribed figure}) \cdot (\text{remainder of segment}) &> \triangle ABB' \cdot S \\ &> BE \cdot EO \\ &> KF \cdot FH \end{aligned}$$

Suppose  $L$  taken on  $HK$  produced so that the former ratio is equal to the ratio  $LF : FH$

Then, since  $H$  is the centre of gravity of the inscribed figure, and  $F$  that of the segment,  $L$  must be the centre of gravity of all the segments taken together which form the remainder of the original segment [I 8]

But this is impossible, since all these segments lie on one side of the line drawn through  $L$  parallel to  $AO$  (Cf Post 7)

Hence the centre of gravity of the segment cannot but lie on  $AO$

### PROPOSITION 5

If in a parabolic segment a figure be inscribed "in the recognised manner," the centre of gravity of the segment is nearer to the vertex of the segment than the centre of gravity of the inscribed figure is

Let  $BAB'$  be the given segment, and  $AO$  its diameter. First, let  $ABB'$  be the triangle inscribed "in the recognised manner"

Divide  $AO$  in  $F$  so that  $AF = 2FO$ ,  $F$  is then the centre of gravity of the triangle  $ABB'$

Bisect  $AB, AB'$  in  $D, D'$  respectively, and join  $DD'$  meeting  $AO$  in  $E$ . Draw  $DQ, D'Q'$  parallel to  $OA$  to meet the curve.  $QD, Q'D'$  will then be the diameters of the segments whose bases are  $AB, AB'$ , and the centres of gravity of those segments will be respectively on  $QD, Q'D'$  [Prop 4]. Let them be  $H, H'$ , and join  $HH'$  meeting  $AO$  in  $K$ .

Now  $QD, Q'D'$  are equal,<sup>1</sup> and therefore the segments of which they are the diameters are equal [On Conoids and Spheroids, Prop 3]

Also, since  $QD, Q'D'$  are parallel, and  $DE = ED'$ ,  $K$  is the middle point of  $HH'$

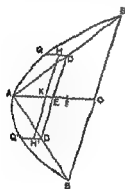
Hence the centre of gravity of the equal segments  $AQB, AQ'B'$  taken together =  $K$ , where  $K$  lies between  $E$  and  $A$ . And the centre of gravity of the triangle  $ABB'$  is  $F$

It follows that the centre of gravity of the whole segment  $BAB'$  lies between  $K$  and  $F$ , and is therefore nearer to the vertex  $A$  than  $F$  is

Secondly take the five-sided figure  $BQAQ'B'$  inscribed "in the recognised manner,"  $QD, Q'D'$  being as before, the diameters of the segments  $AQB, AQ'B'$

Then, by the first part of this proposition, the centre of gravity of the segment  $AQB$  (lying of course on  $QD$ ) is nearer to  $Q$  than the centre of gravity of

<sup>1</sup>This may either be inferred from Lemma (1) above (since  $QQ', DD'$  are both parallel to  $BB'$ ), or from Prop 19 of the *Quadrature of the Parabola*, which applies equally to  $Q$  or  $Q'$



the triangle  $AQB$  is  $I$ . Let the centre of gravity of the segment be  $H$ , and that of the triangle  $I$ .

Similarly let  $H'$  be the centre of gravity of the segment  $AQ'B'$ , and  $I'$  that of the triangle  $AQ'B'$ .

It follows that the centre of gravity of the two segments  $AQB$ ,  $AQ'B'$  taken together is  $K$ , the middle point of  $HH'$ , and that of the two triangles  $AQB$ ,  $AQ'B'$  is  $L$ , the middle point of  $II'$ .

If now the centre of gravity of the triangle  $ABB'$  be  $F$ , the centre of gravity of the whole segment  $BAB'$  (i.e. that of the triangle  $ABB'$  and the two segments  $AQB$ ,  $AQ'B'$  taken together) is a point  $G$  on  $KF$  determined by the proportion

$$(\text{sum of segments } AQB, AQ'B') : \triangle ABB' = FG : GK \quad [I\ 6, 7]$$

And the centre of gravity of the inscribed figure  $BQAQ'B'$  is a point  $F'$  on  $LF$  determined by the proportion

$$(\triangle AQB + \triangle AQ'B') : \triangle ABB' = F'F' : F'L \quad [I\ 6, 7]$$

[Hence

$$FG : GK > F'F' : F'L,$$

or  $GK : FG < F'L : F'F'$ ,

and, *componendo*  $FK : FG < FL : F'F'$ , while  $FK > FL$ ]

Therefore  $FG > F'F'$ , or  $G$  lies nearer than  $F'$  to the vertex  $A$ .

Using this last result, and proceeding in the same way, we can prove the proposition for any figure inscribed "in the recognised manner".

#### PROPOSITION 6

Given a segment of a parabola cut off by a straight line, it is possible to inscribe in it "in the recognised manner" a figure such that the distance between the centres of gravity of the segment and of the inscribed figure is less than any assigned length.

Let  $BAB'$  be the segment,  $AO$  its diameter,  $G$  its centre of gravity, and  $ABB'$  the triangle inscribed "in the recognised manner".

Let  $D$  be the assigned length and  $S$  an area such that

$$AG : D = \triangle ABB' : S$$

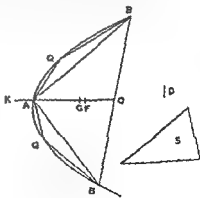
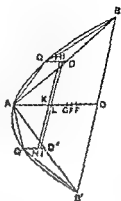
In the segment inscribe "in the recognised manner" a figure such that the sum of the segments left over is less than  $S$ . Let  $F$  be the centre of gravity of the inscribed figure.

We shall prove that  $FG < D$ .

For, if not,  $FG$  must be either equal to, or greater than,  $D$ .

And clearly

$$\begin{aligned} & (\text{inscribed fig}) : (\text{sum of remaining segmts}) \\ & > \triangle ABB' : S \\ & > AG : D \\ & > AG : FG, \text{ by hypothesis (since } FG < D) \end{aligned}$$



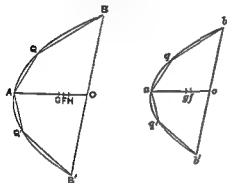
Let the first ratio be equal to the ratio  $KG : FG$  (where  $K$  lies on  $GA$  produced), and it follows that  $K$  is the centre of gravity of the small segments taken together [I 8]

But this is impossible, since the segments are all on the same side of a line drawn through  $K$  parallel to  $BB'$

Hence  $FG$  cannot but be less than  $D$

## PROPOSITION 7

If there be two similar parabolic segments, their centres of gravity divide their diameters in the same ratio



Let  $BAB'$ ,  $bab'$  be the two similar segments,  $AO$ ,  $ao$  their diameters, and  $G$ ,  $g$  their centres of gravity respectively

Then, if  $G$ ,  $g$  do not divide  $AO$ ,  $ao$  respectively in the same ratio, suppose  $H$  to be such a point on  $AO$  that

$$AH : HO = ag : go,$$

and inscribe in the segment  $BAB'$  "in the recognised manner" a figure such that, if  $F$  be its centre of gravity,

$$GF < GH \quad [\text{Prop 6}]$$

Inscribe in the segment  $bab'$  "in the recognised manner" a similar

figure, then if  $f$  be the centre of gravity of this figure,

$$ag < af$$

[Prop 5]

And, by Prop 3,  $af : fo = AF : FO$

But  $AF : FO < AH : HO$

$$< ag : go, \text{ by hypothesis}$$

Therefore  $af : fo < ag : go$ , which is impossible

It follows that  $G$ ,  $g$  cannot but divide  $AO$ ,  $ao$  in the same ratio

## PROPOSITION 8

If  $AO$  be the diameter of a parabolic segment, and  $G$  its centre of gravity, then

$$AG = \frac{3}{8}AO$$

triangle  $ABB'$  "in the recognised

manner"  $DQ'$  parallel to  $OA$  to meet the curve, so that  $QD$ ,  $Q'D'$  are the diameters of the segments  $AQB$ ,  $AQ'B'$  respectively

Let  $H$ ,  $H'$  be the centres of gravity of the segments  $AQB$ ,  $AQ'B'$  respectively. Join  $QQ'$ ,  $HH'$  meeting  $AO$  in  $V$ ,  $K$  respectively

$K$  is then the centre of gravity of the two segments  $AQB$ ,  $AQ'B'$  taken together

Now  $AG : GO = QH : HD$ ,

[Prop 7]

whence

$$AO : OG = QD : HD$$

But  $AO = 4QD$  [as easily proved by means of Lemma (3), p. 511]

Therefore  $OG = 4HD$ ,

and, by subtraction,

$$AG = 4QH$$

Also, by Lemma (2),  $QQ'$  is parallel to  $BB'$  and therefore to  $DD'$ . It follows from Prop 7 that  $HH'$  is also parallel to  $QQ'$  or  $DD'$ ,

and hence  $QH = VK$

Therefore  $AG = 4VK$ ,

and  $AV + KG = 3VK$

Measuring  $VL$  along  $VK$  so that  $VL = \frac{1}{3}AV$ , we have

$$KG = 3LK \quad (1)$$

Again  $AO = 4AV$  [Lemma (3)]

$$= 3AL, \text{ since } AV = 3VL,$$

whence  $AL = \frac{1}{3}AO = OF$  (2)

Now, by I 6, 7,

$\triangle ABB'$  (sum of segmts  $AQB, AQ'B'$ ) =  $KG \cdot GF$ ,

and  $\triangle ABB' = 3(\text{sum of segments } AQB, AQ'B')$

[since the segment  $ABB'$  is equal to  $\frac{1}{3}\triangle ABB'$  (*Quadrature of the Parabola*, Props 17, 24)]

Hence  $KG = 3GF$

But  $KG = 3LK$ , from (1) above

Therefore  $LF = LK + KG + GF$   
 $= 5GF$

And, from (2),

$$LF = (AO - AL - OF) = \frac{1}{3}AO = OF.$$

Therefore  $OF = 5GF$ ,

and  $OG = 6GF$

But  $AO = 3OF = 15GF$

Therefore, by subtraction,

$$AG = 9GF \\ = \frac{3}{2}GO$$

### PROPOSITION 9 (LEMMA)

If  $a, b, c, d$  be four lines in continued proportion and in descending order of magnitude, and if

$$d(a-d) = x \cdot \frac{1}{2}(a-c),$$

and  $(2a+4b+6c+3d)(5a+10b+10c+5d) = y(a-c)$ ,

it is required to prove that

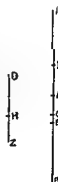
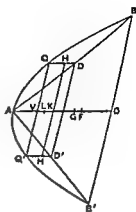
$$x+y = \frac{1}{2}a$$

[The following is the proof given by Archimedes, with the only difference that it is set out in algebraical instead of geometrical notation. This is done in the particular case simply in order to make the proof easier to follow. Archimedes exhibits his lines in the figure reproduced in the margin but, now that it is possible to use algebraical notation there is no advantage in using the figure and the more cumbersome notation which only obscures the course of the proof. The relation between Archimedes' figure and the letters used below is as follows

$AB = a, \Gamma B = b, \Delta B = c, EB = d, ZH = x, H\Theta = y, \Delta O = z$

We have

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} \quad (1)$$



whence

$$\frac{a-b}{b} = \frac{b-c}{c} = \frac{c-d}{d},$$

and therefore

$$\frac{a-b}{b-c} = \frac{b-c}{c-d} = \frac{a}{b} = \frac{b}{c} = \frac{c}{d} \quad (2)$$

Now

$$\frac{2(a+b)}{2c} = \frac{a+b}{c} = \frac{a+b}{b} \cdot \frac{b}{c} = \frac{a-c}{b-c} \cdot \frac{b-c}{c-d} = \frac{a-c}{c-d}$$

And in like manner,

$$\frac{b+c}{d} = \frac{b+c}{c} = \frac{a-c}{c-d}$$

It follows from the last two relations that

$$\frac{a-c}{c-d} = \frac{2a+3b+c}{2c+d} \quad (3)$$

Suppose  $z$  to be so taken that

$$\frac{2a+4b+4c+2d}{2c+d} = \frac{a-c}{z} \quad (4)$$

so that  $z < (c-d)$

Therefore

$$\frac{a-c+z}{a-c} = \frac{2a+4b+6c+3d}{2(a+d)+4(b+c)}$$

And, by hypothesis,

$$\frac{a-c}{y} = \frac{5(a+d)+10(b+c)}{2a+4b+6c+3d},$$

so that

$$\frac{a-c+z}{y} = \frac{5(a+d)+10(b+c)}{2(a+d)+4(b+c)} = \frac{5}{2} \quad (5)$$

Again, dividing (3) by (4) crosswise, we obtain

$$\frac{z}{c-d} = \frac{2a+3b+c}{2(a+d)+4(b+c)},$$

$$\frac{c-d-z}{c-d} = \frac{b+3c+2d}{2(a+d)+4(b+c)} \quad (6)$$

whence

But, by (2),

$$\frac{c-d}{d} = \frac{a-b}{b} = \frac{3(b-c)}{3c} = \frac{2(c-d)}{2d},$$

so that

$$\frac{c-d}{d} = \frac{(a-b)+3(b-c)+2(c-d)}{b+3c+2d} \quad (7)$$

Combining (6) and (7), we have

$$\frac{c-d-z}{d} = \frac{(a-b)+3(b-c)+2(c-d)}{2(a+d)+4(b+c)},$$

whence

$$\frac{z-z}{d} = \frac{3a+6b+3c}{2(a+d)+4(b+c)} \quad (8)$$

And, since [by (1)]

$$\frac{c-d}{c+d} = \frac{b-c}{b+c} = \frac{a-b}{a+b},$$

we have

$$\frac{c-d}{a-c} = \frac{c+d}{b+c+a+b},$$

whence

$$\frac{a-d}{a-c} = \frac{a+2b+2c+d}{a+2b+c} = \frac{2(a+d)+4(b+c)}{2(a+c)+4b} \quad (9)$$



Thus

$$\frac{a-d}{\frac{2}{3}(a-c)} = \frac{2(a+d)+4(b+c)}{\frac{2}{3}\{2(a+c)+4b\}}$$

and therefore, by hypothesis,

$$\frac{d}{x} = \frac{2(a+d)+4(b+c)}{\frac{2}{3}\{2(a+c)+4b\}}$$

But, by (8),

$$\frac{c-z}{d} = \frac{3a+6b+3c}{2(a+d)+4(b+c)}$$

and it follows, *ex aequali*, that

$$\frac{c-z}{x} = \frac{3(a+c)+6b}{\frac{2}{3}\{2(a+c)+4b\}} = \frac{5}{3} \cdot \frac{3}{2} = \frac{5}{2}$$

And, by (5),

$$\frac{a-c+z}{y} = \frac{5}{2}$$

Therefore

$$\frac{5}{2} = \frac{a}{x+y'}$$

or

$$x+y' = \frac{2}{5}a$$

## PROPOSITION 10

If  $PP'B'B$  be the portion of a parabola intercepted between two parallel chords  $PP'$ ,  $BB'$  bisected respectively in  $N$ ,  $O$  by the diameter  $ANO$  ( $N$  being nearer than  $O$  to  $A$ , the vertex of the segments), and if  $NO$  be divided into five equal parts of which  $LM$  is the middle one ( $L$  being nearer than  $M$  to  $N$ ), then, if  $G$  be a point on  $LM$  such that

$$LG \cdot GM = BO^2 (2PN + BO) \quad PN^2 (2BO + PN),$$

$G$  will be the centre of gravity of the area  $PP'B'B$

Take a line  $ao$  equal to  $AO$ , and  $an$  on it equal to  $AN$ . Let  $p, q$  be points on the line  $ao$  such that

$$ao \cdot aq = aq \cdot an, \quad (1)$$

$$ao \cdot an = aq \cdot ap, \quad (2)$$

[whence  $ao \cdot aq = aq \cdot an = an \cdot ap$ , or  $ao, aq, an, ap$  are lines in continued proportion and in descending order of magnitude]

Measure along  $GA$  a length  $GF$  such that

$$op \cdot ap = OL \cdot GF \quad (3)$$

Then, since  $PN, BO$  are ordinates to  $ANO$ ,

$$BO^2 \cdot PN^2 = AO \cdot AN$$

$$= ao \cdot an$$

$$= ao^2 \cdot aq^2, \text{ by (1),}$$

$$\frac{BO}{BO^2} \cdot \frac{PN}{PN^2} = \frac{ao}{ao^2} \cdot \frac{aq}{aq^2}, \quad (4)$$

$$= \frac{(ao \cdot aq)}{ao^2} \cdot \frac{(aq \cdot an)}{aq^2}$$

$$= \frac{(ao \cdot aq)}{ao} \cdot \frac{(aq \cdot an)}{aq}$$

$$= ao \cdot ap \quad (5)$$

$$= (\text{segment } BAB') \cdot (\text{segment } PAP')$$

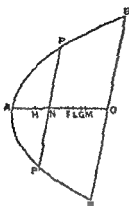
$$= \triangle BAB' \cdot \triangle PAP'$$

$$= BO^2 \cdot PN^2$$

$$= ao \cdot ap,$$

whence

$$(\text{area } PP'B'B) \cdot (\text{segment } PAP) = op \cdot ap$$



$$\begin{aligned} &= OL \quad GF, \text{ by (3),} \\ &= 20 \setminus GF \end{aligned} \quad (6)$$

$$\begin{aligned} BO^* (2PV + BO) & BO^* \\ & = (2PN + BO) BO \\ & = (2aq + ao) ao \text{ by (4),} \\ BO^* P^* & \\ & = ao ap, \text{ by (5),} \end{aligned}$$

$$\begin{aligned} P\lambda^3 &= P\lambda^2 (2BO + PN) \\ &= P\lambda^2 (2BO + PN) \\ &= aq (2ao + aq) \text{ by (4)} \\ &= ap (2an + ap) \text{ by (2)} \end{aligned}$$

Hence, ex æquali,  
 $BO^2 (2PV + BO) : PA^2$   
 $(2BO + PV) = (2aq + ao)$   
 $(2an + ap),$

so that by hypothesis

$$LG \quad CV = (2aq + ao) \quad (2an + ap)$$

Componendo, and multiplying the antecedents by 5

$$O_V \quad GM = \{5(ao+ap)+10(ao+an)\} \quad (2an+ap)$$

But

$$OV \quad OV = 5 \quad 2 = \{5(a_0 + a_p) + 10(a_0 + a_n)\} \quad \{2(a_0 + a_p) + 4(a_0 + a_n)\}$$

It follows that

$$OY \quad OG = 15(ao + ay) + 10(ag + an) \quad (2ao + 4aq + 6an + 3ap)$$

Therefore

$$\begin{array}{rcl} (2ao+4aq+6an+3ap) & \{5(ao+ap)+10(aq+an)\} & =0G \quad ON \\ & & =0G \quad on \end{array}$$

And  $ap \ (ao \rightarrow ap) = ap \quad op$   
 $\quad \quad \quad = GF \quad OL \text{ by hypothesis}$   
 $\quad \quad \quad = GF \quad \text{?on}$

while  $a_0, a_1, a_2, a_3$  are in continued proportion

Therefore by Prop 9

$$GF + OG = OF = \frac{1}{2}OD = \frac{1}{2}OA$$

Thus  $F$  is the centre of gravity of the segment  $BAB$  [Prop 8]

Let  $H$  be the centre of gravity of the segment  $PAP$  so that  $AH = \frac{1}{3}AN$

And since  $AF = \frac{2}{3}AO$

we have by subtraction

But by (6) above

$$\begin{aligned} (\text{area } PPBB) \quad (\text{segment } PAP) &= \frac{1}{2} ON \cdot GF \\ &= HF \cdot EG \end{aligned}$$

Thus since  $F$ ,  $H$  are the centres of gravity of the segments  $BAB$ ,  $PAP$  respectively, it follows [by 1.6.7] that  $G$  is the centre of gravity of the area  $PLRR$ .

## THE SAND-RECKONER

"THERE are some, King Gelon, who think that the number of the sand is infinite in multitude, and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed its multitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the earth, including in it all the seas and the hollows of the earth filled up to a height equal to that of the highest of the mountains, would be many times further still from recognising that any number could be expressed which exceeded the multitude of the sand so taken. But I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuvippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described, but also that of a mass equal in magnitude to the universe. Now you are aware that 'universe' is the name given by most astronomers to the sphere whose centre is the centre of the earth and whose radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account (*τα γυαφόμενα*), as you have heard from astronomers. But Aristarchus of Samos brought out a book consisting of some hypotheses, in which the premisses lead to the result that the universe is many times greater than that now so called. His hypotheses are that the fixed stars and the sun remain unmoved, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface. Now it is easy to see that this is impossible, for, since the centre of the sphere has no magnitude, we cannot conceive it to bear any ratio whatever to the surface of the sphere. We must however take Aristarchus to mean this: since we conceive the earth to be, as it were, the centre of the universe, the ratio which the earth bears to what we describe as the 'universe' is the same as the ratio which the sphere containing the circle in which he supposes the earth to revolve bears to the sphere of the fixed stars. For he adapts the proofs of his results to a hypothesis of this kind, and in particular he appears to suppose the magnitude of the sphere in which he represents the earth as moving to be equal to what we call the 'universe'.

"I say then that, even if a sphere were made up of the sand, as great as Aristarchus supposes the sphere of the fixed stars to be, I shall still prove that,

of the numbers named in the *Principles*,<sup>1</sup> some exceed in multitude the number of the sand which is equal in magnitude to the sphere referred to, provided that the following assumptions be made "

1 "*The perimeter of the earth is about 3,000,000 stadia and not greater*

"It is true that some have tried, as you are of course aware, to prove that the said perimeter is about 300,000 stadia. But I go further and, putting the magnitude of the earth at ten times the size that my predecessors thought it, I suppose its perimeter to be about 3,000,000 stadia and not greater "

2 "*The diameter of the earth is greater than the diameter of the moon, and the diameter of the sun is greater than the diameter of the earth*

"In this assumption I follow most of the earlier astronomers "

3 "*The diameter of the sun is about 30 times the diameter of the moon and not greater*

"It is true that, of the earlier astronomers, Eudoxus declared it to be about nine times as great, and Pheidias my father twelve times, while Aristarchus tried to prove that the diameter of the sun is greater than 18 times but less than 20 times the diameter of the moon. But I go even further than Aristarchus, in order that the truth of my proposition may be established beyond dispute, and I suppose the diameter of the sun to be about 30 times that of the moon and not greater "

4 "*The diameter of the sun is greater than the side of the chiliagon inscribed in the greatest circle in the (sphere of the) universe*

"I make this assumption because Aristarchus discovered that the sun appeared to be about  $\frac{1}{180}$ th part of the circle of the zodiac, and I myself tried, by a method which I will now describe, to find experimentally (*δορυανδρῶς*) the angle subtended by the sun and having its vertex at the eye "

[Up to this point the treatise has been literally translated because of the historical interest attaching to the *ipsisima verba* of Archimedes on such a subject. The rest of the work can now be more freely reproduced, and, before proceeding to the mathematical contents of it, it is only necessary to remark that Archimedes next describes how he arrived at a higher and a lower limit for the angle

fastening  
tion of the

it), then putting the cylinder at such a distance that it just concealed, and just failed to conceal, the sun, and lastly measuring the angles subtended by the cylinder. He explains also the correction which he thought it necessary to make because "the eye does not see from one point but from a certain area"]

The result of the experiment was to show that the angle subtended by the diameter of the sun was less than  $\frac{1}{180}$ th part, and greater than  $\frac{1}{180}$ th part, of a right angle

To prove that (on this assumption) the diameter of the sun is greater than the side of a chiliagon, or figure with 1000 equal sides, inscribed in a great circle of the "universe "

Suppose the plane of the paper to be the plane passing through the centre of the sun, the centre of the earth and the eye, at the time when the sun has

<sup>1</sup>A lost work of Archimedes.

just risen above the horizon. Let the plane cut the earth in the circle  $EHL$  and the sun in the circle  $FKG$ , the centres of the earth and sun being  $C, O$  respectively, and  $E$  being the position of the eye.

Further, let the plane cut the sphere of the 'universe' (i.e. the sphere whose centre is  $C$  and radius  $CO$ ) in the great circle  $AOB$ .

Draw from  $E$  two tangents to the circle  $FKG$  touching it at  $P, Q$  and from  $C$  draw two other tangents to the same circle touching it in  $F, G$  respectively.

Let  $CO$  meet the sections of the earth and sun in  $H, K$  respectively, and let  $CF, CG$  produced meet the great circle  $AOB$  in  $A, B$ .

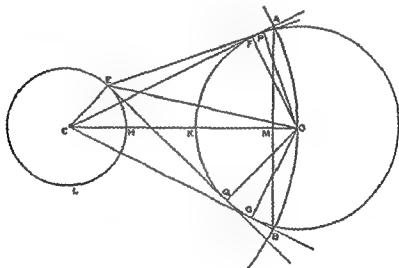
Join  $EO, OF, OG, OP, OQ, AB$  and let  $AB$  meet  $CO$  in  $M$ .

Now  $CO > EO$ , since the sun is just above the horizon.

Therefore

$$\angle PEQ > \angle FCG$$

And  $\angle PEQ > \frac{1}{128}R$   
but  $< \frac{1}{128}R$  } where  $R$  represents a right angle



Thus

$$\angle FCG < \frac{1}{128}R \text{ a fortiori}$$

and the chord  $AB$  subtends an arc of the great circle which is less than  $\frac{1}{128}$ th of the circumference of that circle i.e.

$$AB < (\text{side of 656-sided polygon inscribed in the circle})$$

Now the perimeter of any polygon inscribed in the great circle is less than  $\frac{1}{2}CO$  [ Cf Measurement of a circle Prop 3 ]

Therefore

$$AB \cdot CO < 11 \cdot 148$$

and a fortiori

$$AB < \frac{1}{128}CO$$

Again since  $C \neq CO$  and  $AM$  is perpendicular to  $CO$  while  $OF$  is perpendicular to  $CA$

$$AM = OF$$

Therefore

$$AB = 2 \cdot AM = (\text{diameter of sun})$$

Thus

$$(\text{diameter of sun}) < \frac{1}{128}CO \text{ by } (\alpha),$$

and a fortiori

$$(\text{diameter of earth}) < \frac{1}{128}CO$$

Hence

$$CH + OH < \frac{1}{128}CO$$

so that

$$HH > \frac{1}{128}CO$$

[Assumption 2]

or  $CO \quad HK < 100 \quad 99$   
 And  $CO > CF$ ,  
 while  $HK < EQ$   
 Therefore  $CF \quad EQ < 100 \quad 99$  ( $\beta$ )  
 Now in the right-angled triangles  $CFO$ ,  $EQO$ , of the sides about the right angles,  
 $OF = OQ$ , but  $EQ < CF$  (since  $EO < CO$ )  
 Therefore  $\angle OEQ \quad \angle OCF > CO \quad EO$ ,  
 but  $< CF \quad EQ$ <sup>1</sup>

Doubling the angles,  
 $\angle PEQ \quad \angle ACB < CF \quad EQ$   
 $< 100 \quad 99$ , by ( $\beta$ ) above  
 But  $\angle PEQ > \frac{1}{100} R$ , by hypothesis  
 Therefore  $\angle ACB > \frac{1}{10000} R$   
 $> \frac{1}{100} R$

It follows that the arc  $AB$  is greater than  $\frac{1}{100}$ th of the circumference of the great circle  $AOB$

Hence, *a fortiori*,

$AB >$  (side of chiliagon inscribed in great circle),

and  $AB$  is equal to the diameter of the sun, as proved above

The following results can now be proved

(diameter of "universe")  $< 10,000$  (diameter of earth),

and (diameter of "universe")  $< 10,000,000,000$  stadia

(1) Suppose, for brevity, that  $d_u$  represents the diameter of the "universe,"  $d_s$  that of the sun,  $d_e$  that of the earth and  $d_m$  that of the moon

By hypothesis,  $d_s > 30d_m$ , [Assumption 3]

and  $d_s > d_m$ , [Assumption 2]

therefore  $d_s < 30d_s$

Now, by the last proposition,

$d_s >$  (side of chiliagon inscribed in great circle),

so that (perimeter of chiliagon)  $< 1000d_s$ ,

$< 30\,000d_s$ .

But the perimeter of any regular polygon with more sides than 6 inscribed in a circle is greater than that of the inscribed regular hexagon, and therefore greater than three times the diameter Hence

(perimeter of chiliagon)  $> 3d_s$

It follows that  $d_u < 10,000d_s$

(2) (Perimeter of earth)  $> 3,000,000$  stadia [Assumption 1]

and (perimeter of earth)  $> 3d_s$

Therefore  $d_s < 1,000,000$  stadia,

whence  $d_u < 10,000,000\,000$  stadia

Assumption 5

Suppose a quantity of sand taken not greater than a poppy-seed, and suppose that it contains not more than 10 000 grains

<sup>1</sup>The proposition here assumed is of course equivalent to the trigonometrical formula which states that, if  $\alpha \quad \beta$  are the circular measures of two angles, each less than a right angle, of which  $\alpha$  is the greater then

$$\frac{\tan \alpha}{\tan \beta} > \frac{\alpha}{\beta} > \frac{\sin \alpha}{\sin \beta}$$

Next suppose the diameter of the poppy-seed to be not less than  $\frac{1}{16}$ th of a finger-breadth

### ORDERS AND PERIODS OF NUMBERS

I. We have traditional names for numbers up to a myriad (10,000) and therefore express numbers up to a myriad in terms of the unit. Numbers greater than a myriad be called numbers of the first order.

Suppose the unit of the first order be the myriad. The order of numbers ending with (100,000,000) is the second order.

Let the unit of the second order be the myriad of the first order. The order of numbers ending with (100,000,000) is the second order. We will call  $P$  the unit of the second order.

II. Suppose the numbers from 1 to  $P$  just described to form the first period. Let  $P$  be the unit of the first order of the second period, and let this consist of the numbers from  $P$  up to  $100,000,000P$ .

Let the last number be the unit of the second order of the second period, and let this end with  $(100,000,000)^2P$ .

We can go on in this way till we reach the  $100,000,000$ th order of the second period ending with  $(100,000,000)^{100,000,000}P$ , or  $P^2$ .

III. Taking  $P^2$  as the unit of the first order of the third period, we proceed in the same way till we reach the  $100,000,000$ th order of the third period ending with  $P^3$ .

IV. Taking  $P^3$  as the unit of the first order of the fourth period, we continue the same process until we arrive at the  $100,000,000$ th order of the fourth period ending with  $P^{100,000,000}$ . This last number is expressed by Archimedes as "a myriad myriad units of the myriad-myriad-th order of the myriad-myriad-th period (αἱ μυριάσις μυριάστων περιέχον μυριάσις μυριάστων ἀριθμῶν μυριάσις μυριάδες)," which is easily seen to be  $100,000,000$  times the product of  $(100,000,000)^{100,000,000}$  and  $P^{100,000,000}$ , i.e.  $P^{100,000,000}$ .

### OCTADS

Consider the series of terms in continued proportion of which the first is 1 and the second 10 [i.e. the geometrical progression 1,  $10^1$ ,  $10^2$ ,  $10^3$ , ...]. The first octad of these terms [i.e. 1,  $10^1$ ,  $10^2$ , ...,  $10^7$ ] fall accordingly under the first order of the first period above described, the second octad [i.e.  $10^8$ ,  $10^9$ , ...,  $10^{15}$ ] under the second order of the first period, the first term of the octad being the unit of the corresponding order in each case. Similarly for the third octad, and so on. We can, in the same way, place any number of octads.

### THEOREM

If there be any number of terms of a series in continued proportion, say  $A_1, A_2, A_3, \dots, A_n, A_{n+1}, A_{n+2}, \dots, A_{m+n-1}$ , of which  $A_1=1, A_2=10$  [so that the series forms the geometrical progression 1,  $10^1$ ,  $10^2$ , ...,  $10^{m+n-1}$ ], and if any two terms as  $A_m, A_n$  be taken and multiplied, the product  $A_m \cdot A_n$  will be a term in the same series and will be as many terms distant from  $A_1$  as  $A_m$  is distant from  $A_1$ , also it will be distant from  $A_1$  by a number of terms less by one than the sum of the numbers of terms by which  $A_m$  and  $A_n$  respectively are distant from  $A_1$ .

Take the term which is distant from  $A_1$  by the same number of terms as  $A_n$

is distant from  $A_1$ . This number of terms is  $m$  (the first and last being both counted). Thus the term to be taken is  $m$  terms distant from  $A_n$ , and is therefore the term  $A_{m+n-1}$ .

We have therefore to prove that

$$A_m \cdot A_n = A_{m+n-1}$$

Now terms equally distant from other terms in the continued proportion are proportional

$$\text{Thus} \quad \frac{A_m}{A_1} = \frac{A_{m+n-1}}{A_n}$$

$$\text{But} \quad A_m = A_m \cdot A_1, \text{ since } A_1 = 1$$

$$\text{Therefore} \quad A_{m+n-1} = A_m \cdot A_n \quad (1)$$

The second result is now obvious, since  $A_m$  is  $m$  terms distant from  $A_1$ ,  $A_n$  is  $n$  terms distant from  $A_1$ , and  $A_{m+n-1}$  is  $(m+n-1)$  terms distant from  $A_1$ .

#### APPLICATION TO THE NUMBER OF THE SAND

By Assumption 5 [p. 523],

(diam. of poppy seed)  $< \frac{1}{10}$  (finger breadth),

and, since spheres are to one another in the triplicate ratio of their diameters, it follows that

$$\begin{aligned} (\text{sphere of diam. 1 finger breadth}) &> 64,000 \text{ poppy-seeds} \\ &> 64,000 \times 10,000 \\ &> 640,000,000 \\ &> 6 \text{ units of second order} \left. \begin{array}{l} \text{order} + 40,000,000 \\ \text{units of first order} \end{array} \right\} \begin{array}{l} \text{grains} \\ \text{of} \\ \text{sand} \end{array} \\ (\text{a fortiori}) &< 10 \text{ units of second} \\ &\quad \text{order of numbers} \end{aligned}$$

We now gradually increase the diameter of the supposed sphere, multiplying it by 100 each time. Thus, remembering that the sphere is thereby multiplied by  $100^3$  or 1,000,000, the number of grains of sand which would be contained in a sphere with each successive diameter may be arrived at as follows

Diameter of sphere	Corresponding number of grains of sand
(1) 100 finger breadths	$< 1,000,000 \times 10$ units of second order $< (7\text{th term of series}) \times (10\text{th term of series})$ $< 16\text{th term of series}$ [i.e. $10^{16}$ ] $< [10^8 \text{ or } 10,000,000]$ units of the second order
(2) 10,000 finger breadths	$< 1,000,000 \times (\text{last number})$ $< (7\text{th term of series}) \times (16\text{th term})$ $< 22\text{nd term of series}$ [i.e. $10^{22}$ ] $< [10^8 \text{ or } 100,000]$ units of third order $< 100,000$ units of third order
(3) 1 stadium ( $< 10,000$ finger breadths)	$< 1,000,000 \times (\text{last number})$ $< (7\text{th term of series}) \times (22\text{nd term})$ $< 28\text{th term of series}$ [i.e. $10^{28}$ ] $< [10^8 \text{ or } 1,000]$ units of fourth order
(4) 100 stadia	$< 1,000,000 \times (\text{last number})$ $< (7\text{th term of series}) \times (28\text{th term})$ $< 34\text{th term of series}$ [i.e. $10^{34}$ ] $< 1,000,000 \times (\text{last number})$ $< (7\text{th term of series}) \times (34\text{th term})$ $< 40\text{th term of series}$ [i.e. $10^{40}$ ] $< 10$ units of fifth order
(5) 10,000 stadia	$< 1,000,000 \times (\text{last number})$ $< (7\text{th term of series}) \times (40\text{th term})$ $< 46\text{th term of series}$ [i.e. $10^{46}$ ] $< [10^8 \text{ or } 10,000]$ units of sixth order



(6) 1,000,000 stadia	<(7th term of series)×(34th term) <40th term <[10 <sup>7</sup> or] 10,000,000 units of <i>fifth order</i>	[10 <sup>39</sup> ]
(7) 100,000,000 stadia	<(7th term of series)×(40th term) <46th term <[10 <sup>8</sup> or] 100,000 units of <i>sixth order</i>	[10 <sup>41</sup> ]
(8) 10,000,000,000 stadia	<(7th term of series)×(46th term) <52nd term of series <[10 <sup>9</sup> or] 1,000 units of <i>seventh order</i>	[10 <sup>43</sup> ]

But, by the proposition above [p 523],

(diameter of "universe") < 10,000,000,000 stadia

Hence *the number of grains of sand which could be contained in a sphere of the size of our "universe" is less than 1,000 units of the seventh order of numbers [or 10<sup>61</sup>]*

From this we can prove further that *a sphere of the size attributed by Aristarchus to the sphere of the fixed stars would contain a number of grains of sand less than 10,000,000 units of the eighth order of numbers [or 10<sup>58+7</sup> = 10<sup>65</sup>]*

For, by hypothesis,

(earth) ("universe") = ("universe") (sphere of fixed stars)

And [p 523]

(diameter of "universe") < 10,000 (diam of earth),

whence

(diam of sphere of fixed stars) < 10,000 (diam of "universe")

Therefore

(sphere of fixed stars) < (10,000)<sup>3</sup> ("universe")

It follows that the number of grains of sand which would be contained in a sphere equal to the sphere of the fixed stars

< (10,000)<sup>3</sup> × 1,000 units of *seventh order*

< (13th term of series) × (52nd term of series)

< 64th term of series

< [10<sup>7</sup> or] 10,000,000 units of *eighth order* of numbers

[i.e. 10<sup>65</sup>]

## CONCLUSION

"I conceive that these things, King Gelon, will appear incredible to the great majority of people who have not studied mathematics, but that to those who are conversant therewith and have given thought to the question of the distances and sizes of the earth, the sun and moon and the whole universe, the proof will carry conviction. And it was for this reason that I thought the subject would be not inappropriate for your consideration."

## QUADRATURE OF THE PARABOLA

*"ARCHIMEDES to DOSITHEUS greeting*

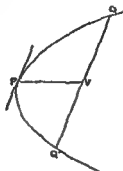
"When I heard that Conon, who was my friend in his lifetime was dead, but that you were acquainted with Conon and withal versed in geometry, while I grieved for the loss not only of a friend but of an admirable mathematician, I set myself the task of communicating to you, as I had intended to send to Conon, a certain geometrical theorem which had not been investigated before but has now been investigated by me, and which I first discovered by means of mechanics and then exhibited by means of geometry. Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle, and after that they endeavoured to square the area bounded by the section of the whole cone and a straight line, assuming lemmas not easily conceded, so that it was recognised by most people that the problem was not solved. But I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a right-angled cone [a parabola], of which problem I have now discovered the solution. For it is here shown that every segment bounded by a straight line and a section of a right-angled cone [a parabola] is four thirds of the triangle which has the same base and equal height with the segment, and for the demonstration of this property the following lemma is assumed: that the excess by which the greater of (two) unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area. The earlier geometers have also used this lemma, for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height, also that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid. And, in the result, each of the aforesaid theorems has been accepted no less than those proved without the lemma. As therefore my work now published has satisfied the same test as the propositions referred to, I have written out the proof and send it to you, first as investigated by means of mechanics, and afterwards too as demonstrated by geometry. Prefixed are, also, the elementary propositions in conics which are of service in the proof. Farewell."

## PROPOSITION 1

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as  $PV$ , and if  $QQ'$  be a chord parallel to the tangent to the parabola at  $P$  and meeting  $PV$  in  $V$ , then

$$QV = VQ'$$

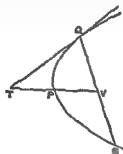
Conversely, if  $QV = VQ'$ , the chord  $QQ'$  will be parallel to the tangent at  $P$



## PROPOSITION 2

If in a parabola  $QQ'$  be a chord parallel to the tangent at  $P$ , and if a straight line be drawn through  $P$  which is either itself the axis or parallel to the axis, and which meets  $QQ'$  in  $V$  and the tangent at  $Q$  to the parabola in  $T$ , then

$$PV = PT$$



## PROPOSITION 3

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as  $PV$ , and if from two other points  $Q, Q'$  on the parabola straight lines be drawn parallel to the tangent at  $P$  and meeting  $PV$  in  $V, V'$  respectively, then

$$PV \cdot PV' = QV^2 \cdot Q'V'^2$$

"And these propositions are proved in the elements of conics 1"

## PROPOSITION 4

If  $Qq$  be the base of any segment of a parabola, and  $P$  the vertex of the segment, and if the diameter through any other point  $R$  meet  $Qq$  in  $O$  and  $QP$  (produced if necessary) in  $F$ , then

$$QV \cdot VO = OF \cdot FR$$

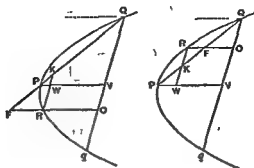
Draw the ordinate  $RW$  to  $PV$ , meeting  $QP$  in  $K$

Then  $PV \cdot PW = QV^2 \cdot RW^2$ ,

where, by parallels  $PQ \cdot PK = PQ^2 \cdot PF^2$

<sup>1</sup> i.e. in the treatises on conics by Euclid and Anstaeus.

In other words,  $PQ, PF, PK$  are in continued proportion; therefore



$$\begin{aligned} PQ \cdot PF &= PF : PK \\ &= PQ \pm PF : PF \pm PK \\ &= QF \cdot KF. \end{aligned}$$

Hence, by parallels,  $QV : VO = OF : FR$ .

PROPOSITION 5

If  $Qq$  be the base of any segment of a parabola,  $P$  the vertex of the segment, and  $PV$  its diameter, and if the diameter of the parabola through any other point  $R$  meet  $Qq$  in  $O$  and the tangent at  $Q$  in  $E$ , then

$$QO \cdot Oq = ER \cdot RO.$$

Let the diameter through  $R$  meet  $QP$  in  $F$

Then, by Prop 4,

$$QV \cdot VO = OF \cdot FR$$

Since  $QV = Vq$ , it follows that

$$QV \cdot qO = OF \cdot OR \quad (1)$$

Also, if  $VP$  meet the tangent in  $T$ ,  $PT = PV$ , and therefore  $EF = OF$ .

Accordingly, doubling the antecedents in (1), we have

$$Qq \cdot qO = OE \cdot OR,$$

whence  $QO \cdot Oq = ER \cdot RO$

PROPOSITIONS 6, 7<sup>1</sup>

Suppose a lever  $AOB$  placed horizontally and supported at its middle point  $O$ . Let a triangle  $BCD$  in which the angle  $C$  is right or obtuse be suspended from  $B$  and  $O$ , so that  $C$  is attached to  $O$  and  $CD$  is in the same vertical line with  $O$ . Then, if  $P$  be such an area as, when suspended from  $A$ , will keep the system in equilibrium,

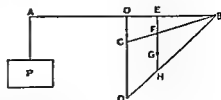
$$P = \frac{1}{2} \Delta BCD$$

<sup>1</sup>In Prop. 6 Archimedes takes the separate case in which the angle  $BCD$  of the triangle is

Take a point  $E$  on  $OB$  such that  $BE = 2OE$ , and draw  $EFH$  parallel to  $CD$  meeting  $BC$ ,  $BD$  in  $F$ ,  $H$  respectively. Let  $G$  be the middle point of  $FH$ .

Then  $G$  is the centre of gravity of the triangle  $BCD$ .

Hence, if the angular points  $B$ ,  $C$  be set free and the triangle be suspended by attaching  $F$  to  $E$ , the triangle will hang in the same position as before, because  $EFG$  is a vertical straight line. "For this is proved"<sup>1</sup>



Therefore, as before, there will be equilibrium. Thus  $P \triangle BCD = OE \cdot AO$

$$= 1 \cdot 3,$$

$$\text{or } P = \frac{1}{3} \triangle BCD$$

### PROPOSITIONS 8, 9

Suppose a lever  $AOB$  placed horizontally and supported at its middle point  $O$ . Let a triangle  $BCD$ , right-angled or obtuse-angled at  $C$ , be suspended from the points  $B$ ,  $E$  on  $OB$ , the angular point  $C$  being so attached to  $E$  that the side  $CD$  is in the same vertical line with  $E$ . Let  $Q$  be an area such that

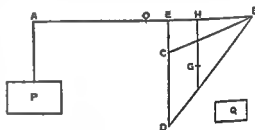
$$AO \cdot OE = \triangle BCD \cdot Q$$

Then, if an area  $P$  suspended from  $A$  keep the system in equilibrium,

$$P < \triangle BCD \text{ but } > Q$$

Take  $G$  the centre of gravity of the triangle  $BCD$ , and draw  $GH$  parallel to  $DC$ , i.e. vertically, meeting  $BO$  in  $H$ .

We may now suppose the triangle  $BCD$  suspended from  $H$ , and, since there is equilibrium



$$\triangle BCD \cdot P = AO \cdot OH,$$

$$P < \triangle BCD$$

$$\triangle BCD \cdot Q = AO \cdot OE$$

$$\triangle BCD \cdot Q > \triangle BCD \cdot P,$$

$$P > Q$$

(1)

whence

Also

Therefore, by (1),

and

### PROPOSITIONS 10 11

Suppose a lever  $AOB$  placed horizontally and supported at  $O$ , its middle point. Let  $CDEF$  be a trapezium which can be so placed that its parallel sides  $CD$ ,  $FE$  are vertical, while  $C$  is vertically below  $O$ , and the other sides  $CF$ ,  $DE$  meet in  $B$ . Let  $EF$  meet  $BO$  in  $H$ , and let the trapezium be suspended by attaching  $F$  to  $H$  and  $C$  to  $O$ . Further, suppose  $Q$  to be an area such that

$$AO \cdot OH = (\text{trapezium } CDEF) \cdot Q$$

Then, if  $P$  be the area which, when suspended from  $A$ , keeps the system in equilibrium,

$$P < Q$$

<sup>1</sup>Doubtless in the lost book *rept. 1000*



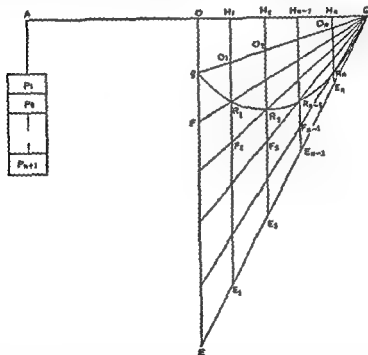
through  $q, O_1, O_2, \dots, O_n$  diameters of the parabola meeting the tangent at  $Q$  in  $E, E_1, E_2, \dots, E_n$  and the parabola itself in  $q, R_1, R_2, \dots, R_n$ . Join  $QR_1, QR_2, \dots, QR_n$  meeting  $qE, O_1E_1, O_2E_2, \dots, O_{n-1}E_{n-1}$  in  $F, F_1, F_2, \dots, F_{n-1}$ .

Let the diameters  $Eg, E_1O_1, E_2O_2, \dots, E_nO_n$  meet a straight line  $QOA$  drawn through  $Q$  perpendicular to the diameters in the points  $O, H_1, H_2, \dots, H_n$  respectively (In the particular case where  $Qq$  is itself perpendicular to the diameters  $q$  will coincide with  $O, O_1$  with  $H_1$ , and so on.)

It is required to prove that

(1)  $\triangle Eqq < 3(\text{sum of trapezia } FO_1, F_1O_2, \dots, F_{n-1}O_n \text{ and } \triangle E_nO_nQ),$

(2)  $\triangle Eqq > 3(\text{sum of trapezia } R_1O_2, R_2O_3, \dots, R_{n-1}O_n \text{ and } \triangle R_nO_nQ)$



Suppose  $AO$  made equal to  $OQ$ , and conceive  $QOA$  as a lever placed horizontally and supported at  $O$ . Suppose the triangle  $Eqq$  suspended from  $OQ$  in the position drawn, and suppose that the trapezium  $EO_1$  in the position drawn is balanced by an area  $P_1$  suspended from  $A$ , the trapezium  $E_1O_2$  in the position drawn is balanced by the area  $P_2$  suspended from  $A$ , and so on, the triangle  $E_nO_nQ$  being in like manner balanced by  $P_{n+1}$ .

Then  $P_1 + P_2 + \dots + P_{n+1}$  will balance the whole triangle  $Eqq$  as drawn, and therefore  $P_1 + P_2 + \dots + P_{n+1} = \frac{1}{3} \triangle Eqq$  [Props 6, 7]

Again  $AO \cdot OH_1 = QO \cdot OH_1$

$$= Qq \cdot qO_1$$

$$= E_1O_1 \cdot O_1R_1 \text{ [by means of Prop 5]}$$

$$= (\text{trapezium } EO_1) (\text{trapezium } FO_1),$$

whence [Props 10, 11]

$$(FO_1) > P_1$$

Next

$$AO \quad OH_1 = E_1O_1 \quad O_1R_1 \\ = (E_1O_2) \quad (R_1O_2), \quad (\alpha)$$

while

$$AO \quad OH_2 = E_2O_2 \quad O_2R_2 \\ = (E_1O_2) \quad (F_1O_2), \quad (\beta)$$

and, since  $(\alpha)$  and  $(\beta)$  are simultaneously true, we have, by Props 12, 13,

$$(F_1O_2) > P_2 > (R_1O_2)$$

Similarly it may be proved that

$$(F_2O_2) > P_3 > (R_2O_2),$$

and so on

$$\text{Lastly [Props 8, 9]} \quad \Delta E_n O_n Q > P_{n+1} > \Delta R_n O_n Q$$

By addition, we obtain

$$(1) \quad (FO_1) + (F_1O_2) + \dots + (F_{n-1}O_n) + \Delta E_n O_n Q > P_1 + P_2 + \dots + P_{n+1} \\ > \frac{1}{3} \Delta E q Q,$$

$$\text{or} \quad \Delta E q Q < 3(FO_1 + F_1O_2 + \dots + F_{n-1}O_n + \Delta E_n O_n Q)$$

$$(2) \quad (R_1O_2) + (R_2O_3) + \dots + (R_{n-1}O_n) + \Delta R_n O_n Q < P_1 + P_2 + \dots + P_{n+1} \\ < P_1 + P_2 + \dots + P_{n+1}, \text{ a fortiori,} \\ < \frac{1}{3} \Delta E q Q,$$

$$\text{or} \quad \Delta E q Q > 3(R_1O_2 + R_2O_3 + \dots + R_{n-1}O_n + \Delta R_n O_n Q)$$

### PROPOSITION 16

Suppose  $Qq$  to be the base of a parabolic segment,  $q$  being not more distant than  $Q$  from the vertex of the parabola. Draw through  $q$  the straight line  $qE$  parallel to the axis of the parabola to meet the tangent at  $Q$  in  $E$ . It is required to prove that

$$(\text{area of segment}) = \frac{1}{3} \Delta E q Q$$

For, if not, the area of the segment must be either greater or less than  $\frac{1}{3} \Delta E q Q$

I Suppose the area of the segment greater than  $\frac{1}{3} \Delta E q Q$ . Then the excess can, if continually added to itself, be made to exceed  $\Delta E q Q$ . And it is possible to find a submultiple of the triangle  $E q Q$  less than the said excess of the segment over  $\frac{1}{3} \Delta E q Q$ .

Let the triangle  $F q Q$  be such a submultiple of the triangle  $E q Q$ . Divide  $E q$  into equal parts each equal to  $qF$ , and let all the points of division including  $F$  be joined to  $Q$  meeting the parabola in  $R_1, R_2, \dots, R_n$  respectively. Through  $R_1, R_2, \dots, R_n$  draw diameters of the parabola meeting  $qQ$  in  $O_1, O_2, \dots, O_n$  respectively.

Let  $O_1R_1$  meet  $QR_1$  in  $F_1$ .

Let  $O_2R_2$  meet  $QR_1$  in  $D_1$  and  $QR_2$  in  $F_2$ .

Let  $O_3R_3$  meet  $QR_2$  in  $D_2$  and  $QR_3$  in  $F_3$ ,

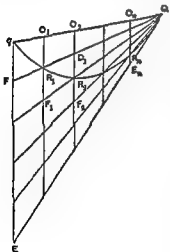
and so on

We have, by hypothesis,

$$\Delta F q Q < (\text{area of segment}) - \frac{1}{3} \Delta E q Q,$$

$$\text{or} \quad (\text{area of segment}) - \Delta F q Q > \frac{1}{3} \Delta E q Q \quad (\alpha)$$

Now, since all the parts of  $qE$ , as  $qF$  and the rest, are equal,  $O_1R_1 = R_1F_1$ ,  $O_2D_1 = D_1R_2 = R_2F_2$ , and so on, therefore





$$\begin{aligned}\Delta FqQ &= (FO_1 + R_1O_2 + D_1O_3 + \dots) \\ &= (FO_1 + F_1D_1 + F_2D_2 + \dots + F_{n-1}D_{n-1} + \Delta E_nR_nQ) \quad (\beta)\end{aligned}$$

But (area of segment)  $< (FO_1 + F_1O_2 + \dots + F_{n-1}O_n + \Delta E_nO_nQ)$

Subtracting, we have

$$(\text{area of segment}) - \Delta FqQ < (R_1O_2 + R_2O_3 + \dots + R_{n-1}O_n + \Delta R_nO_nQ),$$

whence, *a fortiori*, by ( $\alpha$ ),

$$\frac{1}{2}\Delta EqQ < (R_1O_2 + R_2O_3 + \dots + R_{n-1}O_n + \Delta R_nO_nQ)$$

But this is impossible, since [Props 14, 15]

$$\frac{1}{2}\Delta EqQ > (R_1O_2 + R_2O_3 + \dots + R_{n-1}O_n + \Delta R_nO_nQ)$$

Therefore

$$(\text{area of segment}) > \frac{1}{2}\Delta EqQ$$

II If possible, suppose the area of the segment less than  $\frac{1}{2}\Delta EqQ$

Take a submultiple of the triangle  $EqQ$ , as the triangle  $FqQ$ , less than the excess of  $\frac{1}{2}\Delta EqQ$  over the area of the segment, and make the same construction as before

Since  $\Delta FqQ < \frac{1}{2}\Delta EqQ - (\text{area of segment})$ ,  
it follows that

$$\begin{aligned}\Delta FqQ + (\text{area of segment}) &< \frac{1}{2}\Delta EqQ \\ &< (FO_1 + F_1O_2 + \dots + F_{n-1}O_n + \Delta E_nO_nQ) \quad [\text{Props 14, 15}]\end{aligned}$$

Subtracting from each side the area of the segment, we have

$$\begin{aligned}\Delta FqQ &< (\text{sum of spaces } qFR_1, R_1\Gamma_1R_2, \dots, E_nR_nQ) \\ &< (FO_1 + F_1D_1 + \dots + F_{n-1}D_{n-1} + \Delta E_nR_nQ), \text{ a fortiori,}\end{aligned}$$

which is impossible, because by ( $\beta$ ) above,

$$\Delta FqQ = FO_1 + F_1D_1 + \dots + F_{n-1}D_{n-1} + \Delta E_nR_nQ$$

Hence (area of segment)  $< \frac{1}{2}\Delta EqQ$

Since then the area of the segment is neither less nor greater than  $\frac{1}{2}\Delta EqQ$ , it is equal to it

#### PROPOSITION 17

It is now manifest that the area of any segment of a parabola is four-thirds of the triangle which has the same base as the segment and equal height.

Let  $Qq$  be the base of the segment,  $P$  its vertex. Then  $PQq$  is the inscribed triangle with the same base as the segment and equal height.

Since  $P$  is the vertex of the segment, the diameter through  $P$  bisects  $Qq$ . Let  $V$  be the point of bisection.

Let  $VP$ , and  $qE$  drawn parallel to it, meet the tangent at  $Q$  in  $T$ ,  $E$  respectively.

Then, by parallels,

$$qE = 2VT,$$

$$\text{and } PV = PT, \quad [\text{Prop 2}]$$

so that

$$VT = 2PV$$

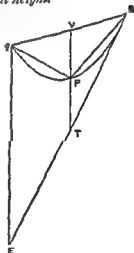
Hence

$$\Delta EqQ = 4\Delta PQq$$

But, by Prop 16 the area of the segment is equal to

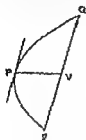
$$\frac{1}{2}\Delta EqQ$$

$$\text{Therefore } (\text{area of segment}) = \frac{1}{2}\Delta PQq$$



DEF "In segments bounded by a straight line and any curve I call the

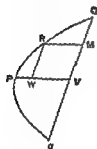
straight line the *base*, and the *height* the greatest perpendicular drawn from the curve to the base of the segment, and the *vertex* the point from which the greatest perpendicular is drawn "



PROPOSITION 18

If  $Qq$  be the base of a segment of a parabola, and  $V$  the middle point of  $Qq$ , and if the diameter through  $V$  meet the curve in  $P$ , then  $P$  is the vertex of the segment

For  $Qq$  is parallel to the tangent at  $P$  [Prop 1] Therefore, of all the perpendiculars which can be drawn from points on the segment to the base  $Qq$ , that from  $P$  is the greatest. Hence, by the definition  $P$  is the vertex of the segment



PROPOSITION 19

If  $Qq$  be a chord of a parabola bisected in  $V$  by the diameter  $PV$ , and if  $Rm$  be a diameter bisecting  $QV$  in  $M$ , and  $Rv$  be the ordinate from  $R$  to  $PV$ , then

$$PV = \frac{1}{4} RM^2$$

For, by the property of the parabola,

$$\begin{aligned} PV \cdot PV &= QV^2 \cdot RV^2 \\ &= 4RV^2 \cdot RV^2, \end{aligned}$$

so that  
whence

$$\begin{aligned} PV &= 4RV^2, \\ PV &= \frac{1}{4} RM^2 \end{aligned}$$

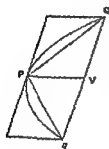
PROPOSITION 20

If  $Qq$  be the base, and  $P$  the vertex, of a parabolic segment, then the triangle  $PQq$  is greater than half the segment  $PQq$

For the chord  $Qq$  is parallel to the tangent at  $P$ , and the triangle  $PQq$  is half the parallelogram formed by  $Qq$ , the tangent at  $P$ , and the diameters through  $Q$ ,  $q$

Therefore the triangle  $PQq$  is greater than half the segment

Con. It follows that it is possible to inscribe in the segment a polygon such that the segments left over are together less than any assigned area



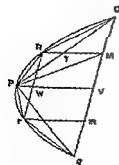
PROPOSITION 21

If  $Qq$  be the base and  $P$  the vertex of any parabolic segment and if  $R$  be the vertex of the segment cut off by  $PQ$ , then

$$\Delta PQq = 8 \Delta PRQ$$

The diameter through  $R$  will bisect the chord  $PQ$ , and therefore also  $QV$ , where  $PV$  is the diameter bisecting  $Qq$ . Let the diameter through  $R$  bisect  $PQ$  in  $Y$  and  $QV$  in  $M$ . Join  $PM$

By Prop 19,  $PV = \frac{1}{4} RM^2$ .  
Also  $PV = 2YM$   
Therefore  $YM = 2RY$ ,



and

$$\Delta PQM = 2\Delta PRQ$$

Hence

$$\Delta PQV = 4\Delta PRQ,$$

and

$$\Delta PQq = 8\Delta PRQ$$

Also, if  $RW$ , the ordinate from  $R$  to  $PV$ , be produced to meet the curve again in  $r$ ,

$$RW = rW,$$

and the same proof shows that

$$\Delta PQq = 8\Delta Prq$$

## PROPOSITION 22

If there be a series of areas  $A, B, C, D$ , each of which is four times the next in order, and if the largest,  $A$ , be equal to the triangle  $PQq$  inscribed in a parabolic segment  $PQq$  and having the same base with it and equal height, then

$$(A+B+C+D+\dots) < (\text{area of segment } PQq)$$

For, since  $\Delta PQq = 8\Delta PRQ = 8\Delta Prq$ , where  $R, r$  are the vertices of the segments cut off by  $PQ, Pq$ , as in the last proposition,

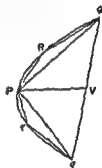
$$\Delta PQq = 4(\Delta PQR + \Delta Prq)$$

Therefore, since  $\Delta PQq = A$ ,

$$\Delta PQR + \Delta Prq = B$$

In like manner we prove that the triangles similarly inscribed in the remaining segments are together equal to the area  $C$ , and so on

Therefore  $A+B+C+D+\dots$  is equal to the area of a certain inscribed polygon, and is therefore less than the area of the segment



## PROPOSITION 23

Given a series of areas  $A, B, C, D, \dots, Z$ , of which  $A$  is the greatest, and each is equal to four times the next in order, then

$$A+B+C+\dots+Z+\frac{1}{3}Z = \frac{1}{3}A$$

Take areas  $b, c, d, \dots$  such that

$$b = \frac{1}{3}B,$$

$$c = \frac{1}{3}C,$$

$$d = \frac{1}{3}D, \text{ and so on}$$

Then, since  $b = \frac{1}{3}B$ ,

and

$$B = \frac{1}{3}A,$$

$$B+b = \frac{1}{3}A$$

Similarly  $C+c = \frac{1}{3}B$

Therefore

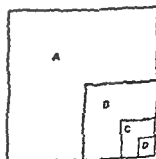
$$B+C+D+\dots+Z+b+c+d+\dots+z = \frac{1}{3}(A+B+C+\dots+Y)$$

But

$$b+c+d+\dots+y = \frac{1}{3}(B+C+D+\dots+Y)$$

Therefore, by subtraction,

$$\begin{aligned} B+C+D+\dots+Z+z &= \frac{1}{3}A \\ A+B+C+\dots+Z+\frac{1}{3}Z &= \frac{1}{3}A \end{aligned}$$



or

PROB 21

thirds of the

Suppose  
where  $P$  is the vertex of the segment,  $K$  the area  
of the segment  $\equiv$  equal to  $K$

For, if the segment be not equal to  $K$ , it must either be greater or less.

I Suppose the area of the segment greater than  $K$   
If then we inscribe in the segments cut off by  $PQ$ ,  $Pq$

angles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment  $PQq$  exceeds  $K$

Therefore the polygon so formed must be greater than the area  $K$ , which is impossible, since [Prop 23]

$$A+B+C+\dots+Z < \frac{1}{2}A,$$

$$A = \Delta PQq$$

where

Thus the area of the segment cannot be greater than  $K$

It is not possible for the area of the segment to be greater than  $K$

But

$$= K$$

Now, since  $K$  exceeds  $A+B+C+\dots+X$  by an area less than  $X$ , and the area of the segment by an area greater than  $X$  it follows that

$$A+B+C+\dots+X > (\text{the segment}),$$

which is impossible, by Prop 22 above



Therefore the parts along  $PQ, QR$  will be unequally compressed and the part which is compressed the less will be set in motion by that which is compressed the more

Therefore there will not be rest, which is contrary to the hypothesis

Hence the section of the surface will be the circumference of a circle whose centre is  $O$ , and so will all other sections by planes through  $O$

Therefore the surface is that of a sphere with centre  $O$

### PROPOSITION 3

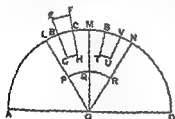
*Of solids those which, size for size, are of equal weight with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface but do not sink lower*

If possible, let a certain solid  $EFHG$  of equal weight, volume for volume, with the fluid remain immersed in it so that part of it,  $EBCF$ , projects above the surface

Draw through  $O$ , the centre of the earth, and through the solid a plane cutting the surface of the fluid in the circle  $ABCD$

Conceive a pyramid with vertex  $O$  and base a parallelogram at the surface of the fluid, such that it includes the immersed portion of the solid. Let this pyramid be cut by the plane of  $ABCD$  in  $QL$ ,  $OM$ . Also let a sphere within the fluid and below  $GH$  be described with centre  $O$ , and let the plane of  $ABCD$  cut this sphere in  $PQR$

Conceive also another pyramid in the fluid with vertex  $O$ , continuous with the former pyramid and equal and similar to it. Let the pyramid so described be cut in  $OM, ON$  by the plane of  $ABCD$



not be at rest, which is contrary to the hypothesis

Therefore the solid will not stand out above the surface

Nor will it sink further, because all the parts of the fluid will be under the same pressure

### PROPOSITION 4

*A solid lighter than a fluid will if immersed in it not be completely submerged, but part of it will project above the surface*

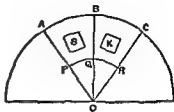
In this case after the manner of the previous proposition we assume the solid if possible to be completely submerged and the fluid to be at rest in that

this latter pyramid equal to the immersed solid in the other pyramid (4) a sphere with centre  $O$  whose surface is below the immersed solid and the part of the fluid in the second pyramid corresponding thereto. We suppose a plane to be drawn through the centre  $O$  cutting the surface of the fluid in the circle

$ABC$ , the solid in  $S$ , the first pyramid in  $OA$ ,  $OB$ , the second pyramid in  $OB$ ,  $OC$ , the portion of the fluid in the second pyramid in  $K$ , and the inner sphere in  $PQR$

Then the pressures on the parts of the fluid at  $PQ$ ,  $QR$  are unequal, since  $S$  is lighter than  $K$ . Hence there will not be rest, which is contrary to the hypothesis.

Therefore the solid  $S$  cannot, in a condition of rest, be completely submerged.



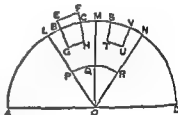
### PROPOSITION 5

*Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.*

For let the solid be  $EGHF$ , and let  $BGHC$  be the portion of it immersed

pose a portion of the fluid  $STUV$  at the base of the second pyramid to be equal and similar to the immersed portion of the solid, and let the construction be the same as in Prop. 3.

Then, since the pressure on the parts of the fluid at  $PQ$ ,  $QR$  must be equal in order that the fluid may be at rest, it follows that the weight of the portion  $STUV$  of the fluid must be equal to the weight of the solid  $EGHF$ . And the former is equal to the weight of the fluid displaced by the immersed portion of the solid  $BGHC$ .



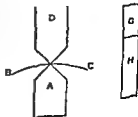
### PROPOSITION 6

*If a solid lighter than a fluid be forcibly immersed in it, the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced.*

For let  $A$  be completely immersed in the fluid, and let  $G$  represent the weight of  $A$ , and  $(G+H)$  the weight of an equal volume of fluid. Then a solid  $D$ , that will

therefore the weight of the fluid displaced is  $(G+H)$  and hence the volume of the fluid displaced is the volume of the solid  $A$ . There will accordingly be rest with  $A$  immersed and  $D$  projecting.

Thus the weight of  $D$  balances the upward force exerted by the fluid on  $A$  and therefore the latter force is equal to  $H$ , which is the difference between the weight of  $A$  and the weight of the fluid which  $A$  displaces.



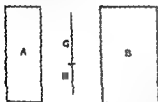
## PROPOSITION 7

A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced

(1) The first part of the proposition is obvious, since the part of the fluid

nd let

$(G+H)$  represent its weight, while  $G$  represents the weight of the same volume of the fluid



Take a solid  $B$  lighter than the same volume of the fluid, and such that the weight of  $B$  is  $G$ , while the weight of the same volume of the fluid is  $(G+H)$

Let  $A$  and  $B$  be now combined into one solid and immersed. Then, since  $(A+B)$  will be of the same weight as the same volume of fluid, both weights being equal to  $(G+H)+G$ , it follows that  $(A+B)$  will remain stationary in the fluid

Therefore the force which causes  $A$  by itself to sink must be equal to the upward force exerted by the fluid on  $B$  by itself. Thus latter  $\equiv$  equal to the difference between  $(G+H)$  and  $G$  [Prop 6]. Hence  $A$  is depressed by a force equal to  $H$ , i.e. its weight in the fluid is  $H$ , or the difference between  $(G+H)$  and  $G$

## POSTULATE 2

"Let it be granted that bodies which are forced upwards in a fluid are forced upwards along the perpendicular [to the surface] which passes through their centre of gravity"

## PROPOSITION 8

If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base does not touch the surface, the solid will rest in such a position that its axis is perpendicular to the surface, and, if the solid be forced into such a position that its base touches the fluid on one side and be then set free, it will not remain in that position but will return to the symmetrical position

## PROPOSITION 9

If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base is completely below the surface, the solid

Suppose, first, that the segment is greater than a hemisphere. Let it be cut by a plane through its axis and the centre of the earth, and, if possible, let it be at rest in the position shown in the figure, where  $AB$  is the intersection of



the plane with the base of the segment,  $DE$  its axis,  $C$  the centre of the sphere of which the segment is a part,  $O$  the centre of the earth

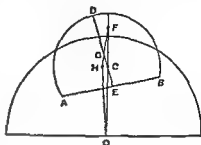
The centre of gravity of the portion of the segment outside the fluid, as  $F$ , lies on  $OC$  produced, its axis passing through  $C$

Let  $G$  be the centre of gravity of the segment. Join  $FG$ , and produce it to  $H$  so that

$FG : GH = (\text{volume of immersed portion}) : (\text{rest of solid})$

Join  $OH$

Then the weight of the portion of the solid outside the fluid acts along  $FO$ , and the pressure of the fluid on the immersed portion acts along  $OH$ , while the weight of the immersed portion acts along  $HO$  and is by hypothesis less than



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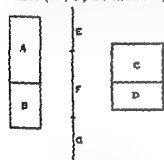
# ON FLOATING BODIES

## BOOK TWO

### PROPOSITION 1

If a solid lighter than a fluid be at rest in it, the weight of the solid will be to that of the same volume of the fluid as the immersed portion of the solid is to the whole

Let  $(A+B)$  be the solid,  $B$  the portion immersed in the fluid



Let  $(C+D)$  be an equal volume of the fluid,  $C$  being equal in volume to  $A$  and  $B$  to  $D$

Further suppose the line  $E$  to represent the weight of the solid  $(A+B)$  ( $F+G$ ) to represent the weight of  $(C+D)$ , and  $G$  that of  $D$

Then

$$\begin{array}{lcl} \text{weight of } (A+B) & \text{weight of } (C+D) = & \\ & E & (F+G) \end{array} \quad (1)$$

And the weight of  $(A+B)$  is equal to the weight of a volume  $B$  of the fluid [I 5], i.e. to the weight of  $D$

That is to say,  $E=G$

Hence, by (1),

$$\begin{array}{lcl} \text{weight of } (A+B) & \text{weight of } (C+D) = & G & F+G \\ & & = D & C+D \\ & & = B & A+B \end{array}$$

### PROPOSITION 2

If a right segment of a paraboloid of revolution whose axis is not greater than  $\frac{2}{3}p$  (where  $p$  is the principal parameter of the generating parabola) and whose specific gravity is less than that of a fluid be placed in the fluid with its axis inclined to the vertical at any angle but so that the base of the segment does not touch the surface of the fluid the segment of the paraboloid will not remain in that position but will return to the position in which its axis is vertical

Let the axis of the segment of the paraboloid be  $AN$  and through  $AN$  draw a plane perpendicular to the surface of the fluid. Let the plane intersect the paraboloid in the parabola  $BAB$  the base of the segment of the paraboloid in  $BB$  and the plane of the surface of the fluid in the chord  $QQ$  of the parabola

Then since the axis  $AN$  is placed in a position not perpendicular to  $QQ$ ,  $BB$  will not be parallel to  $QQ$

Draw the tangent  $PT$  to the parabola which is parallel to  $QQ'$ , and let  $P$  be the point of contact

<sup>1</sup>The rest of the proof is given in brackets as supplied by Commandinus

[From  $P$  draw  $PV$  parallel to  $AN$  meeting  $QQ'$  in  $V$ . Then  $PV$  will be a diameter of the parabola, and also the axis of the portion of the paraboloid immersed in the fluid]

Let  $C$  be the centre of gravity of the paraboloid  $BAB'$ , and  $F$  that of the portion immersed in the fluid. Join  $FC$  and produce it to  $H$  so that  $H$  is the centre of gravity of the remaining portion of the paraboloid above the surface.

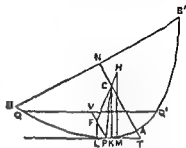
Then, since  $AN = \frac{1}{2}AC$ ,

and  $AN > \frac{1}{2}p$ ,

it follows that  $AC > \frac{p}{2}$

Therefore, if  $CP$  be joined, the angle  $CPT$  is acute. Hence, if  $CK$  be drawn perpendicular to  $PT$ ,  $K$  will fall between  $P$  and  $T$ . And, if  $FL$ ,  $HM$  be drawn parallel to  $CK$  to meet  $PT$ , they will each be perpendicular to the surface of the fluid.

Now the force acting on the immersed portion of the segment of the parabola



will rise and  $B'$  will fall, until  $AN$  takes the vertical position.]

### PROPOSITION 3

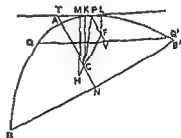
If a right segment of a paraboloid of revolution whose axis is not greater than  $\frac{1}{2}p$  (where  $p$  is the parameter), and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined at any angle to the vertical, but so that its base is entirely submerged, the solid will not remain in that position but will return to the position in which the axis is vertical.

fluid in the chord  $QQ'$  of the parabola.

Then, since  $AN$ , as placed, is not perpendicular to the surface of the fluid,  $QQ'$  and  $BB'$  will not be parallel.

Draw  $PT$  parallel to  $QQ'$  and touching the parabola at  $P$ . Let  $PT$  meet  $NA$  produced in  $T$ . Draw the diameter  $PV$  bisecting  $QQ'$  in  $V$ .  $PV$  is then the axis of the portion of the paraboloid above the surface of the fluid.

Let  $C$  be the centre of gravity of the whole segment of the paraboloid,  $F$  that of the portion above the surface. Join  $FC$  and produce it to  $H$  so that  $H$  is the centre of gravity of the immersed portion.



Then, since  $AC > \frac{p}{2}$ , the angle  $CPT$  is an acute angle, as in the last proposition.

Hence, if  $CK$  be drawn perpendicular to  $PT$ ,  $K$  will fall between  $P$  and  $T$ . Also, if  $HM$ ,  $FL$  be drawn parallel to  $CK$ , they will be perpendicular to the surface of the fluid

And the force acting on the submerged portion will act upwards along  $HM$ , while the weight of the rest will act downwards along  $LF$  produced

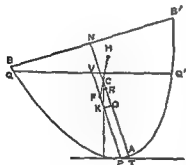
Thus the paraboloid will turn until it takes the position in which  $AN$  is vertical

## PROPOSITION 4

Given a right segment of a paraboloid of revolution whose axis  $AN$  is greater than

its base does not touch the surface of the fluid, it will not remain in that position but will return to the position in which its axis is vertical

Let the axis of the segment of the paraboloid be  $AN$ , and let a plane be drawn through  $AN$  perpendicular to the surface of the fluid and intersecting the segment in the parabola  $BAB'$ , the base of the segment in  $BB'$ , and the surface of the fluid in the chord  $QQ'$  of the parabola



Then  $AN$ , as placed, will not be perpendicular to  $QQ'$

Draw  $PT$  parallel to  $QQ'$  and touching the parabola at  $P$ . Draw the diameter  $PV$  bisecting  $QQ'$  in  $V$ . Thus  $PV$  will be the axis of the submerged portion of the solid

Let  $C$  be the centre of gravity of the whole solid.  $F$  that of the immersed portion. Join  $FC$  and produce it to  $H$  so that  $H$  is the centre of gravity of the remaining portion

$$AN = \frac{3}{2}AC,$$

$$AN > \frac{3}{2}p,$$

$$AC > \frac{p}{2}$$

Now, since  
and  
it follows that

Measure  $CO$  along  $CA$  equal to  $\frac{p}{2}$  and  $OR$  along  $OC$  equal to  $\frac{1}{3}AO$

Then, since  
and  
we have, by subtraction,  
That is,

$$AN = \frac{3}{2}AC,$$

$$AR = \frac{1}{3}AO,$$

$$NR = \frac{1}{3}OC$$

$$AN - AR = \frac{1}{3}OC$$

$$= \frac{1}{3}p,$$

$$AR = (AN - \frac{1}{3}p)$$

$$(AN - \frac{1}{3}p)^2 = AR^2 = AN^2$$

or  
Thus  
and therefore the ratio of the specific gravity of the solid to that of the fluid is, by the enunciation, not less than the ratio  $AR^2 : AN^2$

But, by Prop 1, the former ratio is equal to the ratio of the immersed portion to the whole solid, i.e. to the ratio  $PV^2 : AN^2$  [On Conoids and Spheroids, Prop 24]

Hence

$$PV^2 : AN^2 < AR^2 : AN^2,$$

or

It follows that

$$\begin{aligned} PV &< AR \\ PF(\approx \frac{2}{3}PV) &< \frac{2}{3}AR \\ &< AO \end{aligned}$$

If, therefore,  $OK$  be drawn from  $O$  perpendicular to  $OA$ , it will meet  $PF$  between  $P$  and  $F$

Also, if  $CK$  be joined, the triangle  $\triangle CO$  is equal and similar to the triangle formed by the normal, the subnormal and the ordinate at  $P$  (since  $CO = \frac{1}{2}p$  or the subnormal, and  $KO$  is equal to the ordinate)

Therefore  $CK$  is parallel to the normal at  $P$ , and therefore perpendicular to the tangent at  $P$  and to the surface of the fluid

Hence, if parallels to  $CK$  be drawn through  $F$ ,  $H$ , they will be perpendicular to the surface of the fluid, and the force acting on the submerged portion of the solid will act upwards along the former, while the weight of the other portion will act downwards along the latter

Therefore the solid will not remain in its position but will turn until  $AN$  assumes a vertical position

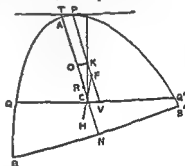
## PROPOSITION 5

Given a right segment of a paraboloid of revolution such that its axis  $AN$  is greater than  $\frac{1}{2}p$  (where  $p$  is the parameter), and its specific gravity is less than that of a fluid but in a ratio to it not greater than the ratio  $\{AN^2 - (AN - \frac{1}{2}p)^2\} : AN^2$ , if the segment be placed in the fluid with its axis inclined at any angle to the vertical, but so that its base is completely submerged, it will not remain in that position but will return to the position in which  $AN$  is vertical

Let a plane be drawn through  $AN$ , as placed perpendicular to the surface of the fluid and cutting the segment of the paraboloid in the parabola  $BAB'$ , the base of the segment in  $BB'$ , and the plane of the surface of the fluid in the chord  $QQ'$  of the parabola

Draw the tangent  $PT$  parallel to  $QQ'$ , and the diameter  $PV$ , bisecting  $QQ'$ , will accordingly be the axis of the portion of the paraboloid above the surface of the fluid

Let  $F$  be the centre of gravity of the portion above the surface  $C$  that of the whole solid, and produce  $FC$  to  $H$ , the centre of gravity of the immersed portion



As in the last proposition,  $AC > \frac{p}{2}$  and we measure  $CO$  along  $CA$  equal to  $\frac{p}{2}$  and  $OR$  along  $OC$  equal to  $\frac{1}{2}AO$

Then  $AR = \frac{2}{3}AC$ , and  $AR = \frac{2}{3}AO$ , and we derive as before,  $AR = (AN - \frac{1}{2}p)$

Now, by hypothesis

$$\begin{aligned} &(\text{spec gravity of solid}) < (\text{spec gravity of fluid}) \\ &\triangleright \{AN^2 - (AN - \frac{1}{2}p)^2\} : AN^2 \\ &\triangleright (AN^2 - AR^2) : AN^2 \end{aligned}$$

Therefore

$$\begin{aligned} &(\text{portion submerged}) < (\text{whole solid}) \\ &\triangleright (AN^2 - AR^2) : AN^2, \end{aligned}$$

and (whole solid) (portion above surface)

$$\begin{aligned} & > AN^2 > AR^2 \\ \text{Thus} & AN^2 > PV^2 > AN^2 > AR^2, \\ \text{whence} & PV < AR, \\ \text{and} & PF < \frac{3}{2}AR \\ & < AO \end{aligned}$$

Therefore, if a perpendicular to  $AC$  be drawn from  $O$ , it will meet  $PF$  in some point  $K$  between  $P$  and  $F$

And, since  $CO = \frac{1}{2}p$ ,  $CK$  will be perpendicular to  $PT$ , as in the last proposition

Now the force acting on the submerged portion of the solid will act upwards through  $H$ , and the weight of the other portion downwards through  $F$ , in directions parallel in both cases to  $CK$ , whence the proposition follows

## PROPOSITION 6

If a right segment of a paraboloid lighter than a fluid be such that its axis  $AM$  is greater than  $\frac{3}{2}p$ , but  $AM < 15 \frac{1}{2}p$ , and if the segment be placed in the fluid with its axis so inclined to the vertical that its base touches the fluid, it will never remain in such a position that the base touches the surface in one point only

Suppose the segment of the paraboloid to be placed in the position described, and let the plane through the axis  $AM$  perpendicular to the surface of the fluid intersect the segment of the paraboloid in the parabolic segment  $BAB'$  and the plane of the surface of the fluid in  $BQ$

Take  $C$  on  $AM$  such that  $AC = 2CM$  (or so that  $C$  is the centre of gravity of the segment of the paraboloid), and measure  $CK$  along  $CA$  such that

$$AM : CK = 15 : 4$$

Thus  $AM : CK > AM : \frac{1}{2}p$ , by hypothesis, therefore  $CK < \frac{1}{2}p$

Measure  $CO$  along  $CA$  equal to  $\frac{1}{2}p$ . Also draw  $KR$  perpendicular to  $AC$  meeting the parabola in  $R$

Draw the tangent  $PT$  parallel to  $BQ$ , and through  $P$  draw the diameter  $PV$  bisecting  $BQ$  in  $V$  and meeting  $KR$  in  $I$

$$PV : PI = AM : AK, \text{ or } >$$

Then  
"for this is proved"

$$\begin{aligned} \text{And} & CK = \frac{1}{2}AM = \frac{1}{2}AC, \\ \text{whence} & AK = AC - CK = \frac{1}{2}AC = \frac{1}{2}AM \end{aligned}$$

$$\text{Thus } KV = \frac{1}{2}AM$$

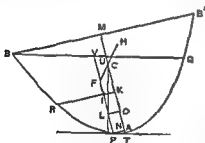
$$\text{Therefore } KM = \frac{1}{2}AK$$

$$\text{It follows that } PV = \frac{2}{3}PI, \text{ or } >$$

$$\text{so that } PI < 2IV$$

Let  $F$  be the centre of gravity of the immersed portion of the paraboloid, so that  $PF = 2FV$ . Produce  $FC$  to  $H$ , the centre of gravity of the portion above the surface

Draw  $OL$  perpendicular to  $PV$

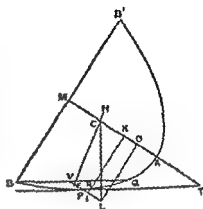


Then, since  $CO = \frac{1}{2}p$ ,  $CL$  must be perpendicular to  $PT$  and therefore to the surface of the fluid

And the forces acting on the immersed portion of the paraboloid and the portion above the surface act respectively upwards and downwards along lines through  $F$  and  $H$  parallel to  $CL$

Hence the paraboloid cannot remain in the position in which  $B$  just touches the surface, but must turn in the direction of increasing the angle  $PTM$

The proof is the same in the case where the point  $I$  is not on  $VP$  but on  $VP$  produced, as in the second figure



### PROPOSITION 7

Given a right segment of a paraboloid of revolution lighter than a fluid and such that its axis  $AM$  is greater than  $\frac{1}{2}p$ , but  $AM - \frac{1}{2}p < 15/4$ , if the segment be placed in the fluid so that its base is entirely submerged, it will never rest in such a position that the base touches the surface of the fluid at one point only

Suppose the solid so placed that one point of the base only ( $B$ ) touches the surface of the fluid. Let the plane through  $B$  and the axis  $AM$  cut the solid in the parabolic segment  $BAB'$  and the plane of the surface of the fluid in the chord  $BQ$  of the parabola

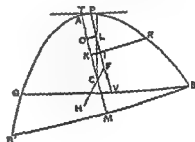
Let  $C$  be the centre of gravity of the segment, so that  $AC = 2CM$ , and measure  $CK$  along  $CA$  such that

$$AM : CK = 15 : 4$$

It follows that  $CK < \frac{1}{2}p$

Measure  $CO$  along  $CA$  equal to  $\frac{1}{2}p$ . Draw  $KR$  perpendicular to  $AM$  meeting the parabola in  $R$

Let  $PT$ , touching at  $P$ , be the tangent to the parabola which is parallel to  $BQ$ , and  $PV$  the diameter bisecting  $BQ$ , i.e. the axis of the portion of the paraboloid above the surface



Then as in the last proposition we prove that

$$PV_{\text{or}} > \frac{1}{2}PI,$$

and

$$PI_{\text{or}} < 2IV.$$

Let  $F$  be the centre of gravity of the portion of the solid above the surface, join  $FC$  and produce it to  $H$ , the centre of gravity of the portion submerged

Draw  $OL$  perpendicular to  $PV$ , and, as before, since  $CO = \frac{1}{2}p$ ,  $CL$  is perpendicular to the tangent  $PT$ . And the lines through  $H$ ,  $F$  parallel to  $CL$  are perpendicular to the surface of the fluid, thus the proposition is established as before

The proof is the same if the point  $I$  is not on  $VP$  but on  $VP$  produced.

### PROPOSITION 8

Given a solid in the form of a right segment of a paraboloid of revolution whose axis  $AM$  is greater than  $\frac{2}{3}p$ , but such that  $AM - \frac{2}{3}p < 15/4$ , and whose specific gravity bears to that of a fluid a ratio less than  $(AM - \frac{2}{3}p)^2 / AM^2$ , then, if the solid be placed in the fluid so that its base does not touch the fluid and its axis is inclined at an angle to the vertical, the solid will not return to the position in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid a certain angle to be described.

Let  $am$  be taken equal to the axis  $AM$ , and let  $c$  be a point on  $am$  such that  $ac = 2cm$ . Measure  $co$  along  $ca$  equal to  $\frac{1}{2}p$ , and  $or$  along  $oc$  equal to  $\frac{1}{2}ao$ .

Let  $X + Y$  be a straight line such that

(spec gr of solid) (spec gr of fluid) =  $(X+Y)^2$  am<sup>2</sup>, (α)  
and suppose  $X=2Y$

$$\begin{aligned}\text{Now } ar &= \frac{1}{2}ao = \frac{1}{2}\left(\frac{1}{2}am - \frac{1}{2}p\right) \\ &= am - \frac{1}{2}p \\ &= AM - \frac{1}{2}p\end{aligned}$$

Therefore, by hypothesis,  
 $(X+Y)^2 \leq ar^2 < ar^2 \leq am^2$ ,  
whence  $(X+Y) < ar$ , and therefore  
 $X < ar$

Measure  $ob$  along  $oa$  equal to  $X$ , and draw  $bd$  perpendicular to  $ab$  and of such length that  $bd^2 = \frac{1}{2}co \cdot ab$  (8)

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Draw the tangent  $PT$  parallel to  $QQ$ , touching at  $P$ , and let  $PV$  be the diameter bisecting  $QQ$  in  $V$  (or the axis of the immersed portion of the solid), and  $PN$  the ordinate from  $P$

Measure  $AO$  along  $AM$  equal to  $ao$ , and  $OC$  along  $OM$  equal to  $oc$ , and draw  $OL$  perpendicular to  $PV$ .

I Suppose the angle  $OTP$  greater than the angle  $dab$

Thus

$$PN^2 \quad NT^2 > db^2 \quad ba^2$$

But

$$PN^2 \quad NT^2 = p \quad 4AN$$
$$=co NT.$$

and

$$db^2 \quad ba^2 = \frac{1}{2}ca \quad ab, \text{ by } (\beta)$$

Therefore

$$NT < 2ab,$$

or

 $AN < ab.$ 

whence

$$NO > b_0 \text{ (since } a_0 = A_0)$$
 $\geq X$ 

Now  $(Y+Y)^2 \text{ am}^2 = \frac{\text{spec gr of solid}}{\text{portion immersed}} \frac{\text{spec gr of fluid}}{\text{rest of solid}}$



$$= PV^2 \cdot AM^2,$$

so that

$$X + Y = PV$$

But

$$\begin{aligned} PL (=NO) &> X \\ &> \frac{2}{3}(X + Y), \text{ since } X = 2Y, \\ &> \frac{2}{3}PV, \end{aligned}$$

or

$$PV < \frac{3}{2}PL,$$

and therefore

$$PL > 2LV$$

Take  $\square$  point  $F$  on  $PV$  so that  $PF = 2FV$ , i.e. so that  $F$  is the centre of gravity of the immersed portion of the solid

Also  $AC = ac = \frac{2}{3}am = \frac{2}{3}AM$ , and therefore  $C$  is the centre of gravity of the whole solid

Join  $FC$  and produce it to  $H$ , the centre of gravity of the portion of the solid above the surface

Now, since  $CO = \frac{1}{3}p$ ,  $CL$  is perpendicular to the surface of the fluid, therefore so are the parallels to  $CL$  through  $F$  and  $H$ . But the force on the immersed portion acts upwards through  $F$  and that on the rest of the solid downwards through  $H$

Therefore the solid will not rest but turn in the direction of diminishing the angle  $MTP$

II Suppose the angle  $OTP$  less than the angle  $dab$ . In this case, we shall have, instead of the above results, the following,

$$AN > ab,$$

$$NO < X$$

Also

$$PV > \frac{3}{2}PL,$$

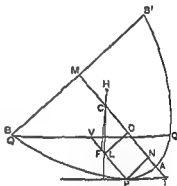
and therefore

$$PL < 2LV$$

Make  $PF$  equal to  $2FV$ , so that  $F$  is the centre of gravity of the immersed portion

And, proceeding as before, we prove in this case that the solid will turn in the direction of increasing the angle  $MTP$

III When the angle  $MTP$  is equal to the angle  $dab$ , equalities replace



described

### PROPOSITION 9

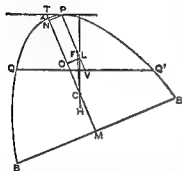
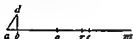
*Cone solid with its axis inclined at an angle to the vertical but so that its base is entirely below the surface, the solid will not return to the position in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid an angle equal to that described in the last proposition*

the solid be placed in the fluid with its axis inclined at an angle to the vertical but so that its base is entirely below the surface, the solid will not return to the position in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid an angle equal to that described in the last proposition

Take  $am$  equal to  $AM$ , and take  $c$  on  $am$  such that  $ac = 2cm$ . Measure  $co$  along  $ca$  equal to  $\frac{1}{3}p$ , and  $or$  along  $ac$  such that  $or = \frac{2}{3}ao$

Let  $X + Y$  be such a line that

(spec gr of solid) (spec gr of fluid) =  $\{am^2 - (X+Y)^2\} am^2$ ,  
and suppose  $X = 2Y$



$$\begin{aligned}\text{Now } ar &= \frac{2}{3}ao \\ &= \frac{2}{3}(\frac{2}{3}am - \frac{1}{3}p) \\ &= AM - \frac{1}{3}p\end{aligned}$$

Therefore, by hypothesis,  
 $am^2 - ar^2 \quad am^2 < \{am^2 - (X+Y)^2\} am^2$ ,  
whence  $X+Y < ar$ ,  
and therefore  $X < ao$

Make  $ob$  (measured along  $oa$ ) equal to  $X$ , and draw  $bd$  perpendicular to  $ba$  and of such length that

$$bd^2 = \frac{1}{3}co \ ab$$

Join  $ad$

Now suppose the solid placed as in the figure with its axis  $AM$  inclined to the vertical. Let the plane through  $AM$  perpendicular to the surface of the fluid cut the solid in the parabola  $BAB'$  and the surface of the fluid in  $QQ'$

Let  $PT$  be the tangent parallel to  $QQ'$ ,  $PV$  the diameter bisecting  $QQ'$  (or the axis of the portion of the paraboloid above the surface),  $PN$  the ordinate from  $P$

I Suppose the angle  $MTP$  greater than the angle  $dab$ . Let  $AM$  be cut as before in  $C$  and  $O$  so that  $AC = 2CM$ ,  $OC = \frac{1}{3}p$ , and accordingly  $AM$ ,  $am$  are equally divided. Draw  $OL$  perpendicular to  $PV$

Then, we have, as in the last proposition

$$\begin{aligned}PN^2 \quad NT^2 &> db^2 \quad ba^2, \\ co \quad NT &> \frac{1}{3}co \ ab, \\ AN &< ab \\ NO &> bo \\ &> X\end{aligned}$$

whence  
and therefore

It follows that

Again, since the specific gravity of the solid is to that of the fluid as the immersed portion of the solid to the whole

$$\begin{aligned}AM^2 - (Y+Y)^2 \quad AM^2 &= AM^2 - PV^2 \quad AM^2, \\ \text{or } (Y+Y)^2 \quad AM^2 &= PV^2 \quad AM^2\end{aligned}$$

That is  $Y+Y = PV$

And  $PL$  (or  $NO$ )  $> X$   
 $> \frac{1}{3}PV$ ,

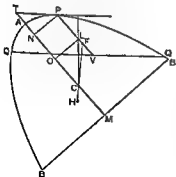
so that  $PL > 2LV$

Take  $\Gamma$  on  $PV$  so that  $PF = 2FV$ . Then  $F$  is the centre of gravity of the portion of the solid above the surface

Also  $C$  is the centre of gravity of the whole solid. Join  $FC$  and produce it to  $H$ , the centre of gravity of the unimmersed portion

Then since  $CO = \frac{1}{3}p$ ,  $CL$  is perpendicular to  $PT$  and to the surface of the fluid, and

the force acting on the immersed portion of the solid acts upwards along the





$$\begin{aligned}
 &= CM + CK \quad AC - CK \\
 &= \left(\frac{1}{2} + \frac{1}{2}\right) AM \quad \left(\frac{1}{2} - \frac{1}{2}\right) AM \\
 &= 0 \quad \blacksquare \\
 &= MA \quad AC
 \end{aligned}
 \tag{\beta}$$

Thus  $C$  is seen to be on the parabola  $BA_1B_2$  by the converse of Prop 4 of the *Quadrature of the Parabola*]

Also if a perpendicular to  $AM$  be drawn from  $O$  it will meet the parabola

$$\begin{aligned}
 \text{And} \quad BM \quad MB_1 &= BM \quad (2BM_1 - BM) && \text{by means of } (\beta) \text{ above} \\
 &= 5 \quad (6 - 5) && \text{by means of } (\beta), \\
 &= 5 \quad 1 \\
 \text{It follows that} \quad Q_1Q_2 \quad Q_1Q_2 &= 2 \quad 1 \\
 \text{or} \quad Q_1Q_2 &= 2Q_1Q_2 \quad \left. \begin{aligned} Q_1Q_2 &= 2Q_1Q_2 \\ P_1P_2 &= 2P_1P_2 \end{aligned} \right\} \\
 \text{Similarly} \quad P_1P_2 &= 2P_1P_2 \\
 \text{Also since} \quad MR &\approx \frac{1}{2}CO - \frac{1}{2}p \\
 &AR \approx AM - MR \\
 &= AM - \frac{1}{2}p
 \end{aligned}$$

#### (ENUNCIATION)

If the segment of the paraboloid be placed in the fluid with its base entirely above the surface then

(I) if

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) < AR^2 \quad AM^2 \quad [ < (AM - \frac{1}{2}p)^2 \quad AM^2 ],$$

the solid will rest in the position in which its axis  $AM$  is vertical,

(II) if

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) < AR^2 \quad AM^2 \quad \text{but} > Q_1Q_2^2 \quad AM^2$$

the solid will not rest with its base touching the surface of the fluid in one point only but in such a position that its base does not touch the surface at any point and its axis makes with the surface an angle greater than  $U$ ,

(III a) if

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) = Q_1Q_2^2 \quad AM^2,$$

the solid will rest and remain in the position in which the base touches the surface of the fluid at one point only and the axis makes with the surface an angle equal to  $U$

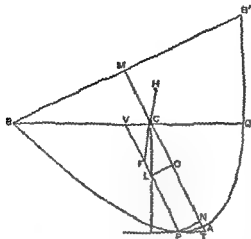
(III b) if

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) = P_1P_2^2 \quad AM^2,$$

the solid will rest with its base touching the surface of the fluid at one point only and with its axis inclined to the surface at an angle equal to  $T_1$ ,



Suppose the segment of the paraboloid placed in the fluid, as described, with its axis inclined at an angle to the vertical, and with its base touching the surface at one point  $B$  only. Let the solid be cut by a plane through the axis and perpendicular to the surface of the fluid, and let the plane intersect the solid in the parabolic segment  $BAB'$  and the plane of the surface of the fluid in  $BQ$ .



Take the points  $C, O$  on  $AM$  as before described. Draw the tangent parallel to  $BQ$  touching the parabola in  $P$  and meeting  $AM$  in  $T$ , and let  $PV$  be the diameter bisecting  $BQ$  (i.e. the axis of the immersed portion of the solid).

$$\begin{aligned} \text{Then } P' AM^2 &= (\text{spec gr of solid}) (\text{spec gr of fluid}) \\ &= (\text{portion immersed}) (\text{whole solid}) \\ &= PV^2 AM^2, \\ P'V' &= l = PV \end{aligned}$$

whence

Thus the segments in the two figures, namely  $BP'Q$ ,  $BPQ$ , are equal and similar.

Therefore

$$\angle PTN = \angle P'T'N'$$

Also

$$AT = AT', AN = AN', PN = P'N'$$

Now, in the first figure,  $P'I < 2IV'$

Therefore, if  $OL$  be perpendicular to  $PV$  in the second figure,

$$PL < 2LV$$

Take  $F$  on  $LV$  so that  $PF = 2LV$ , i.e. so that  $F$  is the centre of gravity of the immersed portion of the solid. And  $C$  is the centre of gravity of the whole solid. Join  $FC$  and produce it to  $H$ , the centre of gravity of the portion above the surface.

Now, since  $CO = \frac{1}{2}p$ ,  $CL$  is perpendicular to the tangent at  $P$  and to the surface of the fluid. Thus, as before, we prove that the solid will not rest with  $B$  touching the surface, but will turn in the direction of increasing the angle  $PTN$ .

Hence, in the position of rest the axis  $AM$  must make with the surface of the fluid an angle greater than the angle  $U$  which the tangent at  $Q_1$  makes with  $AM$ .

(III a) In this case

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) = Q_1 Q_1^2 AM^2$$

Let the segment of the paraboloid be placed in the fluid so that its base nowhere touches the surface of the fluid, and its axis is inclined at an angle to the vertical.

Let the plane through  $AM$  perpendicular to the surface of the fluid cut the paraboloid in the parabola  $BAB'$  and the plane of the surface of the fluid  $QQ'$ . Let  $PT$  be the tangent parallel to  $QQ'$ ,  $PV$  the diameter bisecting  $QQ'$ ,  $PV$  the ordinate at  $P$ .

Divide  $AM$  as before at  $C$   $O$

In the other figure let  $Q_1N$  be the ordinate at  $Q_1$ . Join  $BQ_2$  and produce it to meet the outer parabola in  $q$ . Then  $BQ_2 = Q_2q$  and the tangent  $QU$  is parallel to  $Bq$ . Now

$$\begin{aligned} Q Q_2^2 \cdot AM^2 &= (\text{spec gr of solid}) (\text{spec gr of fluid}) \\ &= (\text{portion immersed}) (\text{whole solid}) \\ &= PV^2 \cdot AM^2 \end{aligned}$$

Therefore  $Q Q_2 = PV$  and the segments  $QPQ$   $BQ_1q$  of the paraboloid are equal in volume. And the base of one passes through  $B$  while the base of the other passes through  $Q$  a point nearer to  $A$  than  $B$  is.

It follows that the angle between  $QQ$  and  $BB$  is less than the angle  $B_1Bq$ .

Therefore

$$\angle U < \angle PTN$$

whence  $AN > AN$ ,  
and therefore

$$NO (\text{or } Q Q_2) < PL$$

where  $OL$  is perpendicular to  $PV$ .

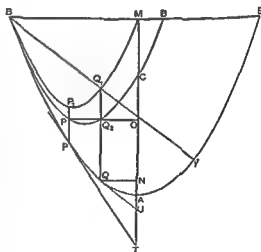
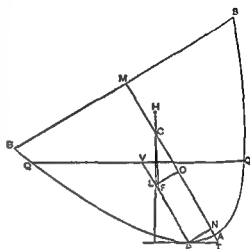
It follows since  $Q Q_2 = 2Q_2Q_3$  that

$$PL > 2LV$$

Therefore  $F$  the centre of gravity of the immersed portion of the solid is between  $P$  and  $L$  while as before  $CL$  is perpendicular to the surface of the fluid.

Producing  $FC$  to  $H$  the centre of gravity of the portion of the solid above the surface we see that the solid must turn in the direction of diminishing the angle  $PTN$  until one point  $B$  of the base just touches the surface of the fluid.

When this is the case we shall



so the angle  $PTV$

$C, H$  are all in

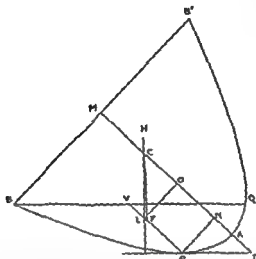
one vertical straight line

Thus the paraboloid will remain in the position in which one point  $B$  of the base touches the surface of the fluid and the axis makes with the surface an angle equal to  $U$ .



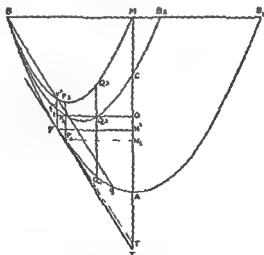


Thus  $PL > 2LV$ , so that  $F$ , the centre of gravity of the immersed portion of the solid, is between  $L$  and  $P$ , while  $CL$  is perpendicular to the surface of the fluid



If then we produce  $FC$  to  $H$ , the centre of gravity of the portion of the solid above the surface, we prove that the solid will not rest but turn in the direction of diminishing the angle  $PTN$

2 Next let the paraboloid be so placed in the fluid that its base touches the surface of the fluid at one point  $B$  only, and let the construction proceed as before



Then  $PV = P'V'$ , and the segments  $BPQ$ ,  $BP'q$  are equal and similar, so that

$$\begin{aligned}\angle PTN &= \angle P'T'N' \\ AN &= AN', NO = N'O, \\ P'I &= PL, \\ PL &> 2LV\end{aligned}$$

It follows that  
and therefore  
whence

Thus  $F$  again lies between  $P$  and  $L$ , and, as before, the paraboloid will turn in the direction of diminishing the angle  $PTN$ , i.e. so that the base will be more submerged.

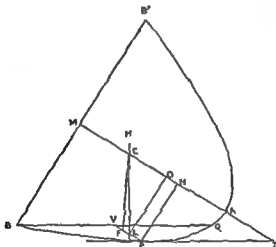
(V) In this case

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) < P_1 P_2 \cdot 2 \text{ } AM^2$$

If then the ratio is equal to  $P^2 \cdot AM^2$ ,  $l < P_1P_2$ . Place  $P'V'$  between the parabolas  $BP_1Q_1$  and  $BP_2Q_2$  equal in length to  $l$  and parallel to  $AM$ . Let  $P'V'$  meet the intermediate parabola in  $P''$  and  $OP_2$  in  $I$ .

Join  $BV'$  and produce it to meet the outer parabola in  $q$ . Then, as before,  $BV' = V'q$ , and the tangent  $P'T'$  is parallel to  $Bq$ .

1 Let the paraboloid be so placed in the fluid that its base touches the surface at one point only



Let the plane through  $AM$  perpendicular to the surface of the fluid cut the paraboloid in the parabolic section  $BAB'$  and the plane of the surface of the fluid in  $BQ$

Making the usual construction, we find

$$PV = l = P' V',$$

and the segments  $BPQ$ ,  $BP_1q$  are equal and similar

Therefore

$$\angle PTN = \angle P'T'N',$$

and

$$AN \cong AN', N'O \cong NO$$

Therefore

$$PL \subseteq P'I.$$

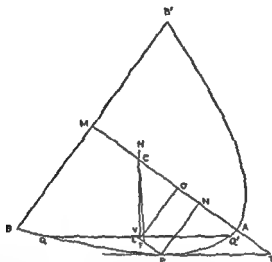
whence it follows that

$$PL < 2LY$$

Thus  $F$ , the centre of gravity of the unimmersed portion of the solid, lies between  $L$  and  $V$ , while  $CL$  is perpendicular to the surface of the fluid.

Producing  $FC$  to  $H$  the centre of gravity of the portion above the surface, we prove, as usual, that there will not be rest but the solid will turn in the direction of increasing the angle  $PTN$ , so that the base will not anywhere touch the surface.

2 The solid will however rest in a position where its axis makes with the surface of the fluid an angle less than  $T_1$



For let it be placed so that the angle  $PTN$  is not less than  $T_1$

Then, with the same construction as before,  $PV = P'V'$ .

And, since

$$\angle T < \angle T_1,$$

$$AN > AN_1,$$

and therefore  $NO < N_1O$ , where  $P_1N_1$  is the ordinate of  $P_1$

Hence

$$PL < P_1P_2,$$

But

$$P_1P_2 > P'F',$$

Therefore

$$PL > \frac{1}{2}PV,$$

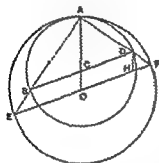
so that  $F$ , the centre of gravity of the immersed portion of the solid, lies between  $P$  and  $L$

Thus the solid will turn in the direction of diminishing the angle  $PTN$  until that angle becomes less than  $T_1$

## BOOK OF LEMMAS

### PROPOSITION 1

*If two circles touch at A, and if BD, EF be parallel diameters in them, ADF is a straight line*



Let  $O, C$  be the centres of the circles and let  $OC$  be joined and produced to  $A$ . Draw  $DH$  parallel to  $AO$  meeting  $OF$  in  $H$ .

Then since  $OH = CD = CA$ ,  
and  $OF = OA$ ,

we have, by subtraction  $HF = CO = DH$   
Therefore  $\angle HDF = \angle HFD$

Thus both the triangles  $CAD, HDF$  are isosceles and the third angles  $ACD, DHF$  in each are equal. Therefore the equal angles in each are equal to one another, and

$$\angle ADC = \angle DFH$$

Add to each the angle  $CDF$ , and it follows that

$$\begin{aligned}\angle ADC + \angle CDF &= \angle CDF + \angle DFH \\ &= (\text{two right angles})\end{aligned}$$

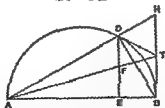
Hence  $ADF$  is a straight line

The same proof applies if the circles touch externally

### PROPOSITION 2

*Let AB be the diameter of a semicircle, and let the tangents to it at B and at any other point D on it meet in T. If now DE be drawn perpendicular to AB, and if AT, DE meet in F,*

$$DF = FE$$



Produce  $AD$  to meet  $BT$  produced in  $H$ . Then the angle  $ADB$  in the semicircle is right, therefore the angle  $BDH$  is also right. And  $TB, TD$  are equal.

Therefore  $T$  is the centre of the semicircle on  $BH$  as diameter, which passes through  $D$ .

Hence

$$HT = TB$$

And, since  $DE \parallel HB$  are parallel, it follows that  $DF = FE$

## PROPOSITION 3

Let  $P$  be any point on a segment of a circle whose base is  $AB$ , and let  $PN$  be perpendicular to  $AB$ . Take  $D$  on  $AB$  so that  $AN = ND$ . If now  $PQ$  be an arc equal to the arc  $PA$ , and  $BQ$  be joined,

$BQ, BD$  shall be equal

Join  $PA, PQ, PD, DQ$

Then, since the arcs  $PA, PQ$  are equal,

$$PA = PQ$$

But, since  $AN = ND$ , and the angles at  $N$  are right,

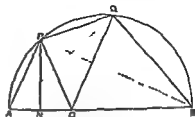
$$PA = PD$$

Therefore

$$PQ = PD,$$

and

$$\angle PQD = \angle PDQ$$



Now, since  $A, P, Q, B$  are concyclic,

$$\angle PAD + \angle PQB = (\text{two right angles}),$$

whence

$$\angle PDA + \angle PQB = (\text{two right angles})$$

$$= \angle PDA + \angle PDB$$

Therefore

$$\angle PQB = \angle PDB,$$

and, since the parts, the angles  $PQD, PDQ$ , are equal,

$$\angle BQD = \angle BDQ,$$

and

$$BQ = BD$$

## PROPOSITION 4

If  $AB$  be the diameter of a semicircle and  $N$  any point on  $AB$ , and if semicircles be described within the first semicircle and having  $AN, NB$  as diameters respectively, the figure included between the circumferences of the three semicircles is "what Archimedes called an  $\alpha\rho\beta\eta\lambda\omicron\varsigma$ ",<sup>1</sup> and its area is equal to the circle on  $PN$  as diameter, where  $PN$  is perpendicular to  $AB$  and meets the original semicircle in  $P$ .

For

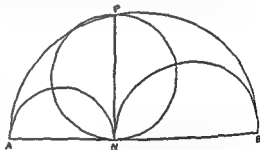
$$AB^2 = AN^2 + NB^2 + 2AN \cdot NB$$

$$= AN^2 + NB^2 + 2PN^2$$

But circles (or semicircles) are to one another as the squares of their radii (or diameters)

Hence

$$\begin{aligned} (\text{semicircle on } AB) &= (\text{sum of} \\ &\text{semicircles on } AN, NB) \\ &+ 2(\text{semicircle on } PN) \end{aligned}$$



That is the circle on  $PN$  as diameter is equal to the difference between the semicircle on  $AB$  and the sum of the semicircles on  $AN, NB$ , i.e. is equal to the area of the  $\alpha\rho\beta\eta\lambda\omicron\varsigma$ .

## PROPOSITION 5

Let  $AB$  be the diameter of a semicircle,  $C$  any point on  $AB$ , and  $CD$  perpendicular to it, and let semicircles be described within the first semicircle and having  $AC, CB$  as diameters. Then, if two circles be drawn touching  $CD$  on different sides and each touching two of the semicircles, the circles so drawn will be equal.

Let one of the circles touch  $CD$  at  $E$ , the semicircle on  $AB$  in  $F$ , and the semicircle on  $AC$  in  $G$ .

<sup>1</sup> $\alpha\rho\beta\eta\lambda\omicron\varsigma$  is literally a shoemaker's knife

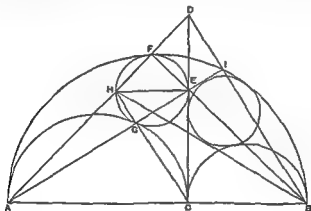
Draw the diameter  $EH$  of the circle, which will accordingly be perpendicular to  $CD$  and therefore parallel to  $AB$

Join  $FH$ ,  $HA$ , and  $FE$ ,  $EB$  Then, by Prop 1,  $FHA$ ,  $FEB$  are both straight lines since  $EH$ ,  $AB$  are parallel

For the same reason  $AGE$ ,  $CGH$  are straight lines

Let  $AF$  produced meet  $CD$  in  $D$ , and let  $AE$  produced meet the outer semi-circle in  $I$  Join  $BI$ ,  $ID$

Then, since the angles  $AFB$ ,  $ACD$  are right, the straight lines  $AD$ ,  $AB$  are such that the perpendiculars on each from the extremity of the other meet in the point  $E$  Therefore, by the properties of triangles,  $AE$  is perpendicular to the line joining  $B$  to  $D$



But  $AE$  is perpendicular to  $BI$

Therefore  $BID$  is a straight line

Now, since the angles at  $G$ ,  $I$  are right,  $CH$  is parallel to  $BD$

Therefore

$$\begin{aligned} AB \cdot BC &= AD \cdot DH \\ &= AC \cdot HE, \end{aligned}$$

so that

$$AC \cdot CB = AB \cdot HE$$

In like manner, if  $d$  is the diameter of the other circle, we can prove that

$$AC \cdot CB = AB \cdot d$$

Therefore  $d = HE$ , and the circles are equal

#### PROPOSITION 6

Two circles are equal if their diameters are equal

Let two circles be given, with centers  $O$  and  $O'$ , and diameters  $AB$  and  $A'B'$  respectively. If  $AB = A'B'$ , then the circles are equal.

Let

produced meet  $AB$  in  $N$ ,  $P$  respectively

Now, in the triangle  $AGC$ , the perpendiculars from  $A, C$  on the opposite sides meet in  $L$ . Therefore, by the properties of triangles,  $GLN$  is perpendicular to  $AC$ .

Similarly  $HMP$  is perpendicular to  $CB$ .

Again, since the angles at  $I, K, D$  are right,  $CK$  is parallel to  $AD$ , and  $CI$  to  $BD$ .

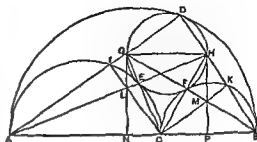
Therefore  $AC : CB = AL : LH$   
 $\quad \quad \quad = AN : NP,$   
 and  $BC : CA = BM : MG$   
 $\quad \quad \quad = BP : PN$

Hence  $AN : NP = NP : PB$ ,  
 or  $AN, NP, PB$  are in continued proportion.

Now, in the case where  $AC = \frac{1}{2}CB$ ,  
 $AN = \frac{1}{2}NP = \frac{1}{2}PB$ ,  
 whence  $BP : PN : NA : AB = 4 : 6 : 9 : 19$

Therefore  $GH = NP = \frac{1}{19}AB$

And similarly  $GH$  can be found when  $AC : CB$  is equal to any other given ratio.



### PROPOSITION 7

If circles be circumscribed about and inscribed in a square, the circumscribed circle is double of the inscribed circle.

For the ratio of the circumscribed to the inscribed circle is equal to that of the square on the diagonal to the square itself, i.e. to the ratio 2 : 1.

### PROPOSITION 8

If  $AB$  be any chord of a circle whose centre is  $O$ , and if  $AB$  be produced to  $C$  so that  $BC$  is equal to the radius, if further  $CO$  meet the circle in  $D$  and be produced to meet the circle a second time in  $E$ , the arc  $AE$  will be equal to three times the arc  $BD$ .

Draw the chord  $EF$  parallel to  $AB$ , and join  $OB, OF$ .

Then since the angles  $OEF, OFE$  are equal,

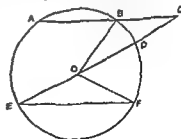
$$\begin{aligned}\angle COF &= 2\angle OEF \\ &= 2\angle BCO \text{ by parallels,} \\ &= 2\angle BOD, \text{ since } BC = BO\end{aligned}$$

Therefore

$$\angle BOF = 3\angle BOD,$$

so that the arc  $BF$  is equal to three times the arc  $BD$ .

Hence the arc  $AE$ , which is equal to the arc  $BF$ , is equal to three times the arc  $BD$ .

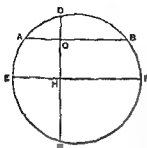


### PROPOSITION 9

If in a circle two chords  $AB, CD$  which do not pass through the centre intersect at right angles, then

$$(\text{arc } AD) + (\text{arc } CB) = (\text{arc } AC) + (\text{arc } DB)$$

Let the chords intersect at  $O$ , and draw the diameter  $EF$  parallel to  $AB$  intersecting  $CD$  in  $H$ .  $EF$  will thus bisect  $CD$  at right angles in  $H$ , and



$$(\text{arc } ED) = (\text{arc } EC)$$

Also  $EDF$ ,  $ECF$  are semicircles, while

$$(\text{arc } ED) = (\text{arc } EA) + (\text{arc } AD)$$

Therefore

$$(\text{sum of arcs } CF, EA, AD) = (\text{arc of a semicircle})$$

And the arcs  $AE$ ,  $BF$  are equal

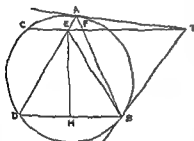
Therefore

$$(\text{arc } CB) + (\text{arc } AD) = (\text{arc of a semicircle})$$

Hence the remainder of the circumference, the sum of the arcs  $AC$ ,  $DB$ , is also equal to a semicircle, and the proposition is proved

### PROPOSITION 10

Let  $AB$  meet  $TC$  in  $F$ , and join  $BE$



Now the angle  $TAB$  is equal to the angle in the alternate segment i.e.

$$\angle TAB = \angle ADB$$

$$= \angle AET, \text{ by parallels}$$

Hence the triangles  $EAT$ ,  $AFT$  have one angle equal and another (at  $T$ ) common. They are therefore similar and

$$FT : AT = AT : ET$$

$$\text{Therefore } ET : TT = TA^2 = TB^2$$

It follows that the triangles  $EBT$ ,  $BFT$  are similar

$$\angle TEB = \angle TBF = \angle TAB$$

But the angle  $TEB$  is equal to the angle  $EBD$ , and the angle  $TAB$  was proved equal to the angle  $EDB$

$$\text{Therefore } \angle EDB = \angle EBD$$

And the angles at  $H$  are right angles

$$\text{It follows that } BH = HD$$

### PROPOSITION 11

If two chords  $AB$ ,  $CD$  in a circle intersect at right angles in a point  $O$ , not being the centre, then

$$AO^2 + BO^2 + CO^2 + DO^2 = (\text{diameter})^2$$

Draw the diameter  $CE$ , and join  $AC$ ,  $CB$ ,  $AD$ ,  $BE$

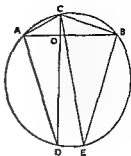


Then the angle  $CAO$  is equal to the angle  $CEB$  in the same segment, and the angles  $AOC$ ,  $EBC$  are right; therefore the triangles  $AOC$ ,  $EBC$  are similar, and

$$\angle ACO = \angle ECB$$

It follows that the subtended arcs, and therefore the chords  $AD$ ,  $BE$ , are equal

$$\begin{aligned} \text{Thus } (AO^2 + DO^2) + (BO^2 + CO^2) &= AD^2 + BC^2 \\ &= BE^2 + BC^2 \\ &= CE^2 \end{aligned}$$



### PROPOSITION 12

If  $AB$  be the diameter of a semicircle, and  $TP$ ,  $TQ$  the tangents to it from any point  $T$ , and if  $AQ$ ,  $BP$  be joined meeting in  $R$ , then  $TR$  is perpendicular to  $AB$

Let  $TR$  produced meet  $AB$  in  $M$ , and join  $PA$ ,  $QB$

Since the angle  $APB$  is right,

$$\begin{aligned} \angle PAB + \angle PBA &= (\text{a right angle}) \\ &= \angle AQB \end{aligned}$$

Add to each side the angle  $RBQ$ , and

$$\angle PAB + \angle QBA = (\text{exterior}) \angle PRQ$$

But

$$\angle TPR = \angle PAB, \text{ and } \angle TQR = \angle QBA,$$

in the alternate segments,

$$\text{therefore } \angle TPR + \angle TQR = \angle PRQ$$

It follows from this that

$$TP = TQ = TR$$

[For, if  $PT$  be produced to  $O$  so that  $TO = TQ$ , we have

$$\angle TOQ = \angle TQO$$

And, by hypothesis,  $\angle PRQ = \angle TPR + \angle TQR$

By addition,  $\angle POQ + \angle PRQ = \angle TPR + \angle OQR$

It follows that, in the quadrilateral  $OPRQ$ , the opposite angles are together equal to two right angles. Therefore a circle will go round  $OPQR$ , and  $T$  is its centre, because  $TP = TO = TQ$ . Therefore  $TR = TP$ ]

Thus  $\angle TRP = \angle TPR = \angle PAM$

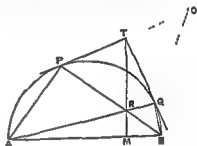
Adding to each the angle  $PRM$ ,

$$\begin{aligned} \angle PAM + \angle PRM &= \angle TRP + \angle PRM \\ &= (\text{two right angles}) \end{aligned}$$

Therefore  $\angle APR + \angle AMR = (\text{two right angles})$ ,

whence

$$\angle AMR = (\text{a right angle})$$



### PROPOSITION 13

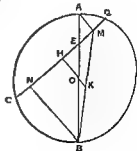
If a diameter  $AB$  of a circle meet any chord  $CD$ , not a diameter, in  $E$ , and if  $AM$ ,  $BN$  be drawn perpendicular to  $CD$ , then

$$CN = DM$$

Let  $O$  be the centre of the circle, and  $OH$  perpendicular to  $CD$ . Join  $BM$ , and produce  $HO$  to meet  $BM$  in  $K$

Then

$$CH = HD$$



And, by parallels, since

$$\begin{aligned} BO &= OA, \\ BK &= KM, \\ NH &= HM \\ CN &= DM \end{aligned}$$

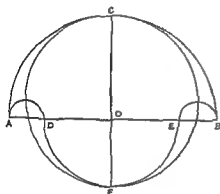
Therefore

Accordingly

### PROPOSITION 14

Let  $ACB$  be a semicircle on  $AB$  as diameter, and let  $AD, BE$  be equal lengths measured along  $AB$  from  $A, B$  respectively. On  $AD, BE$  as diameters describe semicircles on the side towards  $C$ , and on  $DE$  as diameter a semicircle on the opposite side. Let the perpendicular to  $AB$  through  $O$ , the centre of the first semicircle,

intersect the arcs of all the semi-



By Eucl II 10, since  $ED$  is bisected at  $O$  and produced to  $A$ ,

$$EA^2 + AD^2 = 2(EO^2 + OA^2),$$

and  $CF = OA + OE = EA$

Therefore

$$AB^2 + DE^2 = 4(EO^2 + OA^2) = 2(CF^2 + AD^2)$$

But circles (and therefore semicircles) are to one another as the squares on their radii (or diameters)

Therefore

(sum of semicircles on  $AB, DE$ )

= (circle on  $CF$ ) + (sum of semicircles on  $AD, BE$ )

Therefore

(area of "sahnun") = (area of circle on  $CF$  as diam)

### PROPOSITION 15

Let  $AB$  be the diameter of a circle,  $AC$  a side of an inscribed regular pentagon,  $D$  the middle point of the arc  $AC$ . Join  $CD$  and produce it to meet  $BA$  produced in  $E$ , join  $AC, DB$  meeting in  $F$ , and draw  $FM$  perpendicular to  $AB$ . Then

$EM = (\text{radius of circle})$

Let  $O$  be the centre of the circle and join  $DA, DM, DO, CB$

Now

$$\angle ABC = \frac{1}{2}(\text{right angle}),$$

and

whc

$\Gamma$

$\pi$

$BC$

being equal and

But

$$\begin{aligned} \angle BCD + \angle BAD &= (\text{two right angles}) \\ &= \angle BAD + \angle DAE \\ &= \angle BMD + \angle DMA, \end{aligned}$$

so that

and

$$\begin{aligned} \angle DAE &= \angle BCD, \\ \angle BAD &= \angle AMD \end{aligned}$$

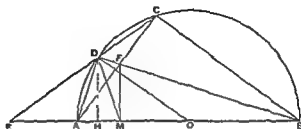
Therefore

$$AD = MD$$

Now, in the triangle  $DMO$ ,

$$\angle MOD = \frac{2}{3}(\text{right angle}),$$

$$\angle DMO = \frac{1}{3}(\text{right angle})$$



Therefore

$$\angle ODM = \frac{1}{3}(\text{right angle}) = \angle AOD,$$

whence

$$OM = MD$$

Again

$$\angle EDA = (\text{supplement of } \angle ADC)$$

$$= \angle CBA$$

$$= \frac{1}{3}(\text{right angle})$$

$$= \angle ODM$$

Therefore, in the triangles  $EDA$ ,  $ODM$ ,

$$\angle EDA = \angle ODM,$$

$$\angle EAD = \angle OMD,$$

and the sides  $AD$   $MD$  are equal

Hence the triangles are equal in all respects, and

$$EA = MO$$

Therefore

$$EM = AO$$

Mo

inscrib

at  $D$  :

in the

side of the decagon inscribed in the same circle be put together, the whole straight line is divided in extreme and mean ratio and the greater segment is the side of the hexagon ]

## THE METHOD TREATING OF MECHANICAL PROBLEMS

*"Archimedes to Eratosthenes greeting*

"I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciations and inviting you to discover the proofs which at the moment I did not give. The enunciations of the theorems which I sent were as follows

drawn and the other the plane in which the base of the cylinder is and the surface being that which is between the said planes, and the segment cut off from the cylinder is one sixth part of the whole prism

Now these theorems differ in character from those communicated before, for we compared the figures then in question conoids and spheroids and seg-

by planes. The proofs then of these theorems I have written in this book and

means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves, for certain things first became clear to me by a mechanical method although they had to be demonstrated by geom-

<sup>1</sup>The parallelograms are apparently squares

<sup>2</sup>ie squares

city afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously

namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard

at I am myself in the position of  
orem now to be published [by the  
to expound the method partly because I have already spoken of it and I do not want to be thought to have uttered vain words, but equally because I am persuaded that it will be of no little service to mathematics, for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me

"First then I will set out the very first theorem which became known to me by means of mechanics, namely that

*"Any segment of a section of a right-angled cone (i.e. a parabola) is four thirds*

the propositions]

[I premise the following propositions which I shall use in the course of the work]

1 "If from [one magnitude another magnitude be subtracted which has not the same centre of gravity, the centre of gravity of the remainder is found by] producing [the straight line joining the centres of gravity of the whole magnitude and of the subtracted part in the direction of the centre of gravity of the

■ "If the centres of gravity of any number of magnitudes whatever be on the same straight line, the centre of gravity of the magnitude made up of all of them will be on the same straight line" [Cf *Ibid* 1 5]

3 "The centre of gravity of any straight line is the point of bisection of the straight line" [Cf *Ibid* 1 4]

4 "The centre of gravity of any triangle is the point in which the straight lines drawn from each vertex to the middle points of the opposite sides meet" [Cf *Ibid* 1 13 14]

5 "The centre of gravity of any parallelogram is the point in which the diagonals meet" [Cf *Ibid* 1 10]

6 "The centre of gravity of a circle is the point which is also the centre [of the circle]"

7 "The centre of gravity of any cylinder is the point of bisection of the axis"

8 "The centre of gravity of any cone is [the point which divides its axis so that] the portion [adjacent to the vertex is] triple [of the portion adjacent to the base]"

[All these propositions have already been] proved.<sup>1</sup> [Besides these I require also the following proposition, which is easily proved]

If in two series of magnitudes those of the first series are, in order, propor-

the same ratio which the sum of the magnitudes of the second series has to the sum of the (correspondingly) selected magnitudes of the fourth series.<sup>2</sup> [*On Conoids and Spheroids*, Prop 1]

## PROPOSITION I

Let  $ABC$  be a segment of a parabola bounded by the straight line  $AC$  and the parabola  $ABC$ , and let  $D$  be the middle point of  $AC$ . Draw the straight line  $DBE$  parallel to the axis of the parabola and join  $AB$ ,  $BC$ .

Then shall the segment  $ABC$  be  $\frac{4}{3}$  of the triangle  $ABC$ .

From  $A$  draw  $AKF$  parallel to  $DE$ , and let the tangent to the parabola at  $C$  meet  $DBE$  in  $E$  and  $AKF$  in  $F$ . Produce  $CB$  to meet  $AF$  in  $K$ , and again produce  $CK$  to  $H$ , making  $KH$  equal to  $CK$ .

Consider  $CH$  as the bar of a balance,  $K$  being its middle point.

Let  $MO$  be any straight line parallel to  $ED$ , and let it meet  $CF$ ,  $CK$ ,  $AC$  in  $M$ ,  $N$ ,  $O$  and the curve in  $P$ .

Now, since  $CE$  is a tangent to the parabola and  $CD$  the semi ordinate,

$$EB = BD,$$

"for this is proved in the Elements [of Conics]"<sup>3</sup>

Since  $FA$ ,  $MO$  are parallel to  $ED$  it follows that

$$FK = KA, MN = NO$$

Now, by the property of the parabola, "proved in a lemma,"

$$MO \cdot OP = CA \cdot AO \quad [\text{Of Quadrature of Parabola, Prop 5}]$$

$$= CK \cdot KN$$

[Eucl vi 2]

$$= HK \cdot KN$$

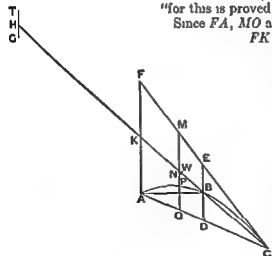
Take a straight line  $TG$  equal to  $OP$ , and place it with its centre of gravity at  $H$ , so that  $TH = HG$ , then since  $N$  is the centre of gravity of the straight line  $MO$ , and

$$MO \cdot TG = HK \cdot KN,$$

it follows that  $TG$  at  $H$  and  $MO$  at  $N$  will be in equilibrium about  $K$  [*On the Equilibrium of Planes*, I 6, 7]

<sup>1</sup>The problem of finding the centre of gravity of a cone is not solved in any extant work of Archimedes.

<sup>2</sup>See the works on conics by Aristaeus and Euclid.



Similarly, for all other straight lines parallel to  $DE$  and meeting the arc of the parabola, (1) the portion intercepted between  $FC$ ,  $AC$  with its middle point on  $AC$  and (2) a length equal to the intercept between the curve and  $AC$  placed with its centre of gravity at  $H$  will be in equilibrium about  $K$ .

Therefore  $K$  is the centre of gravity of the whole system consisting (1) of all the straight lines as  $MO$  intercepted between  $FC$ ,  $AC$  and placed as they are, equal to

ie  $MO$ ,

and the segment  $CBA$  is made up of all the straight lines like  $PO$  within the curve.

It follows that the triangle, placed where it is in the figure, is in equilibrium about  $A$  with the segment  $CBA$  placed with its centre of gravity at  $H$ .

Divide  $KC$  at  $W$  so that  $CK = 3KW$ , then  $W$  is the centre of gravity of the triangle  $ACF$ , 'for this is proved in the books on equilibrium' (*ἐν τοῖς ἰσορρομικοῖς*)

[Cf. *On the Equilibrium of Planes* 1.15]

Therefore  $\triangle ACF$  (segment  $ABC$ )  $= HK : KW$   
 $= 3 : 1$

Therefore segment  $ABC = \frac{1}{3} \triangle ACF$

But  $\triangle ACF = 4 \triangle ABC$

Therefore segment  $ABC = \frac{1}{4} \triangle ABC$

'Now the fact here stated is not actually demonstrated by the argument'

## PROPOSITION 2

We can investigate by the same method the propositions that

(1) Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius and

(2) the cylinder with base equal to a great circle of the sphere and height equal to the diameter is  $1\frac{1}{2}$  times the sphere

(1) Let  $ABCD$  be a great circle of a sphere, and  $AC$ ,  $BD$  diameters at right angles to one another

Let a circle be drawn about  $BD$  as diameter and in a plane perpendicular to  $AC$  and on this circle as base let a cone be described with  $A$  as vertex. Let the surface of this cone be produced and then cut by a plane through  $C$  parallel to its base, the section will be a circle on  $EF$  as diameter. On this circle as base let a cylinder be erected with height and axis  $AC$ , and produce  $CA$  to  $H$ , making  $AH$  equal to  $CA$ .

Let  $CH$  be regarded as the bar of a balance,  $A$  being its middle point.

Draw any straight line  $MN$  in the plane of the circle  $ABCD$  and parallel to  $BD$ . Let  $MN$  meet the circle in  $O$ ,  $P$  the diameter  $AC$  in  $S$ , and the straight lines  $AE$ ,  $AF$  in  $Q$ ,  $R$  respectively. Join  $AO$ .

Through  $MN$  draw a plane at right angles to  $AC$ ,

thus plane will cut the cylinder in a circle with diameter  $MN$ , the sphere in a circle with diameter  $OP$ , and the cone in a circle with diameter  $QR$

Now, since  $MS=AC$ , and  $QS=AS$ ,

$$\begin{aligned} MS \cdot SQ &= CA \cdot AS \\ &= AO^2 \\ &= OS^2 + SQ^2. \end{aligned}$$

And, since  $HA=AC$ ,

$$\begin{aligned} HA \cdot AS &= CA \cdot AS \\ &= MS \cdot SQ \\ &= MS^2 \cdot MS \cdot SQ \\ &= MS^2 (OS^2 + SQ^2), \\ &\quad \text{from above,} \\ &= MN^2 (OP^2 + QR^2) \\ &= (\text{circle, diam } MN) (\text{circle, diam } OP \\ &\quad + \text{circle, diam } QR). \end{aligned}$$

That is,

$$HA \cdot AS = (\text{circle in cylinder}) : (\text{circle in sphere} + \text{circle in cone})$$

Therefore the circle in the cylinder, placed where it is, is in equilibrium, about  $A$ , with the circle in the sphere together with the circle in the cone, if both the latter circles are placed with their centres of gravity at  $H$

Similarly for the three corresponding sections made by a plane perpendicular to  $AC$  and passing through any other straight line in the parallelogram  $LEF$  parallel to  $EF$

If we deal in the same way with all the sets of three circles in which planes perpendicular to  $AC$  cut the cylinder, the sphere and the cone, and which make up those solids respectively, it follows that the cylinder, in the place where it is, will be in equilibrium about  $A$  with the sphere and the cone together,

when both are placed with their centres of gravity at  $H$

Therefore, since  $K$  is the centre of gravity of the cylinder,

$$HA \cdot AK = (\text{cylinder}) (\text{sphere} + \text{cone } AEF)$$

But  $HA=2AK$ ,

therefore cylinder = 2(sphere + cone  $AEF$ ).

Now cone  $AEF=2(\text{sphere})$ ,

[Eucl xii 10]

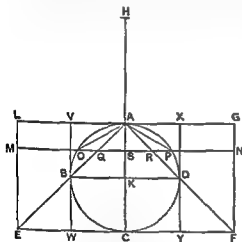
therefore cone  $AEF=2(\text{sphere})$

But, since  $EF=2BD$ ,

$$\text{cone } AEF = 8(\text{cone } ABD);$$

therefore sphere = 4(cone  $ABD$ )

(2) Through  $B, D$  draw  $VBW, XDY$  parallel to  $AC$ , and imagine a cylinder which has  $AC$  for axis and the circles on  $VX, WY$  as diameters for bases





Then

$$\begin{aligned}
 \text{cylinder } VY &= 2(\text{cylinder } VD) \\
 &= 6(\text{cone } ABD) \\
 &= \frac{1}{2}(\text{sphere}), \text{ from above}
 \end{aligned}$$

[Eucl XII 10]

Q E D

"From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it, for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius "

## PROPOSITION 3

By this method we can also investigate the theorem that

*A cylinder with base equal to the greatest circle in a spheroid and height equal to the axis of the spheroid is  $1\frac{1}{2}$  times the spheroid, and, when this is established, it is plain that*

*If any spheroid be cut by a plane through the centre and at right angles to the axis, the half of the spheroid is double of the cone which has the same base and the same axis as the segment (i.e. the half of the spheroid)*

Let a plane through the axis of a spheroid cut its surface in the ellipse  $ABCD$ , the diameters (i.e. axes) of which are  $AC$ ,  $BD$ , and let  $K$  be the centre

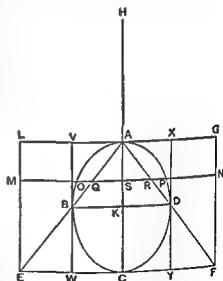
Draw a circle about  $BD$  as diameter and in a plane perpendicular to  $AC$ , imagine a cone with this circle as base and  $A$  as vertex produced and cut by a plane through  $C$  parallel to its base, the section will be a circle in a plane at right angles to  $AC$  and about  $EF$  as diameter

Imagine a cylinder with the latter circle as base and axis  $AC$ , produce  $CA$  to  $H$ , making  $AH$  equal to  $CA$

Let  $HC$  be regarded as the bar of a balance,  $A$  being its middle point

In the parallelogram  $LF$  draw any straight line  $MN$  parallel to  $EF$  meeting the ellipse in  $O$ ,  $P$  and  $AE$ ,  $AF$ ,  $AC$  in  $Q$ ,  $R$ ,  $S$  respectively

If now a plane be drawn through  $MN$  at right angles to  $AC$ , it will cut the cylinder in a circle with diameter  $MN$ , the spheroid in a circle with diameter  $OP$ , and the cone in a circle with diameter  $QR$



Since  $HA = AC$ ,

$$\begin{aligned} HA \cdot AS &= CA \cdot AS \\ &= EA \cdot AQ \\ &= MS \cdot SQ \end{aligned}$$

Therefore

$$HA \cdot AS = MS^2 \cdot MS \cdot SQ$$

But, by the property of the ellipse,

$$\begin{aligned} AS \cdot SC \cdot SO^2 &= AK^2 \cdot KB^2 \\ &= AS^2 \cdot SQ^2, \end{aligned}$$

therefore

$$\begin{aligned} SQ^2 \cdot SO^2 &= AS^2 \cdot AS \cdot SC \\ &= SQ^2 \cdot SQ \cdot QM, \end{aligned}$$

and accordingly

$$SO^2 = SQ \cdot QM$$

Add  $SQ^2$  to each side, and we have

$$SO^2 + SQ^2 = SQ \cdot SM$$

Therefore, from above we have

$$\begin{aligned} HA \cdot AS &= MS^2 \cdot (SO^2 + SQ^2) \\ &= MN^2 \cdot (OP^2 + QR^2) \\ &= (\text{circle, diam } MN) \cdot (\text{circle, diam } OP + \text{circle, diam } QR) \end{aligned}$$

That is,

$$HA \cdot AS = (\text{circle in cylinder}) \cdot (\text{circle in spheroid} + \text{circle in cone})$$

Therefore the circle in the cylinder, in the place where it is, is in equilibrium, about  $A$ , with the circle in the spheroid and the circle in the cone together, if both the latter circles are placed with their centres of gravity at  $H$ .

Similarly for the three corresponding sections made by a plane perpendicular to  $AC$  and passing through any other straight line in the parallelogram  $LF$  parallel to  $EF$ .

If we deal in the same way with all the sets of three circles in which planes perpendicular to  $AC$  cut the cylinder, the spheroid and the cone, and which make up those figures respectively, it follows that the cylinder in the place where it is, will be in equilibrium about  $A$  with the spheroid and the cone together when both are placed with their centres of gravity at  $H$ .

Therefore, since  $K$  is the centre of gravity of the cylinder

$$HA \cdot AK = (\text{cylinder}) \cdot (\text{spheroid} + \text{cone } AEF)$$

But  $HA = 2AK$ ,

therefore

$$\text{cylinder} = 2(\text{spheroid} + \text{cone } AEF)$$

And

$$\text{cylinder} = 3(\text{cone } AEF),$$

[Eucl. xii. 10]

therefore

$$\text{cone } AEF = 2(\text{spheroid})$$

But, since  $EF = 2BD$

$$\text{cone } AEF = 8(\text{cone } ABD),$$

therefore

$$\text{spheroid} = 4(\text{cone } ABD),$$

and

$$\text{half the spheroid} = 2(\text{cone } ABD)$$

Through  $B, D$  draw  $VBW, ADY$  parallel to  $AC$ , and imagine a cylinder which has  $AC$  for axis and the circles on  $VX, WY$  as diameters for bases.

Then

$$\text{cylinder } VY = 2(\text{cylinder } ID)$$

$$= 6(\text{cone } ABD)$$

$$= \frac{1}{2}(\text{spheroid}), \text{ from above}$$

## PROPOSITION 4

Any segment of a right-angled conoid (i.e. a paraboloid of revolution) cut off by a plane at right angles to the axis is  $1\frac{1}{2}$  times the cone which has the same base and the same axis as the segment

This can be investigated by our method, as follows

Let a paraboloid of revolution be cut by a plane through the axis in the parabola  $BAC$ , and let it also be cut by another plane at right angles to the axis and intersecting the former plane in  $BC$ . Produce  $DA$ , the axis of the segment, to  $H$ , making  $HA$  equal to  $AD$ .

Imagine that  $HD$  is the bar of a balance,  $A$  being its middle point

The base of the segment being the circle on  $BC$  as diameter and in a plane perpendicular to  $AD$ ,

imagine (1) a cone drawn with the latter circle as base and  $A$  as vertex, and (2) a cylinder with the same circle as base and  $AD$  as axis

In the parallelogram  $EC$  let any straight line  $MN$  be drawn parallel to  $BC$ , and through  $MN$  let a plane be drawn at right angles to  $AD$ , this plane will cut the cylinder in a circle with diameter  $MN$  and the paraboloid in a circle with diameter  $OP$ .

Now,  $BAC$  being a parabola and  $BD$ ,  $OS$  ordinates

$$DA \quad AS = BD^2 \quad OS^2,$$

$$\text{or} \quad HA \quad AS = MS^2 \quad SO^2$$

Therefore

$$HA \quad AS = (\text{circle, rad } MS) (\text{circle, rad } OS) \\ = (\text{circle in cylinder}) (\text{circle in paraboloid})$$

Therefore the circle in the cylinder, in the place where it is, will be in equilibrium about  $A$  with the circle in the paraboloid, if the latter is placed with its centre of gravity at  $H$ .

Similarly for the two corresponding circular sections made by a plane perpendicular to  $AD$  and passing through any other straight line in the parallelogram which is parallel to  $BC$ .

Therefore, as usual if we take all the circles making up the whole cylinder and the whole segment and treat them in the same way, we find that the cylinder, in the place where it is, is in equilibrium about  $A$  with the segment placed with its centre of gravity at  $H$ .

If  $K$  is the middle point of  $AD$   $K$  is the centre of gravity of the cylinder,

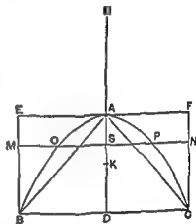
therefore  $HA \quad AK = (\text{cylinder}) (\text{segment})$

Therefore cylinder = 2(segment)

And cylinder = 3(cone  $ABC$ ),

therefore segment =  $\frac{2}{3}$ (cone  $ABC$ )

[Eucl. xii. 10]



## PROPOSITION III

The centre of gravity of a segment of a right-angled conoid (i.e. a paraboloid of revolution) cut off by a plane at right angles to the axis in on the straight line which is the axis of the segment, and divides the said straight line in such a way that the portion of it adjacent to the vertex is double of the remaining portion

This can be investigated by the method, as follows

Let a paraboloid of revolution be cut by a plane through the axis in the parabola  $BAC$ , and let it also be cut by another plane at right angles to the axis and intersecting the former plane in  $BC$

Produce  $DA$ , the axis of the segment, to  $H$ , making  $HA$  equal to  $AD$ , and imagine  $DH$  to be the bar of a balance, its middle point being  $A$

The base of the segment being the circle on  $BC$  as diameter and in a plane perpendicular to  $AD$ , imagine a cone with this circle as base and  $A$  as vertex, so that  $AB$ ,  $AC$  are generators of the cone

In the parabola let any double ordinate  $OP$  be drawn meeting  $AB$ ,  $AD$ ,  $AC$  in  $Q$ ,  $S$ ,  $R$  respectively

Now, from the property of the parabola,

$$\begin{aligned} BD^2 \cdot OS^2 &= DA \cdot AS \\ &= BD \cdot QS \\ &= BD^2 \cdot BD \cdot QS \end{aligned}$$

Therefore  $OS^2 = BD \cdot QS$ ,

or  $BD \cdot OS = OS^2 \cdot QS$ ,

whence  $BD \cdot QS = OS^2 \cdot QS^2$

$$\begin{aligned} \text{But } BD \cdot QS &= AD \cdot AS \\ &= HA \cdot AS \end{aligned}$$

$$\begin{aligned} \text{Therefore } HA \cdot AS &= OS^2 \cdot QS^2 \\ &= OP^2 \cdot QR^2 \end{aligned}$$

If now through  $OP$  a plane be drawn at right angles to  $AD$ , this plane cuts the paraboloid in a circle with diameter  $OP$  and the cone in a circle with diameter  $QR$

$$\begin{aligned} \text{We see therefore that } HA \cdot AS &= (\text{circle, diam } OP) \cdot (\text{circle, diam } QR) \\ &= (\text{circle in paraboloid}) \cdot (\text{circle in cone}), \end{aligned}$$

and the circle in the paraboloid, in the place where it is, is in equilibrium about  $A$  with the circle in the cone placed with its centre of gravity at  $H$

Similarly for the two corresponding circular sections made by a plane perpendicular to  $AD$  and passing through any other ordinate of the parabola

Dealing therefore in the same way with all the circular sections which make up the whole of the segment of the paraboloid and the cone respectively, we see that the segment of the paraboloid, in the place where it is, is in equilibrium about  $A$  with the cone placed with its centre of gravity at  $H$

Now, since  $A$  is the centre of gravity of the whole system as placed, and the centre of gravity of part of it, namely the cone, as placed, is at  $H$ , the centre of gravity of the rest, namely the segment, is at a point  $K$  on  $HA$  produced such that

$$HA \cdot AK = (\text{segment}) \cdot (\text{cone})$$

But

$$\text{segment} = \frac{1}{2}(\text{cone})$$

[Prop 4]

Therefore  $HA = \frac{3}{2}AK$ ,  
that is,  $K$  divides  $AD$  in such a way that  $AK = 2KD$

## PROPOSITION 6

*The centre of gravity of any hemisphere [is on the straight line which] is its axis, and divides the said straight line in such a way that the portion of it adjacent to the surface of the hemisphere has to the remaining portion the ratio which 3 has to 2*

Let a sphere be cut by a plane through its centre in the circle  $ABCD$ , let  $AC$ ,  $BD$  be perpendicular diameters of this circle, and through  $BD$  let a plane be drawn at right angles to  $AC$

The latter plane will cut the sphere in a circle on  $BD$  as diameter

Imagine a cone with the latter circle as base and  $A$  as vertex

Produce  $CA$  to  $H$ , making  $AH$  equal to  $CA$ , and let  $HC$  be regarded as the bar of a balance,  $A$  being its middle point

In the semicircle  $BAD$ , let any straight line  $OP$  be drawn parallel to  $BD$  and cutting  $AC$  in  $E$  and the two generators,  $AB$ ,  $AD$  of the cone in  $Q$ ,  $R$  respectively. Join  $AO$

Through  $OP$  let a plane be drawn at right angles to  $AC$ ,

this plane will cut the hemisphere in a circle with diameter  $OP$  and the cone in a circle with diameter  $QR$

Now

$$\begin{aligned} HA \cdot AE &= AC \cdot AE \\ &= AO^2 - AE^2 \\ &= (OE^2 + AE^2) - AE^2 \\ &= (OE^2 + QE^2) - QE^2 \\ &= (\text{circle, diam } OP) - (\text{circle, diam } QR) \end{aligned}$$

Therefore the circles with diameters  $OP$ ,  $QR$ , in the places where they are are in equilibrium about  $A$  with the circle with diameter  $QR$  if the latter is placed with its centre of gravity at  $H$

And, since the centre of gravity of the two circles with diameters  $OP$ ,  $QR$  taken together, in the place where they are, is

[There is a lacuna here, but the proof can easily be completed on the lines of the corresponding but more difficult case in Prop 8

We proceed thus from the point where the circles with diameters  $OP$ ,  $QR$ , in the place where they are, balance, about  $A$ , the circle with diameter  $QR$  placed with its centre of gravity at  $H$

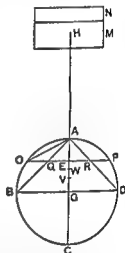
A similar relation holds for all the other sets of circular sections made by other planes passing through points on  $AG$  and at right angles to  $AG$

Taking then all the circles which fill up the hemisphere  $BAD$  and the cone  $ABD$  respectively, we find that

the hemisphere  $BAD$  and the cone  $ABD$ , in the places where they are, together balance, about  $A$ , as cone equal to  $ABD$  placed with its centre of gravity at  $H$

Let the cylinder  $M+N$  be equal to the cone  $ABD$

Then, since the cylinder  $M+N$  placed with its centre of gravity at  $H$  balances the hemisphere  $BAD$  and the cone  $ABD$  in the places where they are, suppose that the portion  $M$  of the cylinder, placed with its centre of gravity at





der the segment of the sphere and the cone with the common height  $AG$  are all cut by any plane perpendicular to  $AC$

$$AW = WV \quad AV = OV$$

Therefore  $W$  will be the centre of gravity of the cylinder and  $V$  will be the centre of gravity of the cone

Since now the bodies are in equilibrium as described

$$(\text{cylinder}) (\text{cone } AEF + \text{segment } BAD \text{ of sphere}) = HA \quad AW$$

[The rest of the proof is lost but it can easily be supplied thus

We have

$$\begin{aligned} (\text{cone } AEF + \text{segmt } BAD) (\text{cylinder}) &= AW \quad AC \\ &= AW \quad AC \quad AC^2 \end{aligned}$$

$$\begin{aligned} \text{But} \quad (\text{cylinder}) (\text{cone } AEF) &= AC^2 \quad \frac{1}{2} EG^2 \\ &= AC^2 \quad \frac{1}{2} AG^2 \end{aligned}$$

Therefore *ex aequali*

$$\begin{aligned} (\text{cone } AEF + \text{segmt } BAD) (\text{cone } AEF) &= AW \quad AC \quad \frac{1}{2} AG^2 \\ &= \frac{1}{2} AC \quad \frac{1}{2} AG^2, \end{aligned}$$

$$\text{whence} \quad (\text{segmt } BAD) (\text{cone } AEF) = \left( \frac{1}{2} AC - \frac{1}{2} AG \right) \frac{1}{2} AG$$

$$\begin{aligned} \text{Again} \quad (\text{cone } AEF) (\text{cone } ABD) &= EG^2 \quad DG^2 \\ &= AG^2 \quad AG \quad GC \\ &= AG \quad GC \\ &= \frac{1}{2} AG \quad \frac{1}{2} GC \end{aligned}$$

Therefore *ex aequali*

$$\begin{aligned} (\text{segment } BAD) (\text{cone } ABD) &= \left( \frac{1}{2} AC - \frac{1}{2} AG \right) \frac{1}{2} GC \\ &= \left( \frac{1}{2} AC - AG \right) GC \\ &= \left( \frac{1}{2} AC + GC \right) GC \quad \text{Q.E.D.} \end{aligned}$$

#### PROPOSITION 8

[The enunciation the setting out and a few words of the construction are missing

It is enunciation however can be supplied from that of Prop. 7 with which it must be identical except that it cannot refer to any segment and the presumption therefore is that the proposition was enunciated with reference to one kind of segment only i.e. either a segment greater than a hemisphere or a segment less than a hemisphere

the sphere

Draw  $KL$ , through any point  $Q$  on  $AG$ , parallel to  $EF$  and cutting the segment in  $K, L$ , and  $AE, AF$  in  $R, P$  respectively Join  $AK$

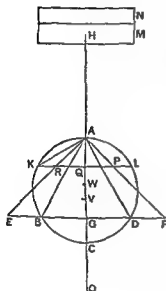
Now

$$\begin{aligned} HA \cdot AQ &= CA \cdot AQ \\ &= AK^2 - AQ^2 \\ &= (KQ^2 + QA^2) - QA^2 \\ &= (KQ^2 + PQ^2) - PQ^2 \\ &= (\text{circle, diam } KL) - (\text{circle, diam } PR) \quad (\text{circle, diam } PR) \end{aligned}$$

Imagine a circle equal to the circle with diameter  $PR$  placed with its centre of gravity at  $H$ , therefore the circles on diameters  $KL, PR$ , in the places where they are, are in equilibrium about  $A$  with the circle with diameter  $PR$  placed with its centre of gravity at  $H$

Similarly for the corresponding circular sections made by any other plane perpendicular to  $AG$ .

Therefore, taking all the circular sections



$AEF$ , in the places where they are, are in equilibrium with the cone  $AEF$  assumed to be placed with its centre of gravity at  $H$

Let the cylinder  $M+N$  be equal to the cone  $AEF$  which has  $A$  for vertex and the circle on  $EF$  as diameter for base

Divide  $AG$  at  $V$  so that

$$AG = 4VG,$$

therefore  $V$  is the centre of gravity of the cone  $AEF$ , "for this has been proved before"

Let the cylinder  $M+N$  be cut by a plane perpendicular to the axis in such a way that the cylinder  $M$  (alone) placed with its centre of gravity at  $H$  is in equilibrium with the cone  $AEF$

Since  $M+N$  suspended at  $H$  is in equilibrium with the segment  $ABD$  of the sphere and the cone  $AEF$  in the places where they are, while  $M$ , also at  $H$ , is in equilibrium with the cone  $AEF$  in the place where it is, it follows that

$N$  at  $H$  is in equilibrium with the segment  $ABD$  of the sphere in the place where it is

Now (segment  $ABD$  of sphere) (cone  $ABD$ ) =  $OG \cdot GC$ ,  
'for this is already proved [Cf. *On the Sphere and Cylinder* II 2 Cor. as well as Prop. 7 ante]

And

$$\begin{aligned} &(\text{cone } ABD) \quad (\text{cone } AEF) \\ &= (\text{circle, diam } BD) \quad (\text{circle, diam } EF) \\ &= BD^2 \quad EF^2 \\ &= BG^2 \quad GE^2 \\ &= CG \cdot GA \quad GA^2 \\ &= CG \cdot GA \end{aligned}$$



Therefore, *ex aequali*,

$$(\text{segment } ABD \text{ of sphere}) (\text{cone } AEF) = OG \cdot GA$$

Take  $\equiv$  point  $W$  on  $AG$  such that

$$AW : WG = (GA + 4GC) : (GA + 2GC)$$

We have then, inversely,

$$GW : WA = (2GC + GA) : (4GC + GA),$$

and, *componendo*,

$$GA : AW = (GGC + 2GA) : (4GC + GA)$$

But

$$GO = \frac{1}{2}(GGC + 2GA),$$

$$[\text{for } GO - GC = \frac{1}{2}(CG + GA)]$$

and

$$CV = \frac{1}{2}(4GC + GA),$$

therefore

$$GA : AW = OG : CV,$$

and, alternately and inversely,

$$OG \cdot GA = CV \cdot WA$$

It follows, from above, that

$$(\text{segment } ABD \text{ of sphere}) (\text{cone } AEF) = CV \cdot WA$$

Now, since the cylinder  $M$  with its centre of gravity at  $H$  is in equilibrium about  $A$  with the cone  $AEF$  with its centre of gravity at  $V$ ,

$$(\text{cone } AEF) (\text{cylinder } M) = HA \cdot AV$$

$$= CA \cdot AV,$$

and, since the cone  $AEF$  = the cylinder  $M + N$ , we have, *dividendo* and *inter-tendo*,

$$(\text{cylinder } M) (\text{cylinder } N) = AV \cdot CV$$

Hence, *componendo*,

$$(\text{cone } AEF) (\text{cylinder } N) = CA \cdot CV$$

$$= HA \cdot CV$$

But it was proved that

$$(\text{segment } ABD \text{ of sphere}) (\text{cone } AEF) = CV \cdot WA,$$

therefore, *ex aequali*,

$$(\text{segment } ABD \text{ of sphere}) (\text{cylinder } N) = HA \cdot AW$$

And it was above proved that the cylinder  $N$  at  $H$  is in equilibrium about  $A$  with the segment  $ABD$ , in the place where it is, therefore, since  $H$  is the centre of gravity of the cylinder  $N$ ,  $W$  is the centre of gravity of the segment  $ABD$  of the sphere

### PROPOSITION 9

In the same way we can investigate the theorem that

The centre of gravity of any segment of a sphere is on the straight line which is the axis of the segment and divides this straight line in such a way that the part of it adjacent to the vertex of the segment has to the remaining part the ratio which the sum of the axis of the segment and four times the axis of the complementary segment has to the sum of the axis of the segment and double the axis of the complementary segment

[As this theorem relates to "any segment" but states the same result as that proved in the preceding proposition, it follows that Prop 8 must have related to one kind of segment either a segment greater than a semicircle (as in Heiberg's figure of Prop 8) or a segment less than a semicircle, and the present proposition completed the proof for both kinds of segments. It would only require a slight change in the figure, in any case.]

## PROPOSITION 10

By this method too we can investigate the theorem that

[A segment of an obtuse-angled conoid (*i.e.* a hyperboloid of revolution) has to the cone which has] the same base [as the segment and equal height the same ratio as the sum of the axis of the segment and three times] the "annex to the axis" (*i.e.* half the transverse axis of the hyperbolic section through the axis of the hyperboloid, or, in other words, the distance between the vertex of the segment and the vertex of the enveloping cone) has to the sum of the axis of the segment and double of the "annex" [this is the theorem proved in *On Conoids and Spheroids*, Prop 25], "and also many other theorems, which, as the method has been made clear by means of the foregoing examples, I will omit, in order that I may now proceed to compass the proofs of the theorems mentioned above"

## PROPOSITION 11

If in a right prism with square bases a cylinder be inscribed having its bases in opposite square faces and touching with its surface the remaining four parallelogrammic faces, and if through the centre of the circle which is the base of the cylinder and one side of the opposite square face a plane be drawn, the figure cut off by the plane so drawn is one sixth part of the whole prism

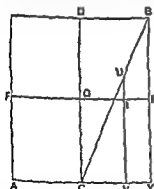
"This can be investigated by the method, and, when it is set out, I will go back to the proof of it by geometrical considerations"

[The investigation by the mechanical method is contained in the two Propositions, 11, 12 Prop 13 gives another solution which, although it contains no mechanics, is still of the character which Archimedes regards as inconclusive, since it assumes that the solid is actually made up of parallel plane sections and that an auxiliary parabola is actually made up of parallel straight lines in it Prop 14 added the conclusive geometrical proof]

Let there be a right prism with a cylinder inscribed as stated

Let the prism be cut through the axis of the prism and cylinder by a plane perpendicular to the plane which cuts off the portion of the cylinder, let this plane make, as section, the parallelogram  $AB$ , and let it cut the plane cutting off the portion of the cylinder (which plane is perpendicular to  $AB$ ) in the straight line  $BC$

Let  $CD$  be the axis of the prism and cylinder, let  $EF$  bisect it at right angles, and through  $EF$



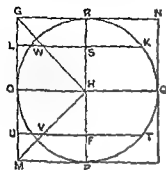
let a plane be drawn at right angles to  $CD$ , this plane will cut the prism in a square and the cylinder in a circle

Let  $MN$  be the square and  $OPQR$  the circle, and let the circle touch the sides of the square in  $O, P, Q, R$  [ $F, E$  in the first figure are identical with  $O, Q$  respectively] Let  $H$  be the centre of the circle

Let  $KL$  be the intersection of the plane through  $EF$  perpendicular to the axis of the cylinder and the plane cutting off the portion of the cylinder,  $KL$  is bisected by  $OHQ$  [and passes through the middle point of  $HQ$ ]



these planes produce as sections in the half cylinder  $PQR$  and in the prism  $GHI$  four parallelograms in which the heights are equal to the axis of the cylinder, and the other sides are equal to  $KS$ ,  $TF$ ,  $LW$ ,  $UV$  respectively



[The rest of the proof is missing but, as Zeuthen says, the result obtained and the method of arriving at it are plainly indicated by the above

Archimedes wishes to prove that the half cylinder  $PQR$ , in the place where it is, balances the prism  $GHI$ , in the place where it is, about  $H$  as fixed point

He has first to prove that the elements (1) the parallelogram with side  $=KS$  and (2) the parallelogram with side  $=LW$ , in the places where they are, balance about  $S$ , or, in other words that the straight lines  $SK$ ,  $LW$ , in the places where they are, balance about  $S$

Now (radius of circle  $OPQR$ )<sup>2</sup>  $= SK^2 + SH^2$ ,  
or  $SL^2 = SH^2 + SW^2$   
Therefore  $LS^2 - SW^2 = SK^2$ ,  
and accordingly  $(LS + SW) LW = SK^2$ ,  
whence  $\frac{1}{2}(LS + SW) \frac{1}{2}SK = SK LW$

And  $\frac{1}{2}(LS + SW)$  is the distance of the centre of gravity of  $LW$  from  $S$ , while  $\frac{1}{2}SK$  is the distance of the centre of gravity of  $SK$  from  $S$

Therefore  $SK$  and  $LW$ , in the places where they are, balance about  $S$

Similarly for the corresponding parallelograms

Taking all the parallelogrammic elements in the half cylinder and prism respectively, we find that the half cylinder  $PQR$  and the prism  $GHI$ , in the places where they are respectively, balance about  $H$

From this result and that of Prop 11 we can at once deduce the volume of the portion cut off from the cylinder For in Prop 11 the portion of the cylinder, placed with its centre of gravity at  $O$ , is shown to balance (about  $H$ ) the half cylinder in the place where it is By Prop 12 we may substitute for the half-cylinder in the place where it is the prism  $GHI$  of that proposition turned the opposite way relatively to  $RP$  The centre of gravity of the prism as thus placed is at a point  $G$  such that  $HO : HG :: 2 : 1$

therefore (portion of cylinder)  $= \frac{2}{3}$ (prism  $GHI$ )  
 $= \frac{2}{3}$ (original prism)

### PROPOSITION 13

Let there be a right prism with square bases one of which is  $ABCD$ , in the prism let a cylinder be inscribed the base of which is the circle  $EFGH$  touching the sides of the square  $ABCD$  in  $E$ ,  $F$ ,  $G$ ,  $H$

Through the centre and through the side corresponding to  $CD$  in the square

face *opposite* to  $ABCD$  let a plane be drawn, this will cut off a prism equal to  $\frac{1}{2}$  of the original prism and formed by three parallelograms and two triangles the triangles forming opposite faces

In the semicircle  $EFG$  describe the parabola which has  $FK$  for axis and passes through  $E, G$ , draw  $MN$  parallel to  $KF$  meeting  $GE$  in  $M$ , the parabola in  $L$ , the semicircle in  $O$  and  $CD$  in  $N$

Then  $MN \cdot NL = NF^2$ ,

"for this is clear"

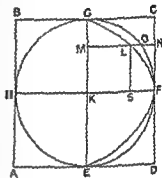
[Cf Apollonius, *Conics* : 11]

[The parameter is of course equal to  $GK$  or  $KF$ ]

Therefore  $MN \cdot NL = GK^2 \cdot LS^2$

Through  $MN$  draw a plane at right angles to  $EG$ ,

this will produce as sections (1) in the prism cut off from the whole prism a right-angled triangle, the base of which is  $MN$ , while the perpendicular is perpendicular at  $N$  to the plane  $ABCD$  and equal to the axis of the cylinder, and the hypotenuse is in the plane cutting the cylinder, and (2) in the portion of the cylinder cut off a right-angled triangle the base of which is  $MO$  while the perpendicular is the generator of the cylinder perpendicular at  $O$  to the plane  $KN$ , and the hypotenuse



[There is a lacuna here to be supplied as follows

Since  $MN \cdot NL = GK^2 \cdot LS^2$

$$= MN^2 \cdot LS^2,$$

it follows that

$$MN \cdot ML = MN^2 \cdot (MN^2 - LS^2)$$

$$= MN^2 \cdot (MN^2 - MK^2)$$

$$= MN^2 \cdot MO^2$$

But the triangle (1) in the prism is to the triangle (2) in the portion of the cylinder in the ratio of  $MN^2 \cdot MO^2$

Therefore  $(\Delta \text{ in prism}) : (\Delta \text{ in portion of cylinder})$

$$= MN : ML$$

$$= (\text{straight line in rect } DG) : (\text{straight line in parabola})$$

We now take all the corresponding elements in the prism, the portion of the cylinder the rectangle  $DG$  and the parabola  $EFG$  respectively], and it will follow that

$$(\text{all the } \Delta \text{ s in prism}) : (\text{all the } \Delta \text{ s in portion of cylinder})$$

$$= (\text{all the str lines in } \square DG) : (\text{all the straight lines between parabola and } EG)$$

But the prism is made up of the triangles in the prism, [the portion of the cylinder is made up of the triangles in it] the parallelogram  $DG$  of the straight lines in it parallel to  $KF$ , and the parabolic segment of the straight lines parallel to  $KF$  intercepted between its circumference and  $EG$ ,

therefore

$$(\text{prism}) : (\text{portion of cylinder})$$

area of Parabola]

Therefore

$$\text{prism} = \frac{1}{2}(\text{portion of cylinder})$$

If then we denote the portion of the cylinder by 2, the prism is 3, and the original prism circumscribing the cylinder is 12 (being 4 times the other prism), therefore the portion of the cylinder =  $\frac{1}{4}$ (original prism)  $\square E D$

[The above proposition and the next are peculiarly interesting for the fact that the parabola is an auxiliary curve introduced for the sole purpose of analytically reducing the required cubature to the known quadrature of the parabola]

#### PROPOSITION 14

Let there be a right prism with square bases [and a cylinder inscribed therein having its base in the square  $ABCD$  and touching its sides at  $E, F, G, H$ , let the cylinder be cut by a plane through  $EG$  and the side corresponding to  $CD$  in the square face opposite to  $ABCD$ ]

This plane cuts off from the prism a prism, and from the cylinder a portion of it

It can be proved that the portion of the cylinder cut off by the plane is  $\frac{1}{2}$  of the whole prism

But we will first prove that it is possible to inscribe in the portion cut off from the cylinder, and to circumscribe about it, solid figures made up of prisms which have equal height and similar triangular bases in such a way that the circumscribed figure exceeds the inscribed by less than any assigned magnitude

But it was proved that  
(prism cut off by  
Now

" " " " "

therefore  $\square DG < \frac{1}{2}(\square \text{ in parabolic segment})$   
which is impossible, since 'it has been proved elsewhere' that the parallelogram  $DG$  is  $\frac{1}{2}$  of the parabolic segment

Consequently

not greater

And

(all the prisms in prism cut off)

(all prisms in circumscrib figure)

= (all  $\square$ s in  $\square DG$ ) (all  $\square$ s in fig circumscrib about parabolic segmt),  
therefore

(prism cut off) (figure circumscrib about portion of cylinder)

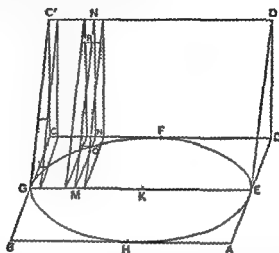
= ( $\square DG$ ) . (figure circumscrib about parabolic segment)

But the prism cut off by the oblique plane is  $> \frac{1}{2}$  of the solid figure circumscribed about the portion of the cylinder

[There are large gaps in the exposition of this geometrical proof, but the way in which the method of exhaustion was applied and the parallelism between this and other applications of it, are clear The first fragment shows that solid figures made up of prisms were circumscribed and inscribed to the portion of the cylinder The parallel triangular faces of these prisms were perpendicular to  $GE$  in the figure of Prop 13, they divided  $GE$  into equal portions of the requisite smallness, each section of the portion of the cylinder by such a

plane was a triangular face common to an inscribed and a circumscribed right prism. The planes also produced prisms in the prism cut off by the same oblique plane as cuts off the portion of the cylinder and standing on  $GD$  as base.

The number of parts into which the parallel planes divided  $GE$  was made great enough to secure that the circumscribed figure exceeded the inscribed figure by less than a small assigned magnitude.



The second part of the proof began with the assumption that the portion of the cylinder is  $> \frac{1}{2}$  of the prism cut off and this was proved to be impossible, by means of the use of the auxiliary parabola and the proportion

$$MN \cdot ML = MN^2 \cdot MO^2$$

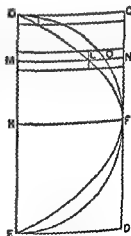
which are employed in Prop. 13.

We may supply the missing proof as follows:

In the accompanying figure are represented (1) the first element-prism circumscribed to the portion of the cylinder (2) two element prisms adjacent to the ordinate  $OM$  of which that on the left is circumscribed and that on the right (equal to the other) inscribed (3) the corresponding element-prisms forming part of the prism cut off ( $CC'GEDD'$ ) which is  $\frac{1}{2}$  of the original prism.

In the second figure are shown element rectangles circumscribed and inscribed to the auxiliary parabola which rectangles correspond exactly to the circumscribed and inscribed element-prisms represented in the first figure (the length of  $GM$  is the same in both figures and the breadths of the element-rectangles are the same as the heights of the element-prisms) the corresponding element-rectangles forming part of the rectangle  $GD$  are similarly shown.

For convenience we suppose that  $GE$  is divided into an even number of equal parts so that  $GA$  contains an integral number of these parts.



For the sake of brevity we will call each of the two element-prisms of which  $OM$  is an edge "el prism ( $O$ )" and each of the element-prisms of which  $MNN'$  is a common face "el prism ( $N$ )". Similarly we will use the corresponding abbreviations "el rect ( $L$ )" and "el rect ( $N$ )" for the corresponding elements in relation to the auxiliary parabola as shown in the second figure

Now it is easy to see that the figure made up of all the inscribed prisms is less than the figure made up of the circumscribed prisms by twice the final circumscribed prism adjacent to  $FK$ , i.e. by twice "el prism ( $N$ )", and, as the height of this prism may be made as small as we please by dividing  $GK$  into sufficiently small parts it follows that inscribed and circumscribed solid figures made up of element prisms can be drawn differing by less than any assigned solid figure

or

such that

$$\begin{aligned} (\text{circumscr fig}) - (\text{inscr fig}) &< \mathcal{V} \\ (\text{inscr fig}) &> (\text{circumscr fig} - \mathcal{V}), \\ &> (\text{portion of cyl} - X) \end{aligned}$$

Therefore  
and *a fortiori*

It follows that

$$(\text{prism cut off}) < \frac{2}{3}(\text{inscribed figure})$$

the prism cut off and those in the

$$\begin{aligned} &IN^2 \quad MO^2 \\ &= MN \quad ML \quad [\text{as in Prop 13}] \\ &= \text{el rect } (N) \quad \text{el rect } (L) \end{aligned}$$

It follows that

$\Sigma\{\text{el prism } (N)\} - \Sigma\{\text{el prism } (O)\} = \Sigma\{\text{el rect } (N)\} - \Sigma\{\text{el rect } (L)\}$   
(There are really two more prisms and rectangles in the first and third than there are in the second and fourth terms respectively, but this makes no difference because the first and third terms may be multiplied by a common factor  $n/(n-2)$  without affecting the truth of the proportion. Cf the proposition from *On Conoids and Spheroids* quoted on p 571 above.)

Therefore

$$\begin{aligned} (\text{prism cut off}) &(\text{figure inscr in portion of cyl}) \\ &= (\text{rect } GD) \quad (\text{fig inscr in parabola}) \end{aligned}$$

But it was proved above that

$$\begin{aligned} (\text{prism cut off}) &< \frac{2}{3}(\text{fig inscr in portion of cyl}), \\ \text{therefore } (\text{rect } GD) &< \frac{2}{3}(\text{fig inscr in parabola}), \\ \text{and a fortiori } (\text{rect } GD) &< \frac{2}{3}(\text{parabolic segmt}) \end{aligned}$$

which is impossible, since

$$(\text{rect } GD) = \frac{2}{3}(\text{parabolic segmt})$$

Therefore (portion of cyl) is not greater than  $\frac{2}{3}$ (prism cut off)

(2) In the second lacuna must have come the beginning of the next *reductio*



(prism cut off)  $> \frac{3}{2}$ (portion of cylinder),

and  $\square$  circumscribe and inscribe figures made up of element-prisms, such that

(prism cut off)  $> \frac{3}{2}$ (fig circumscribed about portion of cyl)

We now consider the element-prisms in the prism cut off and in the circumscribed figure respectively, and the same argument as above gives

(prism cut off) (fig circumscribed about portion of cyl)

$=$  (rect  $GD$ ) (fig circumscribed about parabola),

whence it follows that

(rect  $GD$ )  $> \frac{3}{2}$ (fig circumscribed about parabola),

and, *a fortiori*

(rect  $GD$ )  $> \frac{3}{2}$ (parabolic segment)

which is impossible since

(rect  $GD$ )  $= \frac{3}{2}$ (parabolic segmt)

Therefore

(portion of cyl) is not less than  $\frac{3}{2}$ (prism cut off)

But it was also proved that neither is it greater,

therefore (portion of cyl)  $= \frac{3}{2}$ (prism cut off)  
 $= \frac{3}{2}$ (original prism) ]

### [PROPOSITION 15]

[This proposition, which is lost would be the mechanical investigation of the second of the two special problems mentioned in the preface to the treatise, namely that of the cubature of the figure included between two cylinders each of which is inscribed in one and the same cube so that its opposite bases are in two opposite faces of the cube and its surface touches the other four faces.

Zeuthen has shown how the mechanical method can be applied to this case

In the accompanying figure  $VIVYX$  is a section of the cube by a plane (that of the paper) passing through the axis  $BD$  of one of the cylinders inscribed in the cube and parallel to two opposite faces

The same plane gives the circle  $ABCD$  as the section of the other inscribed cylinder with axis perpendicular to the plane of the paper and extending on each side of the plane to a distance equal to the radius of the circle or half the side of the cube

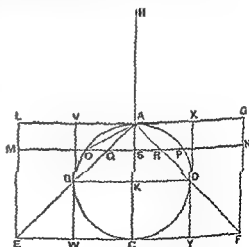
$AC$  is the diameter of the circle which is perpendicular to  $BD$

Join  $AB$   $AD$  and produce them to meet the tangent at  $C$  to the circle in  $E$ ,  $F$

Then  $EC = CF = CA$

Let  $LG$  be the tangent at  $A$  and complete the rectangle  $EFGL$

Draw straight lines from  $A$  to the four corners of the section in which the plane through  $BD$  perpendicular to  $AK$  cuts the cube. These straight lines if



vertex and the latter square for base

Complete the prism (parallelepiped) with the same base and height as the pyramid

Draw in the parallelogram  $LF$  any straight line  $MN$  parallel to  $EF$ , and through  $MN$  draw a plane at right angles to  $AC$

This plane cuts—

- (1) the solid included by the two cylinders in a square with side equal to  $OP$ ,
- (2) the prism in a square with side equal to  $MN$ , and
- (3) the pyramid in a square with side equal to  $QR$

Produce  $CA$  to  $H$ , making  $HA$  equal to  $AC$ , and imagine  $HC$  to be the bar of a balance

$$\begin{aligned}\text{Now, as in Prop 2, since } MS &= AC, QS = AS, \\ MS \cdot SQ &= CA \cdot AS \\ &= AO^2 \\ &= OS^2 + SQ^2\end{aligned}$$

$$\begin{aligned}\text{Also } HA \cdot AS &= CA \cdot AS \\ &= MS \cdot SQ \\ &= MS^2 \cdot MS \cdot SQ \\ &= MS^2 (OS^2 + SQ^2), \text{ from above,} \\ &= MN^2 (OP^2 + QR^2) \\ &= (\text{square, side } MN) (\text{sq, side } OP + \text{sq, side } QR)\end{aligned}$$

Therefore the square with side equal to  $MN$ , in the place where it is, is in equilibrium about  $A$  with the squares with sides equal to  $OP$ ,  $QR$  respectively

two cylinders and

It follows that

$$\begin{aligned}(\text{solid included by cylinders}) &= \frac{1}{3}(\text{prism}) \\ &= \frac{1}{3}(\text{cube})\end{aligned}$$

Q E D

There is no doubt that Archimedes proceeded to and completed, the rigorous geometrical proof by the method of exhaustion

As observed by Prof C Juel (Zeuthen *l c*), the solid in the present proposition is made up of 8 pieces of cylinders of the type of that treated in the preceding proposition. As however the two propositions are separately stated, there is no doubt that Archimedes' proofs of them were distinct

In this case  $AC$  would be divided into a very large number of equal parts and planes would be drawn through the points of division perpendicular to  $AC$ . These planes cut the solid, and also the cube  $VY$ , in square sections. Thus we can inscribe and circumscribe to the solid the requisite solid figures made up of element prisms and differing by less than any assigned solid magnitude, the

prisms have square bases and their heights are the small segments of  $AC$ . The element-prism in the inscribed and circumscribed figures which has the square equal to  $OP^2$  for base corresponds to an element-prism in the cube which has for base a square with side equal to that of the cube, and as the ratio of the element-prisms is the ratio  $OS^2 : BK^2$ , we can use the same auxiliary parabola, and work out the proof in exactly the same way, as in Prop. 14.]

**CONICS**



# BIOGRAPHICAL NOTE

## APOLLONIUS c 262-c 200 B C

APOLLONIUS was born at Perga in Pamphylia Asia Minor some twenty five years after the birth of Archimedes which would place his birth around the year 262 B C He seems to have gone when quite young to Alexandria where according to Pappus the fourth century mathematician he was attracted by the reputation of the astronomer Aristarchus of Samos Apollonius studied under the successors of Euclid at Alexandria and continued to reside there during the reigns of Ptolemy Evergetes and of Ptolemy Philopator (247-203 B C) He was also for some time in Pergamum where he made the acquaintance of the mathematician Eudemus to whom he dedicated the first three books of his *Conics* and of King Attalus I (269-197 B C) to whom the remaining five books of the *Conics* were dedicated

Apollonius appears to have been associated with the leading mathematicians of his day In the dedicatory epistles of the *Conics* he records that he met Philonides while on a trip to Ephesus and that he undertook the composition of

more advanced investigations (Books V-VIII)

*Loci* He wrote on irrationals and like Archimedes devised a system of multiplication for counting large numbers and calculated an approximate value for the ratio of the circumference of a circle to the diameter The ancient writers also record that Apollonius wrote *On the Burning Glass* in which he probably treated the properties of the parabola a work comparing the dodecahedron

emy with an explanation of the motion of the planets by means of epicycles and eccentric circles. He seems to have been especially interested in the theory of the moon, and the Alexandrians are said to have called him Epsilon from the resemblance of that Greek letter to the lunar crescent.

## CONTENTS

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## TRANSLATOR'S NOTE

If on first appearance this treatise should seem to the reader a jumble of propositions, rigorous indeed, but without much rhyme or reason in their sequence, then he can be sure he has not read aright, and as with the planets, he must look further to save the appearances. There are one or two hypotheses at least that can order the apparent wanderings of parabolas, hyperbolas, and ellipses through the first four books. Such hypotheses are the analogies between the three sections, and especially the development of the analogy between the hyperbola and the ellipse reaching its culmination, in the first book, with the final theorem, the construction of conjugate opposite sections.

In First Definitions I 5, Apollonius innocently defines two kinds of diameters, the transverse and the upright. Each one, in a conic section, bisects all the straight lines parallel to the other. But the upright diameter, defined here only as to position, has, in the case of the ellipse, natural bounds fixed by the section itself, and in Proposition I 15 we find it is the mean proportional between the corresponding transverse diameter (or conjugate diameter) and its parameter. The transverse diameter, in turn, is the mean proportional between the upright diameter (or conjugate) and its parameter, so "upright" and "transverse" become meaningless terms, in the case of the ellipse, for something better expressed by the symmetrical relation "conjugate" (First Def I 6). Immediately, in Proposition I 16, as if arbitrarily, the upright diameter of the hyperbola is bounded in the same way, given a definite magnitude, and becomes "the second diameter." But so far transverse and upright diameters, or transverse and second diameters, are distinct things in the case of the hyperbola, and there seems to be little reason for giving this second diameter in magnitude *ρωμος* has not yet become *φωρις*. That the upright diameter should be given even in position for the hyperbola becomes only very significant with two pairs of propositions—Propositions I 37 and 38, and I 39 and 40—where it is shown that certain properties holding for ordinates to the transverse diameter of the hyperbola and ellipse hold also for the ordinates to their conjugates. But it is only with the final proposition of the first book (I 60) that the magnitude of the hyperbola's second diameter is justified in magnitude as well as position. It is the corresponding diameter of the opposite sections conjugate to the first. And this analogy between the hyperbola

and ellipse now stands on the threshold of a vast development. For this theorem, coming as a climax to the first book, makes possible the main theme of the second book: the asymptotes, those strange lines all but touching each opposite section (II 2, 13, 14) and forming a single bound between each adjacent pair (II 15, 17), so making the hyperbola an all but closed section, a puckered ellipse, a mouth turned inside out. And in the third book, the fruits of this analogy are gathered as in the especially nice case of Proposition III 15.

Although this translation is literal, we have not hesitated to use such symbols and abbreviations as, without prejudicing any Greek number theory or introducing any modern theory of symbols, would yet make the reading and the mechanic of study easier and at the same time preserve all the rigor of Greek mathematics.

As for the Greek text, we have used Heiberg, and have constantly referred to the *editio princeps* of Halley. In certain instances we have been glad to consult the very excellent French translation of Paul Ver Eecke (Desclée de Brouwer, Bruges, 1923). We have also deferred, at all relevant points, to the English usage of T. L. Heath's translation of Euclid's *Elements*.

## EXAMPLES OF ABBREVIATIONS AND SYMBOLS USED

- $A=B$  for  $A$  is equal to  $B$   
 $A+B$  for  $A$  added to  $B$   
 $A-B$  for  $B$  subtracted from  $A$   
 $A \ B \ C \ D$  for  $A$  is to  $B$  as  $C$  is to  $D$   
 rect  $AB, BC$  for rectangle  $AB, BC$   
 sq  $AB$  for square on  $AB$   
 ar for area  
 pll $g$  for parallelogram  
 trgl for triangle  
 quadr for quadrilateral  
 rect  $AB, BC$  rect  $CD \ DE$  comp  $AB \ CD, BC \ DE$   
     for ratio of rectangle  $AB, BC$  to rectangle  $CD, DE$  is  
     compounded of the ratio of  $AB$  to  $CD$  and of  $BC$  to  
      $DE$   
 ratio comp  $AB \ BC, CD \ DE$  = ratio comp  $XY \ YZ,$   
      $ZW \ WV$  for ratio compounded of  $AB$  to  $BC$  and of  
      $CD$  to  $DE$  is the same as the ratio compounded of  $XY$   
     to  $YZ$  and of  $ZW$  to  $WV$   
 $A > B$  for  $A$  is greater than  $B$   
 $A < B$  for  $A$  is less than  $B$   
 rt angle for right angle



## BOOK ONE

APOLLONIUS to EUDÆMUS, greetings

If you are restored in body, and other things go with you to your mind well and good, and we too fare pretty well. At the time I was with you in Pergamum, I observed you were quite eager to be kept informed of the work I was doing in conics. And so I have sent you this first book revised, and we shall dispatch the others when we are satisfied with them. For I don't believe you have forgotten hearing from me how I worked out the plan for these conics at the request of Naucrates, the geometer at the time he was with us in Alexandria lecturing and how on arranging them in eight books we immediately communicated them in great haste because of his near departure, not revising them but putting down whatever came to us with the intention of a final going over. And so finding now the occasion of correcting them, one book after another, we publish them. And since it happened that some others among those frequenting us got acquainted with the first and second books before the revision, don't be surprised if you come upon them in a different form.

Of the eight books the first four belong to a course in the elements. The first book contains the generation of the three sections and of the opposite branches, and the principal properties (*τὰ ἀρχικά συμπεράσματα*) in them worked out more fully and universally than in the writings of others. The second book contains the properties (*τὰ συμβαίνοντα*) having to do with the diameters and axes and also the asymptotes and other things of a general and necessary use for limits of possibility (*πρὸς τοὺς διόρισμους*). And what I call diameters and what I call axes you will know from this book. The third book contains many incredible theorems of use for the construction of solid loci and for limits of possibility of which the greatest part and the most beautiful are new. And when we had grasped these we knew that the three-line and four-line locus had not been constructed by Euclid but only a chance part of it and that not very happily. For it was not possible for this construction to be completed without the additional things found by us. The fourth book shows in how many ways the sections of a cone intersect with each other and with the circumference of a circle, and contains other things in addition none of which has been written up by our predecessors that is in how many points the section of a cone or the circumference of a circle and the opposite branches meet the opposite branches. The rest of the books are fuller in treatment. For there is one dealing more fully with maxima and minima and one with equal and similar sections of a cone, and one with limiting theorems and one with determinate conic problems. And so indeed with all of them published those happening upon them can judge them as they see fit. Good bye.

## FIRST DEFINITIONS

1 If from a point a straight line is joined to the circumference of a circle which is not in the same plane with the point, and the line is produced in both directions, and if, with the point remaining fixed, the straight line being rotated about the circumference of the circle returns to the same place from which it began then the generated surface composed of the two surfaces lying vertically opposite one another, each of which increases indefinitely as the generating straight line is produced indefinitely, I call a conic surface, and I call the fixed point the vertex, and the straight line drawn from the vertex to the center of the circle the axis

2 And the figure contained by the circle and by the conic surface between the vertex and the circumference of the circle I call a cone, and the point which is also the vertex of the surface I call the vertex of the cone, and the straight line drawn from the vertex to the center of the circle the axis, and the circle the base of the cone

3 I call right cones those having axes perpendicular to their bases, and oblique those not having axes perpendicular to their bases

4 Of a curve which is situated on a curved line the vertex of the curved line, and I say that each of these parallels is drawn ordinatewise to the diameter

bisects all the straight lines drawn to either of the curved lines parallel to some straight line and I say that each of the parallels is drawn ordinatewise to the diameter

straight lines intercepted between the curved lines and drawn parallel to some straight line and I say that each of the parallels is drawn ordinatewise to the diameter

5 The two straight lines each of which being a diameter bisects the straight lines parallel to the other I call the conjugate diameters (*συζυγείς διαμέτρους*) of a curved line and of two curved lines

7 And I call that straight line the axis of a curved line and of two curved lines which being a diameter of the curved line or lines cuts the parallel straight lines at right angles

to each other at right angles

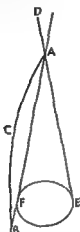
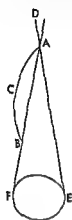
## PROPOSITION 1

The straight lines drawn from the vertex of the conic surface to points on the surface are on that surface

Let there be a conic surface whose vertex is the point A, and let there be

<sup>1</sup>We shall follow modern usage and generally call these parallels ordinates

taken some point  $B$  on the conic surface, and let a straight line  $ACB$  be joined  
I say that the straight line  $ACB$  is on the conic surface



For if possible, let it not be, and let the straight line  $DE$  be the line generating the surface, and  $EF$  be the circle along which  $ED$  is moved. Then if, the point  $A$  remaining fixed, the straight line  $DE$  is moved along the circumference of the circle  $EF$ , it will also go through the point  $B$  (Def 1), and two straight lines will have the same ends. And this is absurd.

Therefore the straight line joined from  $A$  to  $B$  cannot not be on the surface. Therefore it is on the surface.

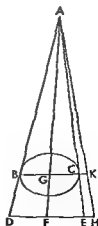
### PORISM

It is also evident that, if a straight line is joined from the vertex to some point among those within the surface it will fall within the conic surface, and if it is joined to some point among those without, it will be outside the surface.

### PROPOSITION 2

If on either one of the two vertically opposite surfaces two points are taken, and the straight line joining the points does not verge to the vertex, then it will fall within the surface, and produced it will fall outside.

Let there be a conic surface whose vertex is the point  $A$ , and a circle  $BC$



along whose circumference the generating straight line is moved, and let two points  $D$  and  $E$  be taken on either one of the two vertically opposite



and let the joining straight line  $DE$  not verge to the point  $A$

I say that the straight line  $DE$  will be within the surface, and produced will be without

Let  $AE$  and  $AD$  be joined and produced. Then they will fall on the circumference of the circle (I. 1). Let them fall to the points  $B$  and  $C$ , and let  $BC$  be joined. Therefore the straight line  $BC$  will be within the circle, and so too within the conic surface.

Then let a point  $F$  be taken at random on  $DE$ , and let the straight line  $AF$  be joined and produced. Then it will fall on the straight line  $BC$ , for the triangle  $BCA$  is in one plane (Eucl. XI. 2). Let it fall to the point  $G$ . Since then the point  $G$  is within the conic surface, therefore the straight line  $AG$  is also within the conic surface (I. 1, porism), and so too the point  $F$  is within the conic surface. Then likewise it will be shown that all the points on the straight line  $DE$  are within the surface. Therefore the straight line  $DE$  is within the surface.

Then let  $DE$  be produced to  $H$ . I say then it will fall outside the conic surface.

For if possible let there be some point  $H$  of it not outside the conic surface and let  $AH$  be joined and produced. Then it will fall either on the circumference of the circle or within (I. 1 and porism). And this is impossible, for it falls on  $BC$  produced as for example to the point  $K$ . Therefore the straight line  $EH$  is outside the surface.

Therefore the straight line  $DE$  is within the conic surface and produced is outside.

### PROPOSITION 3

*If a cone is cut by a plane through the vertex the section is a triangle.*

Let there be a cone whose vertex is the point  $A$  and whose base is the circle  $BC$  and let it be cut by some plane through the point  $A$  and let it make as sections lines  $AB$  and  $AC$  on the surface and the straight line  $BC$  in the base.

I say that  $ABC$  is a triangle.

For since the line joined from  $A$  to  $B$  is the common section of the cutting plane and of the surface of the cone therefore  $AB$  is a straight line. And likewise also  $AC$ . And  $BC$  is also a straight line. Therefore  $ABC$  is a triangle.

If then a cone is cut by some plane through the vertex, the section is a triangle.



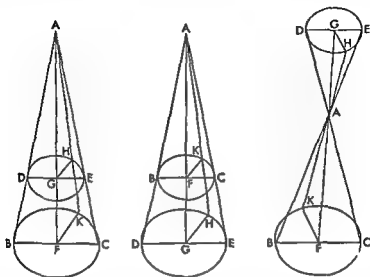
### PROPOSITION 4

*If either one of the vertically opposite surfaces is cut by some plane parallel to the circle along which the straight line generating the surface is moved, the plane cut off within the surface will be a circle having its center on the axis, and the figure contained by the circle and the conic surface intercepted by the cutting plane on the side of the vertex will be a cone.*

Let there be a conic surface whose vertex is the point  $A$  and whose circle along which the straight line generating the surface is moved is  $BC$  and let it be cut by some plane parallel to the circle  $BC$  and let it make on the surface as a section the line  $DE$ .

I say that the line  $DE$  is a circle having its center on the axis

For let the point  $F$  be taken as the center of the circle  $BC$ , and let  $AF$  be joined. Therefore  $AF$  is the axis (Def. 1) and meets the cutting plane. Let it meet it at the point  $G$ , and let some plane be produced through  $AF$ . Then the



section will be the triangle  $ABC$  (1. 3). And since the points  $D, G, E$  are points in the cutting plane, and are also in the plane of the triangle  $ABC$ , therefore  $DGE$  is a straight line (Eucl. xi. 3).

Then let some point  $H$  be taken on the line  $DE$ , and let  $AH$  be joined and produced. Then it falls on the circumference  $BC$  (1. 1). Let it meet it at  $K$ , and let  $GH$  and  $FK$  be joined. And since two parallel planes,  $DE$  and  $BC$ , are cut by a plane  $ABC$ , their common sections are parallel (Eucl. xi. 16). Therefore the straight line  $DE$  is parallel to the straight line  $BC$ . Then for the same reason the straight line  $GH$  is also parallel to the straight line  $KF$ . Therefore

$$FA : AG = FB : DG = FC : GE = FK : GH \text{ (Eucl. vi. 4)}$$

And  $BF = KF = FC$

Therefore also  $DG = GH = GE$  (Eucl. v. 9)

Then likewise we could show also that all the straight lines falling from the point  $G$  on the line  $DE$  are equal to each other.

Therefore the line  $DE$  is a circle having its center on the axis.

And it is evident that the figure contained by the circle  $DE$  and the conic surface cut off by it on the side of the point  $A$  is a cone.

And it is therewith proved that the common section of the cutting plane and of the axial triangle (triangle through the axis) is a diameter of the circle.

#### PROPOSITION 5

If an oblique cone is cut by a plane through the axis at right angles to the base, and is also cut by another plane on the one hand at right angles to the axial, and on the other cutting off on the side of the vertex a triangle similar to the

triangle and lying subcontrariwise then the section is a circle and let such a section be called *subcontrary*

Let there be an oblique cone whose vertex is the point *A* and whose base is the circle *BC* and let it be cut by a plane through the axis perpendicular to the circle *BC* and let it make as a section the triangle *ABC* (1 3) Then let it also be cut by another plane perpendicular to the triangle *ABC* and cutting off on the side of the point *A* the triangle *AHG* similar to the triangle *ABC* and lying subcontrariwise that is so that the angle *AHG* is equal to the angle *ABC* And let it make as a section on the surface the line *GHA*

I say that the line *GHA* is a circle

For let any points *H* and *L* be taken on the lines *GHA* and *BC* and from the points *H* and *L* let perpendiculars be dropped to the plane through the triangle *ABC* Then they will fall to the common sections of the planes (Eucl xi def 6) Let them fall as for example *FH* and *LM* Therefore *FH* is parallel to *LM* (Eucl xi 6)

Then let the straight line *DE* be drawn in the base of the cone (Eucl xi 4) is also parallel to *LM* the base of the cone (Eucl xi 4)

Therefore

rect *DF FE* = sq *FH* (Eucl iii 31 and vi 8 porism)

And since *ED* is parallel to *BC* angle *ADE* is equal to angle *ABC* And angle *AHG* is supposed equal to angle *ABC* And therefore angle *AKG* is equal to angle *ADE* And the vertical angles at the point *F* are also equal Therefore triangle *DFG* is similar to triangle *KFE* and therefore

*EF FK* = *GF FD* (Eucl vi 4)

Therefore

rect *EF FD* = rect *KF FG* (Eucl vi 16)

But it has been shown that

sq *FH* = rect *EF FD*

and therefore

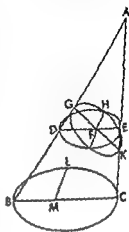
rect *KF FC* = sq *FH*

Likewise then all the perpendiculars drawn from the line *GHA* to the straight line *GA* could also be shown to be equal in square to the rectangle in each case contained by the segments of the straight line *GK*

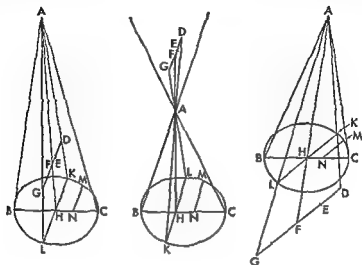
Therefore the section is a circle whose diameter is the straight line *GA*

#### PROPOSITION 6

If a cone is cut by a plane through the axis and some point is taken on the surface of the cone which is not on a side of the axial triangle and from it is drawn a straight line parallel to some straight line which is a perpendicular from the circumference of the circle to the base of the triangle then it meets the axial triangle and on being produced to the other side of the surface it will be bisected by the triangle



Let there be a cone whose vertex is the point  $A$  and whose base is the circle  $BC$ , and let the cone be cut by a plane through the axis, and let it make a common section the triangle  $ABC$  (i 3), and from some point  $M$  of those on the



circumference, let the straight line  $MN$  be drawn perpendicular to the straight line  $BC$ . Then let some point  $D$  be taken on the surface of the cone, and let

$A$   
its surface, will be bisected by the triangle  $ABC$ ,

it will meet  
in  $K$  let the  
Therefore

Let the straight line  $AH$  be joined from  $A$  to  $H$ . Since then in the triangle  $AHK$  the straight line  $DE$  is parallel to the straight line  $HK$ , therefore  $DE$  produced will meet  $AH$ . But  $AH$  is in the plane of  $ABC$ , therefore  $DE$  will meet the plane of the triangle  $ABC$ .

For the same reasons it also meets  $AH$ , let it meet it at  $F$ , and let  $DF$  be produced in a straight line until it meet the surface of the cone. Let it meet it at  $G$ .

I say that  $DF$  is equal to  $FG$ .

For since  $A, G, L$  are points on the surface of the cone, but also in the plane extended through the straight lines  $AH, AK, DG, KL$ , which is a triangle

drawn across them from the point  $A$ , therefore

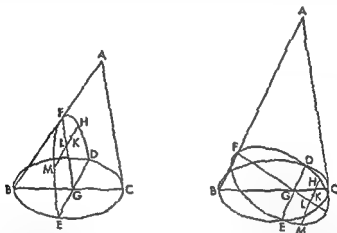
$KH = HL = DF = FG$  (Eucl vi 2)

But  $KH$  is equal to  $HL$ , since  $KL$  is a chord in circle  $BC$  perpendicular to the diameter (Eucl III 3) Therefore  $DF$  is equal to  $FG$ .

## PROPOSITION 7

*If a cone is cut by a plane through the axis, and if it is also cut by another plane cutting the plane the base of the cone is in, in a straight line perpendicular either to the axis or to the diameter of the base, then the straight lines drawn from the vertex to the points where the cutting plane intersects the surface of the cone, parallel to the cutting plane, fall on the common section of the cutting plane and of the axial triangle, and further produced to the other side of the section, are bisected by the common section, and if it is a right cone the straight line in the base will be perpendicular to the common section of the cutting plane and of the axial triangle, and if oblique, it will not always be perpendicular, but whenever the plane through the axis is perpendicular to the base of the cone*

Let there be a cone whose vertex is the point  $A$  and whose base is the circle  $BC$ , and let it be cut by a plane through the axis and let it make as a section



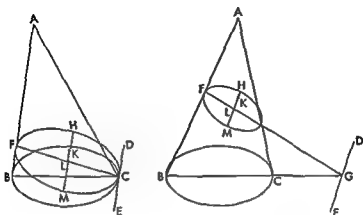
the triangle  $ABC$  (1 3) And let it also be cut by another plane cutting the plane the circle  $BC$  is in, in the straight line  $DE$  perpendicular either to the

straight line  $DE$

I say that the straight line  $HA$  meets the straight line  $FG$ , and, on being produced to the other side of the section  $DFE$ , will be bisected by  $FG$

For since a cone whose vertex is the point  $A$  and whose base is the circle  $BC$  has been cut by a plane through its axis, and makes as a section the triangle  $ABC$ , and some point  $H$  on the surface, not on a side of the triangle  $ABC$ , has been taken, the straight line that is  $HA$ , of the surface, will be bisected by the triangle (1 6)

Then since the straight line drawn through  $H$  parallel to the straight line  $DE$  meets the triangle  $ABC$  and is in the plane of the section  $DFE$ , therefore



—  $FG$  — the straight line and of the triangle  $ABC$

Therefore the

and, if further

produced to the other side of the section  $DFE$ , will be bisected by the straight line  $FG$

Then either the cone is a right cone, or the axial triangle  $ABC$  is perpendicular to the circle  $BC$ , or neither

First let the cone be a right cone. Then the triangle  $ABC$  would be perpendicular to the circle  $BC$  (Def 3, Eucl xi 18). Since then the plane  $ABC$  is

to  $FG$

Then let the axial triangle  $ABC$  not be perpendicular to the circle  $BC$  — I say that  $DE$  is not perpendicular to  $FG$ . For if possible, let it be. And it is also perpendicular to the circle  $BC$ . Therefore  $DE$  is perpendicular to both

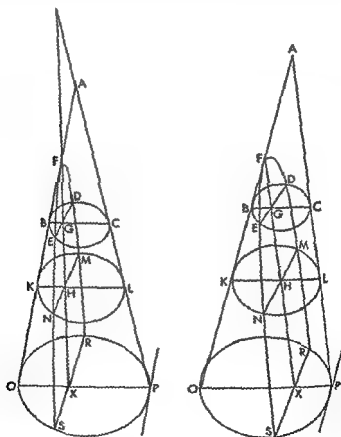
line  $FG$

## PORISMA

Then from this it is evident that the straight line  $FG$  is the diameter of the section  $DFE$ , since it bisects the straight lines drawn parallel to some straight line  $DE$ , and that it is possible for some parallels to be bisected by the diameter  $FG$  and not be perpendicular

## PROPOSITION 8

If a cone is cut by a plane through its axis, and is cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the resulting section on the surface is either parallel to one of the sides of the triangle or meets one of them beyond the vertex of the cone, and the surface of the cone and the cutting plane are produced indefinitely, then the section will also increase indefinitely, and some straight line drawn from the section of the cone parallel to the straight line in the base of the cone will cut off from the diameter on the side of the vertex a straight line equal to any given straight line



Let there be a cone whose vertex is the point  $A$  and whose base is the circle  $BC$ , and let it be cut by a plane through its axis, and let it make as a section the triangle  $ABC$  (1 3). And let it be cut also by another plane cutting the

circle  $BC$  in a straight line  $DE$  perpendicular to the straight line  $BC$ , and let it make as a section on the surface the line  $DFE$ . And let the diameter  $FG$  of the section  $DFE$  be either parallel to the straight line  $AC$  or on being produced meet it beyond the point  $A$  (I 7 and porism)

Since the straight line  $FG$  is either parallel to  $AC$  or produced meets it beyond the point  $A$ , therefore the straight lines  $FG$  and  $AC$  on being produced in the direction of  $C$  and  $G$  will never meet. Then let them be produced and let some point  $H$  be taken at random on the straight line  $FG$ , and let the straight line  $KHL$  be drawn through the point  $H$  parallel to the straight line  $BC$ , and  $MHN$  parallel to  $DE$ . Therefore the plane through  $KL$  and  $MN$  is parallel to the plane through  $BC$  and  $DE$  (Eucl XI 15). Therefore the plane  $KLMN$  is a circle (I 4).

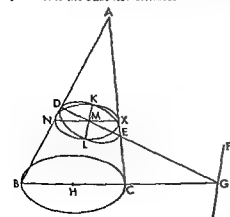
And since the points  $D, E, M, N$  are in the cutting plane and also on the surface of the cone, therefore they are in the common section. Therefore the section  $DFE$  has increased to the points  $M$  and  $N$ . Therefore, with the surface of the cone and the cutting plane increased to the circle  $KLMN$ , the section  $DFE$  has also increased to the points  $M$  and  $N$ . Then likewise we could show also, that if the surface of the cone and the cutting plane are extended indefinitely the section  $MDFEN$  will also increase indefinitely.

And it is evident that some straight line will cut off on straight line  $FH$  on

straight line is drawn meeting the section parallel to  $DE$ , and cutting off on  $FG$  on the side of point  $H$  a straight line equal to the given straight line

### PROPOSITION 9

If a cone is cut by a plane meeting both sides of the axial triangle, and neither parallel to the base nor situated subcontrariwise, then the section will not be a circle



Let there be a cone whose vertex is the point  $A$  and whose base is the circle  $BC$ , and let it be cut by some plane neither parallel to the base nor situated subcontrariwise, and let it make as a section on the surface the line  $DKE$ .

I say that the line  $DKE$  will not be a circle.

For if possible let it be, and let the cutting plane meet the base, and let the straight line  $FG$  be the common section of the planes, and let the point  $H$  be the center of the circle  $BC$ , and let the straight line  $HG$  be



drawn from it perpendicular to the straight line  $FG$ . And let  $\pi$  plane be ex-

therefore  $D, E, G$  are points on the common section of the planes. Therefore  $GED$  is a straight line (Eucl xi 3)

Then let some point  $K$  be taken on the line  $DKE$ , and through  $K$  let the

lines  $BC$  and  $FG$ , that is to the base (Eucl xi 15), and the section will be a circle (I 4). Let it be the circle  $NKX$ .

And since the straight line  $FG$  is perpendicular to the straight line  $BG$ , the straight line  $KM$  is also perpendicular to the straight line  $NX$  (Eucl vi 10). And so

$$\text{rect } NM, MX = \text{sq } KM \text{ (Eucl iii 31, vi 8, porism)}$$

But

$$\text{rect } DM, ME = \text{sq } KM,$$

for the line  $DKE$  is supposed a circle, and the straight line  $DE$  is its diameter. Therefore

$$\text{rect } NM, MX = \text{rect } DM, ME$$

Therefore

$$MN \cdot MD = EM \cdot MX \text{ (Eucl vi 16)}$$

Therefore triangle  $DMN$  is similar to triangle  $XME$  (Eucl vi 6, vi def 1), and angle  $DNM$  is equal to angle  $MEX$ . But angle  $DNM$  is equal to angle  $ABC$ , for the straight line  $NX$  is parallel to the straight line  $BC$ . And therefore angle  $ABC$  is equal to angle  $MEX$ . Therefore the section is subcontrary (I 5). And this is not supposed. Therefore the line  $DKE$  is not a circle.

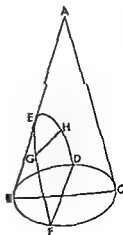
### PROPOSITION 10

If two points are taken on the section of a cone, the straight line joining the two points will fall within the section, and produced in a straight line it will fall outside.

Let there be a cone whose vertex is the point  $A$ , and whose base is the circle  $BC$ , and let it be cut by a plane through the axis, and let it make as a section the triangle  $ABC$  (I 3). Then let it also be cut by another plane, and let it make as a section on the surface of the cone the line  $DEF$ , and let two points  $G$  and  $H$  be taken on the line  $DEF$ .

Let us show that the straight line

For since a cone, whose vertex is the point  $A$  and whose base is the circle  $BC$ , has been cut by a plane through the axis, and some points  $G$  and  $H$  have been taken on its surface which are not on a side of the axial



triangle, and since the straight line joining  $G$  and  $H$  does not verge to the point  $A$ , therefore the straight line joining  $G$  and  $H$  will fall within the cone, and produced in a straight line it will fall outside (1 2), consequently also outside the section  $DFE$

## PROPOSITION 11

If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if further the diameter of the section is parallel to one side of the axial triangle, then any straight line which is drawn from the section of the cone to its diameter parallel to the common section of the cutting plane and of the cone's base, will equal in square the rectangle contained by the straight line cut off by it on the diameter beginning from the section's vertex and by another straight line which has the ratio to the straight line between the angle of the cone and the vertex of the section that the square on the base of the axial triangle has to the rectangle contained by the remaining two sides of the triangle. And let such a section be called a parabola (*παραβολή*).

Let there be a cone whose vertex is the point  $A$ , and whose base is the circle  $BC$ , and let it be cut by a plane through its axis, and let it make as a section the triangle  $ABC$  (1 3). And let it also be cut by another plane cutting the base of the cone in the straight line  $DE$  perpendicular to the straight line  $BC$ , and let it make as a section on the surface of the cone the line  $DFE$ , and let the diameter of the section  $FG$  (1 7, and def 4) be parallel to one side  $AC$  of the axial triangle. And let the straight line  $FH$  be drawn from the point  $F$  perpendicular to the straight line  $FG$ , and let it be contrived that

$$\text{sq } BC \text{ rect } BA, AC = FH \cdot FA$$

And let some point  $K$  be taken at random on the section, and through  $K$  let the straight line  $KL$  be drawn parallel to the straight line  $DE$ .

I say that  $\text{sq } KL = \text{rect } HF, FL$ .

For let the straight line  $MN$  be drawn through  $L$  parallel to the straight line  $BC$ . And the straight line  $DE$  is also parallel to the straight line  $KL$ . Therefore the plane through  $KL$  and  $MN$  is parallel to the plane through  $BC$  and  $DE$  (Eucl xi 15), that is to the base of the cone. Therefore the plane through  $KL$  and  $MN$  is a circle whose diameter is  $MN$  (1 4). And  $KL$  is perpendicular to  $MN$  since  $DE$  is also perpendicular to  $BC$  (Eucl xi 10). Therefore

$$\text{rect } ML, LN = \text{sq } KL \text{ (Eucl iii 31, vi 8, porism)}$$

And since

$$\text{sq } BC \text{ rect } BA, AC = HF \cdot FA,$$

and

$$\text{sq } BC \text{ rect } BA, AC \text{ comp } BC \cdot CA, BC = BA \text{ (Eucl vi 23),}$$

therefore

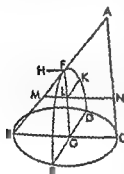
$$HF \cdot FA \text{ comp } BC \cdot CA, BC = BA$$

But

$$BC \cdot CA = MN \cdot NA = ML \cdot LF \text{ (Eucl vi 4),}$$

and

$$BC \cdot BA = MA \cdot MA = LM \cdot MF = NL \cdot FA \text{ (Eucl vi 2)}$$



Therefore

$$HF \cdot FA \text{ comp } ML \cdot LF, NL \cdot FA$$

But

$$\text{rect } ML, LN \text{ rect } LF, FA \text{ comp } ML \cdot LF, LN \cdot FA \text{ (Eucl vi 23)}$$

Therefore

$$HF \cdot FA \text{ rect } ML, LN \text{ rect } LF, FA$$

But, with the straight line  $FL$  taken as common height,

$$HF \cdot FA \text{ rect } HF, FL \text{ rect } LF, FA \text{ (Eucl vi 1),}$$

therefore

$$\text{rect } ML, LN \text{ rect } LF, FA \text{ rect } HF, FL \text{ rect } LF, FA \text{ (Eucl v 11)}$$

Therefore

$$\text{rect } ML, LN = \text{rect } HF, FL \text{ (Eucl v 9)}$$

But

$$\text{rect } ML, LN = \text{sq } AL,$$

therefore also

$$\text{sq } AL = \text{rect } HF, FL$$

And let such a section be called a parabola and let  $HF$  be called the straight line to which the straight lines drawn ordinatewise to the diameter  $FG$  are applied in square ( $\pi\alpha\rho\ \eta\gamma\ \delta\upsilon\tau\alpha\tau\alpha\iota\ \alpha\iota\ \kappa\alpha\tau\alpha\gamma\omicron\mu\epsilon\upsilon\alpha\iota\ \tau\epsilon\tau\alpha\gamma\mu\epsilon\upsilon\omega\varsigma\ \epsilon\pi\iota\ \tau\eta\gamma\ ZH\ \delta\iota\acute{\alpha}\mu\epsilon\tau\epsilon\rho\omega\varsigma$ ), and let it also be called the upright side ( $\delta\epsilon\theta\eta\alpha$ )<sup>1</sup>

### PROPOSITION 12

If a cone is cut by a plane through its axis, and also by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the section produced meets one side of the axial triangle beyond the vertex of the cone, then any straight line which is drawn from the section to its diameter parallel to the common section of the cutting plane and of the cone's base, will equal in square some area applied to a straight line to which the straight line added along the diameter of the section and subtending the exterior angle of the triangle has the ratio that the square on the straight line drawn from the cone's vertex to the triangle's base parallel to the section's diameter has to the rectangle contained by the sections of the base which this straight line makes when drawn, this area having as breadth the straight line cut off on the diameter beginning from the section's vertex by this straight line from the section to the diameter and exceeding ( $\upsilon\pi\epsilon\rho\acute{\epsilon}\lambda\lambda\omicron\upsilon\varsigma$ ) by a figure ( $\epsilon\iota\delta\omicron\varsigma$ ) similar and similarly situated to the rectangle contained by the straight line subtending the exterior angle of the triangle and by the parameter. And let such a section be called an hyperbola ( $\upsilon\pi\epsilon\rho\beta\omicron\lambda\eta$ ).

Let there be a cone whose vertex is the point  $A$  and whose base is the circle  $BC$ , and let it be cut by a plane through its axis and let it make as a section the triangle  $ABC$  (i 3). And let it also be cut by another plane cutting the base of the cone in the straight line  $DE$  perpendicular to  $BC$  the base of the triangle  $ABC$ , and let it make as a section on the surface of the cone the line  $DFE$ , and

<sup>1</sup>The Greek of the phrase  $\delta\iota\acute{\alpha}\mu\epsilon\tau\epsilon\rho\omega\varsigma$  is  $\delta\iota\acute{\alpha}\mu\epsilon\tau\epsilon\rho\omega\varsigma$  as shown ordinatewise

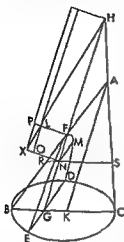
αγώνιος  
ο ή παρ  
we shall  
diameter

The Latin translation of  $\delta\epsilon\theta\eta\alpha$  ( $\pi\lambda\epsilon\upsilon\rho\acute{\epsilon}$ ) is *latus rectum* which has become an English term too

let  $FG$  the diameter of the section (17 and def 4) when produced meet  $AC$  one side of the triangle  $ABC$  beyond the vertex of the cone at the point  $H$ . And let the straight line  $AK$  be drawn through  $A$  parallel to the diameter of the section  $FG$ , and let it cut  $BC$ . And let the straight line  $FL$  be drawn from  $F$  perpendicular to  $FG$ , and let it be contrived that

$$\text{sq } KA \text{ rect } BK, KC \quad FH \quad FL$$

And let some point  $M$  be taken at random on the section and through  $M$  let the straight line  $MN$  be drawn parallel to  $DE$  and through  $N$  let the straight line

to  $FN$ 

I say that  $MN$  is equal in square to the parallelogram  $FX$  which is applied to  $FL$ , having  $FN$  as breadth, and exceeding by a figure  $LX$  similar to the rectangle contained by  $HF$  and  $FL$ .

For let the straight line  $RNS$  be drawn through  $N$  parallel to  $BC$ , and  $NM$  is also parallel to  $DE$ . Therefore the plane through  $MN$  and  $RS$  is parallel to the plane through  $BC$  and  $DE$  that is to the base of the cone (Eucl xi 16). Therefore if the plane is produced through  $MN$  and  $RS$ , the section will be a circle whose diameter is the straight line  $RNS$  (i 4). And  $MN$  is perpendicular to it. Therefore

ject RN NS=sg MN

And since

sq  $AA$  rect  $BK$   $KC$   $FH$   $FL$ .

and

sq  $AA$  rect  $BA$   $AC$  comp  $AK$   $KC$   $AK$   $KB$  (Eucl vi 23),  
therefore also

But

*FH FL comp AK KC AK KB*

and

$AK \perp C \quad HG \perp C \quad HN \perp NS$  (Eucl vi 4).

Therefore

AA KB FG GB GN NR

And

*HF FL comp HN NS FN NR*

Therefore also

$$\text{rect } HN, NF \text{ rect } SN \text{ NR } HF \text{ FL } HN \text{ NX (Eucl vi 4)}$$

But, with the straight line  $FN$  taken as common height

$HN \cdot NX \text{ rect } HN \cdot NF \text{ rect } FN \cdot NX \text{ (Eucl vi 1)}$

Therefore also

$\text{rect } HN \ NF \ \text{rect } SN \ NR \ \text{rect } HN, NF \ \text{rect } XN \ NF \ (\text{Eucl. v. 11})$

Therefore

$\text{rect } SN, NR \sim \text{rect } \backslash N, NF' \text{ (Eucl v 9)}$

But it was shown

sq  $MN = \text{rect } SN, NR.$



ting plane and of the plane the base of the cone is in, be the straight line  $PG$  perpendicular to the  
the straight line  $ED$  (

from  $E$  perpendicular

A parallel to  $ED$ , and let it be contrived that

$$\text{sq } AK = \text{rect } BK, KC$$

And let

drawn

I sa  $LM$  is equal in square to some area which is applied to  $LH$ , having  $EM$  as breadth and deficient by a figure similar to the rectangle contained by  $DE$  and  $EH$

ne hand let the straight  
id on the other let the  
parallel to  $FM$

Since then  $PP$

plane th

to the be

$LM$  and

is perpendicular

And  $LM$

$$\text{rect } PM, MR = \text{sq } LV$$

And since

$$\text{sq } AK = \text{rect } BK, KC = \text{rect } ED, EH,$$

and

$$\text{sq } AK = \text{rect } BK, KC \text{ comp } AK, KB, AK, KC \text{ (Eucl vi 23),}$$

but

$$AK, KB = \text{rect } EG, GB = \text{rect } EM, MP \text{ (Eucl vi 4),}$$

and

$$AK, KC = \text{rect } DG, GC = \text{rect } DV, MR,$$

therefore

$$\text{rect } ED, EH \text{ comp } \text{rect } EM, MP, DV, MR$$

But

$$\text{rect } EM, MD = \text{rect } PM, MR \text{ comp } \text{rect } EM, MP, DV, MR \text{ (Eucl vi 23)}$$

Therefore

$$\text{rect } EM, MD = \text{rect } PM, MR = \text{rect } ED, EH = \text{rect } DV, MY \text{ (Eucl vi 4)}$$

And with the straight line  $ME$  taken as common height,

$$DV, MA = \text{rect } DV, ME = \text{rect } XM, ME \text{ (Eucl vi 1)}$$

Therefore also

$$\text{rect } DV, ME = \text{rect } PM, MR = \text{rect } DV, ME = \text{rect } XM, ME \text{ (Eucl v 11)}$$

Therefore

$$\text{rect } PM, MR = \text{rect } XM, ME \text{ (Eucl v 9)}$$

But it was shown

$$\text{rect } PM, MR = \text{sq } LV,$$

therefore also

$$\text{rect } LV, ME = \text{sq } LV$$

Therefore the straight line  $LV$  is equal in square to the parallelogram  $MO$  which is applied to the straight line  $HE$ , having  $EM$  as breadth and deficient by the figure  $OV$  similar to the rectangle contained by  $DE$  and  $EH$  (Eucl vi 24)

And let such a section be called an ellipse, and let  $EH$  be called the straight line to which the straight lines drawn ordinatewise to  $DE$  are applied in square, and let the same straight line also be called the upright side, and the straight line  $ED$  the transverse side

## PROPOSITION 14

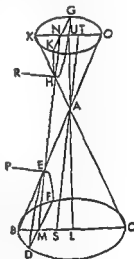
which the straight lines drawn to the diameter parallel to the straight line in the cone's base are applied in square, are equal, and the transverse side of the figure, that between the vertices of the sections, is common. And let such sections be called opposite (*ἀντικείμεναι*)

Let there be the vertically opposite surfaces whose vertex is the point  $A$ , and let them be cut by a plane not through the vertex, and let it make as sections on the surface the lines  $DEF$  and  $GHK$

I say that each of the two sections  $DEF$  and  $GHK$  is the so-called hyperbola

For let there be the circle  $BDCF$  along which the line generating the surface moves, and let the plane  $XGOK$  be extended parallel to it on the vertically opposite surface, and the straight lines  $FD$  and  $GK$  are common sections of the sections  $GHK$  and  $FED$ , and of the circles (1 4). Then they will be parallel (Eucl xi 16). And let the straight line  $LAU$  be the axis of the conic surface, and the points  $L$  and  $U$  be the centers of the circles, and let a straight line drawn from  $L$  perpendicular to the straight line  $FD$  be produced to the points  $B$  and  $C$ , and let a plane be produced through the straight line  $BC$  and the axis. Then it will make as sections in the circles the parallel straight lines  $XO$  and  $BC$  (Eucl xi 16), and on the surface the straight lines  $BAO$  and  $CAX$  (1 1 and Def 4)

Then the straight line  $XO$  will be perpendicular to the straight line  $GK$ , since the straight line  $BC$  is also perpendicular to the straight line  $FD$ , and each of the two is parallel to the other (Eucl xi 10). And since the plane through the axis meets the sections in the points  $M$  and  $N$  within the lines, it is clear that the plane also cuts the lines. Let it cut them at  $H$  and  $E$ , therefore  $M$ ,  $E$ ,  $H$ , and  $N$  are points on the plane through the axis and in the plane the lines are in, therefore the line  $MEHN$  is a straight line (Eucl xi 3). It is also evident both that  $X$ ,  $H$ ,  $A$ , and  $C$  are in a straight line and  $B$ ,  $E$ ,  $A$ , and  $O$  also



$MEHN$ , and let it be contrived that

$$HE \cdot EP = \text{sq } AS = \text{rect } BS, SC,$$

and

$$EH \cdot HR = \text{sq } AT = \text{rect } OT, TY$$

Since then a cone, whose vertex is the point  $A$  and whose base is the circle  $BC$ , has been cut by a plane through its axis, and it has made as a section the triangle  $ABC$ , and it has also been cut by another plane cutting the base of the cone in the straight line  $DMF$  perpendicular to the straight line  $BC$ , and it has made as a section on the surface the line  $DEF$ , and the diameter  $ME$  produced has met one side of the axial triangle beyond the vertex of the cone, and through the point  $A$  the straight line  $AS$  has been drawn parallel to the diameter of the section  $EM$ , and from  $E$  the straight line  $EP$  has been drawn perpendicular to the straight line  $EM$ , and

$$EH \cdot EP \text{ sq } AS \text{ rect } BS, SC,$$

I say that the straight line  $HR$  is equal to the straight line  $EP$ .

For since  $BC$  is parallel to  $XO$ ,

$$AS \cdot SC = AT \cdot TX$$

and

$$AS \cdot SB = AT \cdot TO$$

But

$$\text{sq } AS \text{ rect } BS, SC \text{ comp } AS \cdot SC, AS \cdot SB \text{ (Eucl vi 23)}$$

and

$$\text{sq } AT \text{ rect } XT, TO \text{ comp } AT \cdot TX, AT \cdot TO,$$

therefore

$$\text{sq } AS \text{ rect } BS, SC = \text{sq } AT \text{ rect } XT, TO$$

Also

$$\text{sq } AS \text{ rect } BS, SC = HE \cdot EP,$$

and

$$\text{sq } AT \text{ rect } XT, TO = HE \cdot HR$$

Therefore also

$$HE \cdot EP = HE \cdot HR \text{ (Eucl v 11)}$$

Therefore

$$EP = HR \text{ (Eucl v 9)}$$

#### PROPOSITION 15

If in an ellipse a straight line, drawn ordinatewise from the midpoint of the diameter, is produced both ways to the section, and if it is contrived that, as the straight line

having as breadth the straight line cut off by it beginning from the section and deficient by a figure similar to the rectangle contained by the straight line to which the straight lines are drawn and by the parameter, and if further produced to the other side of the section, will be bisected by the straight line to which it has been drawn

Let there be an ellipse whose diameter is the straight line  $AB$ , and let  $AB$  be bisected at the point  $C$ , and through  $C$  let the straight line  $DCE$  be drawn





Since also the straight line  $DE$  has been cut into equal parts at  $C$ , and into unequal parts at  $H$ , therefore

$$\text{rect } EH, HD + \text{sq } CH = \text{sq } CD \text{ (Eucl II 5),}$$

or

$$\text{rect } EH, HD + \text{sq } XG = \text{sq } CD$$

Therefore

$$\text{sq } CD - \text{sq } XG = \text{rect } EH, HD,$$

but

$$\text{sq } CD - \text{sq } XG = \text{rect } OS, SP,$$

therefore

$$\text{rect } EH, HD = \text{rect } OS, SP$$

And since

$$DE \text{ AB AB DF,}$$

therefore

$$DE \text{ DF sq } DE \text{ sq } AB \text{ (Eucl VI 20),}$$

that is

$$DE \text{ DF sq } CD \text{ sq } CB \text{ (Eucl V 15),}$$

And

$$\text{rect } PC, CA = \text{rect } PC, CB = \text{sq } CD \text{ (I 13),}$$

and since

$$DE \text{ DF EH HL (Eucl VI 4),}$$

or

$$DE \text{ DF rect } EH, HD \text{ rect } DH, HL \text{ (Eucl VI 1),}$$

and since

$$DE \text{ DF rect } PC, CB \text{ sq } CB,$$

and

$$\text{rect } PC, CB \text{ sq } CB \text{ rect } OS \text{ SP sq } OS,<sup>1</sup>$$

therefore also

$$\text{rect } EH, HD \text{ rect } DH, HL \text{ rect } OS \text{ SP sq } OS$$

And

$$\text{rect } EH, HD = \text{rect } OS, SP,$$

therefore

$$\text{rect } DH, HL = \text{sq } OS = \text{sq } GH$$

Therefore the straight line  $GH$  is equal in square to the area  $DL$  which is applied to the straight line  $DF$ , deficient by a figure  $FL$  similar to the rectangle contained by  $ED$  and  $DF$  (Eucl VI 24)

I say then that also if produced to the other side of the section, the straight line  $GH$  will be bisected by the straight line  $DE$

For let it be produced and let it meet the section at  $W$ , and let the straight line  $WY$  be drawn through  $Y$  parallel to  $GX$ , and through  $Y$  let the straight line  $YZ$  be drawn parallel to  $AN$  And since

$$GX = WY,$$

therefore also

$$\text{sq } GX = \text{sq } WY.$$

<sup>1</sup>This follows from the proportions

and  $\frac{PC}{PC} = \frac{CB}{CB} = \frac{PS}{PS} = \frac{OS}{OS}$  (Eucl VI 4),  
and  $\frac{PC}{PS} = \frac{CB}{OS} = \frac{\text{rect } PC, CB}{\text{rect } PS, OS} = \frac{\text{sq } CB}{\text{sq } OS}$  (Eucl VI 1).

But

$$\text{sq } GX = \text{rect } AX, XO \text{ (I 13),}$$

and

$$\text{sq } WY = \text{rect } AY, YZ \text{ (I 13)}$$

Therefore

$$OX \cdot ZY = YA \cdot AX \text{ (Eucl vi 16).}$$

And

$$OX \cdot ZY = XB \cdot BY \text{ (Eucl vi 4);}$$

therefore also

$$YA \cdot AX = XB \cdot BY$$

And *separando*

$$YX \cdot AX = YX \cdot BY \text{ (Eucl v 17).}$$

Therefore

$$AX = YB$$

And also

$$AC = CB,$$

therefore also the remainders

$$XC = CY,$$

and so also

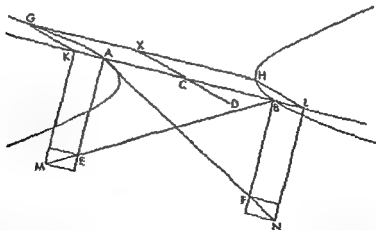
$$GH = HW$$

Therefore the straight line  $HG$ , produced to the other side of the section, is bisected by the straight line  $DH$

#### PROPOSITION 16

*If through the midpoint of the transverse side of the opposite sections a straight line be drawn parallel to a straight line drawn ordinately, it will be a diameter of the opposite sections conjugate to the diameter just mentioned*

Let there be the opposite sections whose diameter is the straight line  $AB$ ,



and let  $AB$  be bisected at  $C$ , and through  $C$  let the straight line  $CD$  be drawn parallel to a straight line drawn ordinately

I say that the straight line  $CD$  is a diameter conjugate to  $AB$

For let the straight lines  $AE$  and  $BF$  be the parameters, and let the straight

lines  $AF$  and  $BE$  be joined and produced, and let some point  $G$  be taken at random on either section, and through  $G$  let the straight line  $GH$  be drawn parallel to  $AB$ , and from  $G$  and  $H$  let the straight lines  $GK$  and  $HL$  be drawn ordinatewise, and through  $K$  and  $L$  let the straight lines  $KM$  and  $LN$  be drawn parallel to  $AE$  and  $BF$ . Since then

$$GK = HL \text{ (Eucl I 34),}$$

therefore also

$$\text{sq } GK = \text{sq } HL,$$

But

$$\text{sq } GK = \text{rect } AK, KM \text{ (I 12),}$$

and

$$\text{sq } HL = \text{rect } BL, LN \text{ (I 12),}$$

therefore

$$\text{rect } AK, KM = \text{rect } BL, LN$$

And since

$$AE = BF,$$

therefore

$$AE \cdot AB = BF \cdot BA \text{ (Eucl V 7)}$$

But

$$AE \cdot AB = MA \cdot KB \text{ (Eucl VI 4),}$$

and as

$$BF \cdot BA = NL \cdot LA \text{ (Eucl VI 4)}$$

And therefore

$$MA \cdot KB = NL \cdot LA$$

But, with  $KA$  taken as common height,

$$MA \cdot KB = \text{rect } MK, KA = \text{rect } BK, KA,$$

and with  $BL$  taken as common height,

$$NL \cdot LA = \text{rect } NL, LB = \text{rect } AL, LB$$

And therefore

$$\text{rect } MA, KA = \text{rect } BK, KA = \text{rect } NL, LB = \text{rect } AL, LB$$

And alternately

$$\text{rect } MK, KA = \text{rect } NL, LB = \text{rect } BK, KA = \text{rect } AL, LB \text{ (Eucl V 16)}$$

And

$$\text{rect } AK, KM = \text{rect } BL, LN,$$

therefore

$$\text{rect } BK, KA = \text{rect } AL, LB,$$

therefore

$$AK = LB^1$$

But also

$$AC = CB,$$

and therefore

$$KC = CL,$$

and so also

$$GX = XH$$

<sup>1</sup>The intermediary steps to this conclusion are as follows II

$$\text{rect } BA, KA = \text{rect } AL, LB$$

then

$$BK \cdot LB = AL \cdot KA$$

or

$$BA + AK \cdot LB = BA + LB \cdot AK$$

and componendo

$$BA + AK + LB \cdot LB = BA + LB + AK \cdot AK$$

Therefore the straight line  $GH$  has been bisected by the straight line  $XCD$ , and is parallel to the straight line  $AB$ . Therefore the straight line  $XCD$  is a diameter and conjugate to the straight line  $AB$  (Def's 4 6)

### SECOND DEFINITIONS

9 Let the midpoint of the diameter of both the hyperbola and the ellipse be called the center of the section and let the straight line drawn from the center to meet the section be called the radius of the section

10 And likewise let the midpoint of the transverse side of the opposite sections be called the center

11 And let the straight line drawn from the center parallel to an ordinate being a mean proportional to the sides of the figure (*τὸ εἶδος*) and bisected by the center be called the second diameter

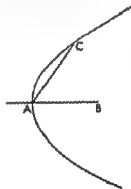
### PROPOSITION 17

*If in a section of a cone a straight line is drawn from the vertex of the line and parallel to an ordinate it will fall outside the section (Cf Eucl III 16)*

Let there be a section of a cone whose diameter is the straight line  $AB$

I say that the straight line drawn from the vertex that is from the point  $A$  parallel to an ordinate will fall outside the section

For if possible let it fall within as  $AC$ . Since then a point  $C$  has been taken at random on a section of a cone therefore the straight line drawn from the point  $C$  within the section parallel to an ordinate will meet the diameter  $AB$  and will be bisected by it (I 7). Therefore the straight line  $AC$  produced will be bisected by the straight line  $AB$ . And this is absurd. For the straight line  $AC$  if produced will fall outside the section (I 10). Therefore the straight line drawn from the point  $A$  parallel to an ordinate will not fall within the line therefore it will fall outside and therefore it is tangent to the section



### PROPOSITION 18

*If a straight line meeting a section of a cone and produced both ways falls outside the section and some point is taken within the section and through it a parallel to the straight line meeting the section is drawn the parallel so drawn if produced both ways will meet the section*

Let there be a section of a cone and the straight line  $AFB$  meeting it and let it fall when produced both ways, outside the section. And let some point  $C$  be taken within the section and through  $C$  let the straight line  $CD$  be drawn parallel to the straight line  $AB$

I say that the straight line  $CD$  produced both ways will meet the section

For let some point  $E$  be taken on the



section, and let the straight line  $EF$  be joined. And since the straight line  $AB$

the section. Then likewise we could show that, produced to the side of points  $F$  and  $B$ , it also meets it. Therefore the straight line  $CD$  produced both ways will meet the section.

### PROPOSITION 19

*In every section of a cone, any straight line drawn from the diameter parallel to an ordinate, will meet the section.*

Let there be a section of a cone whose diameter is the straight line  $AB$  and let some point  $B$  be taken on the diameter, and through  $B$  let the straight line  $BC$  be drawn parallel to an ordinate.

I say that the straight line  $BC$  produced will meet the section.



joined from  $A$  to  $D$  will fall within the section (10). And since the straight line drawn from  $A$  parallel to an ordinate falls outside the section (17), and the straight line  $AD$  meets it and the straight line  $BC$  is parallel to the ordinate, therefore  $BC$  will also meet  $AD$ . And if it meets  $AD$  between the points  $A$  and  $D$ , it will meet the section. If it meets  $AD$  outside the points  $A$  and  $D$ , it will also meet the section.

### PROPOSITION 20

*If in a parabola two straight lines are dropped ordinatewise to the diameter, the squares on them will be to each other as the straight lines cut off by them on the diameter.*

Let  $AB$  be the diameter of a parabola, and from the points  $C$  and  $D$  let the straight lines  $CE$  and  $DF$  be dropped ordinatewise to  $AB$ .

I say that

$$\text{sq } DF : \text{sq } CE :: FA : AE$$

For let  $AG$  be the parameter, therefore

$$\text{sq } DF = \text{rect } FA, AG,$$

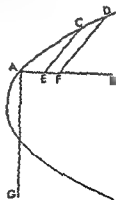
and

$$\text{sq } CE = \text{rect } EA, AG \quad (11)$$

Therefore

$$\text{sq } DF : \text{sq } CE :: \text{rect } FA, AG : \text{rect } EA, AG$$

But

$$\text{rect } FA, AG : \text{rect } EA, AG :: FA : AE \quad (\text{Eucl vi 1}),$$


<sup>1</sup>These are usually called *abscissas* from the Latin *abscindere*, to cut off

and therefore

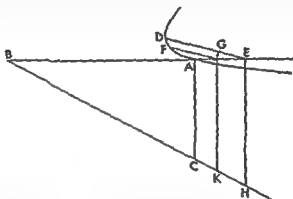
$$\text{sq } DF : \text{sq } CE :: FA : AE.$$

### PROPOSITION 21

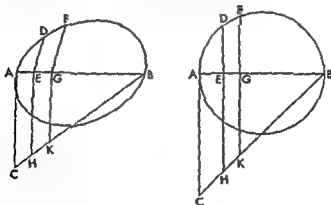
*To find the ratio of the squares of the ordinates of a conic section to the rectangle of the abscissas.*

side of the figure, as the upright side of the figure is to the transverse, and to each other as the areas contained by the straight lines cut off (abscissas), as we have said.

Let there be an hyperbola or ellipse or circumference of a circle whose diam-



eter is  $AB$  and whose parameter is the straight line  $AC$ , and let the straight lines  $DE$  and  $FG$  be dropped ordinately to the diameter.



I say that

$$\text{sq } FG \text{ rect } AG, GB :: AC \cdot AB$$

and

$$\text{sq } FG : \text{sq } DE :: \text{rect } AG, GB : \text{rect } AE, EB$$

For let the straight line  $BC$  determining the figure be joined, and through  $E$  and  $G$  let the straight lines  $EH$  and  $GK$  be drawn parallel to the straight line  $AC$ . Therefore

$$\begin{aligned} \text{sq } FG &= \text{rect } KG, GA \\ \text{sq } DE &= \text{rect } HE EA \quad (\text{r } 12, 13) \end{aligned}$$

And since

$$KG \text{ GB } CA \text{ AB},$$

and with  $AG$  taken as common height,

$$KG \text{ GB } \text{rect } KG, GA \text{ rect } BG, GA,$$

therefore

$$CA \text{ AB } \text{rect } KG, GA \text{ rect } BG, GA$$

or

$$CA \text{ AB } \text{sq } FG \text{ rect } BG, GA$$

Then also for the same reasons

$$CA \text{ AB } \text{sq } DE \text{ rect } BE EA$$

And therefore

$$\text{sq } FG \text{ rect } BG, GA = \text{sq } DE \text{ rect } BE EA,$$

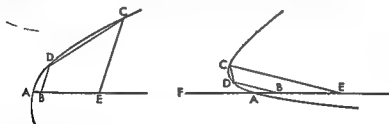
alternately

$$\text{sq } FG \text{ sq } DE \text{ rect } BG, GA \text{ rect } BE EA^1$$

### PROPOSITION 22

*If a straight line cuts a parabola or hyperbola in two points not meeting the diameter inside it will, if produced, meet the diameter of the section outside the section*

Let there be a parabola or hyperbola whose diameter is the straight line  $AB$ , and let some straight line cut the section in two points  $C$  and  $D$



I say that the straight line  $DC$ , if produced, will meet the straight line  $AB$  outside the section

For let the straight lines  $CE$  and  $DB$  be dropped ordinatewise from  $C$  and  $D$ , and first let the section be a parabola. Since then in the parabola

$$\text{sq } CE \text{ sq } DB \text{ EA } AB \quad (\text{r } 20),$$

and

$$EA > AB,$$

<sup>1</sup>Eutocius commenting says It is to be noted that the parameter that is the upright side in the case of the circle is equal to the diameter. For if

$$\text{sq } DE \text{ rect } AK, EB \text{ CA } AB$$

and only in the case of the circle

$$\text{sq } DE = \text{rect. } AE \text{ EB}$$

therefore also

$$CA = AB$$

<sup>2</sup>And this must also be noted that the ordinates on the circumference of the circle are in every case perpendicular to the diameter and are in a straight line with the parameter to  $AC$  (Eucl. III 3 4)



therefore also

$$\text{sq } CE > \text{sq } DB \text{ (Eucl v 14)}$$

And so also

$$CE > DB$$

And they are parallel, therefore  $CD$  produced will meet the diameter  $AB$  outside the section (I 10, Eucl I 33)

But then let it be an hyperbola. Since then in the hyperbola  
 $\text{sq } CE \text{ sq } DB \text{ rect } FE, EA \text{ rect } FB, BA$  (I 21),  
 therefore also

$$\text{sq } CE > \text{sq } DB$$

And they are parallel, therefore the straight line  $CD$  produced will meet the diameter of the section outside the section

### PROPOSITION 23

*If a straight line lying between the two (conjugate) diameters<sup>1</sup> cuts the ellipse, it will, when produced, meet each of the diameters outside the section.*

Let there be an ellipse whose diameters are the straight lines  $AB$  and  $CD$  (I 15), and let some straight line  $EF$  lying between the diameters  $AB$  and  $CD$  cut the section

I say that the straight line  $EF$ , when produced, will meet each of the straight lines  $AB$  and  $CD$  outside the section

For let the straight lines  $GE$  and  $FH$  be dropped ordinatewise from  $E$  and  $F$  to  $AB$ , and the straight lines  $EK$  and  $FL$  ordinatewise to  $CD$ . Therefore

$$\text{sq } EG \text{ sq } FH \text{ rect } BG, GA \text{ rect } BH, HA \text{ (I 21)}$$

and

$$\text{sq } FL \text{ sq } EK \text{ rect } DL, LC \text{ rect } DK, KC \text{ (I 21)}$$

And

$$\text{rect } BG, GA > \text{rect } BH, HA$$

for the point  $G$  is nearer the midpoint (Eucl VI 27 II 5), and

$$\text{rect } DL, LC > \text{rect } DK, KC,$$

therefore also

$$\text{sq } GE > \text{sq } FH,$$

and

$$\text{sq } FL > \text{sq } EK,$$

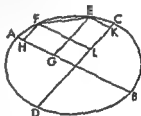
therefore also

$$GE > FH,$$

and

$$FL > EK$$

And  $GE$  is parallel to  $FH$ , and  $FL$  to  $EK$ , therefore the straight line  $EF$  produced will meet each of the diameters  $AB$  and  $CD$  outside the section (I 10 Eucl I 33)



## PROPOSITION 24

*If a straight line, meeting a parabola or hyperbola at a point, when produced both ways, falls outside the section, then it will meet the diameter*

Let there be a parabola or hyperbola whose diameter is the straight line  $AB$ , and let the straight line  $CDE$  meet it at  $D$ , and, when produced both ways, let it fall outside the section

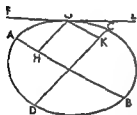
I say that it will meet the diameter  $AB$

For let some point  $F$  be taken on the section, and let the straight line  $DF$  be joined, therefore  $DF$  produced will meet the diameter of the section (1 22) Let it meet it at  $A$ , and

outside the section

## PROPOSITION 25

*If  
pr*



ways fall outside the section

I say that the straight line  $EF$  will meet each of the straight lines  $AB$  and  $CD$

Let the straight lines  $GH$  and  $GK$  be dropped ordinally to the straight lines  $AB$  and  $CD$  respectively Since  $GK$  is parallel to  $AB$  (1 15), and some straight line  $GF$  has met  $GK$ , therefore it will also meet  $AB$  Then likewise  $EF$  will also meet  $CD$

## PROPOSITION 26

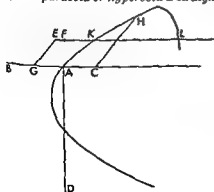
*If in a parabola or hyperbola a straight line is drawn parallel to the diameter of the section, it will meet the section in one point only*

Let there first be a parabola whose diameter is the straight line  $AB$

drawn parallel to  $AB$

I say that the straight line  $FF$  produced will meet the section

For let some point  $E$  be taken on  $FF$ , and from  $E$  let the straight line  $LG$  be drawn parallel to an ordinate, and let



$$\text{rect } DA, AC > \text{sq } GE,$$

and from  $C$  let  $CH$  be erected ordinatewise (I 19) Therefore

$$\text{sq } HC = \text{rect } DA, AC \text{ (I 11)}$$

But

$$\text{rect } DA, AC > \text{sq } EG,$$

therefore

$$\text{sq } HC > \text{sq } EG,$$

therefore

$$HC > EG$$

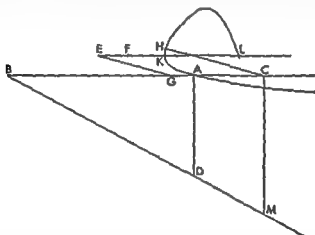
And they are parallel, therefore the straight line  $EF$  produced cuts the straight line  $HC$ , and so it will also meet the section

Let it meet it at the point  $K$

Then I say also that it will meet it in the one point  $K$  only

For if possible, let it also meet it in the point  $L$ . Since then a straight line cuts a parabola in two points, if produced it will meet the diameter of the section (I 22) And this is absurd, for it is supposed parallel. Therefore the straight line  $EF$  produced meets the section in only one point

Next let the section be an hyperbola, and the straight line  $AB$  the transverse



side of the figure, and the straight line  $AD$  the upright side, and let the straight line  $DB$  be joined and produced. Then with the same things being constructed, let the straight line  $CM$  be drawn from  $C$  parallel to  $AD$ . Since then

$$\text{rect } MC, CA > \text{rect } DA, AC,$$

and

$$\text{sq } CH = \text{rect } MC, CA,$$

and

$$\text{rect } DA, AC > \text{sq } GE,$$

therefore also

$$\text{sq } CH > \text{sq } GE$$

And so also

$$CH > GE,$$

and the same things as in the first case will come to pass



because of the section (1 11) Therefore also

$$\text{sq } BC = \text{rect } BA, AM$$

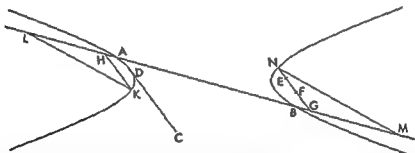
But the straight line  $AM$  is the upright side,<sup>1</sup> and the straight line  $BC$  is parallel to an ordinate Therefore the section passes through the point  $C$  (1 20), and the straight line  $CD$  meets the section at the point  $C$

### PROPOSITION 23

If a straight line touches one of the opposite sections, and some point is taken within the other section and through it a straight line is drawn parallel to the tangent, then produced both ways, it will meet the section

Let there be opposite sections whose diameter is the straight line  $AB$ , and let some straight line  $CD$  touch the section  $A$ , and let some point  $E$  be taken within the other section, and through  $E$  let the straight line  $EF$  be drawn parallel to the straight line  $CD$

I say that the straight line  $EF$  produced both ways will meet the section



Since then it has been proved that the straight line  $CD$  produced will meet the diameter  $AB$  (1 24), and  $EF$  is parallel to it, therefore  $EF$  produced will meet the diameter Let it meet it at  $G$ , and let  $AH$  be made equal to  $GB$ , and

the same straight line And since  $KL$  is parallel to  $MN$  and  $KH$  to  $GN$ , and  $LM$  is one straight line, triangle  $KHL$  is similar to triangle  $HMN$  And

$$LH = GM,$$

therefore

$$KL = MN$$

And so also

$$\text{sq } KL = \text{sq } MN$$

And since

$$LH = GM,$$

and

$$AH = BG,$$

and  $AB$  is common therefore

$$BL = AM,$$

<sup>1</sup>The text reads  $\kappa\lambda\gamma\iota\alpha$  which is impossible I have corrected to  $\alpha\phi\theta\iota\alpha$

therefore

$$\text{rect } BL, LA = \text{rect } AM, MB$$

Therefore

$$\text{rect } BL, LA \text{ sq } LK : \text{rect } AM, MB \text{ sq } MN$$

And

$$\text{rect } BL, LA \text{ sq } LA : \text{the transverse the upright (I 21),}$$

therefore also

$$\text{rect } AM, MB \text{ sq } MN : \text{the transverse the upright}$$

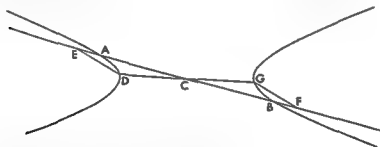
Therefore the point  $N$  is on the section. Therefore the straight line  $EF$  produced will meet the section at the point  $N$  (I 21)

Likewise then it could be shown that produced to the other side it will meet the section

### PROPOSITION 29

*If in opposite sections a straight line is drawn through the center to meet either of the sections then produced it will cut the other section*

Let there be opposite sections whose diameter is the straight line  $AB$  and whose center is the point  $C$ , and let the straight line  $CD$  cut the section  $AD$



I say that it will also cut the other section

For let the straight line  $ED$  be dropped ordinally, and let the straight line  $BF$  be made equal to the straight line  $AE$ , and let the straight line  $FG$  be drawn ordinally (I 19) And since

$$EA = BF,$$

and  $AB$  is common therefore

$$\text{rect } BE, EA = \text{rect } BF, FA$$

And since

$$\text{rect } BE, EA \text{ sq } DE : \text{the transverse the upright (I 21),}$$

but also

$$\text{rect } BF, FA \text{ sq } FG : \text{the transverse the upright (I 21),}$$

therefore also

$$\text{rect } BE, EA \text{ sq } DE : \text{rect } BF, FA \text{ sq } FG \text{ (I 14)}$$

But

$$\text{rect } BE, EA = \text{rect } BF, FA,$$

therefore also

$$\text{sq } DE = \text{sq } FG$$

Since then

$$EC = CF,$$

and

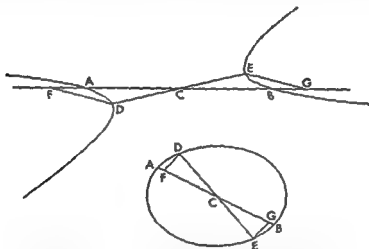
$$DE = FG,$$

and  $EF$  is a straight line, and  $ED$  is parallel to  $FG$ , therefore  $DG$  is also a straight line (Eucl vi 32) And therefore  $CD$  will also cut the other section

### PROPOSITION 30

*If in an ellipse or in opposite sections a straight line is drawn in both directions from the center, meeting the section, it will be bisected at the center*

Let there be an ellipse or opposite sections, and their diameter the straight line  $AB$ , and their center  $C$ , and through  $C$  let some straight line  $DCE$  be drawn (I 29)



I say that the straight line  $CD$  is equal to the straight line  $CE$

For let the straight lines  $DF$  and  $EG$  be drawn ordonatewise And since

rect  $BF, FA$  sq  $FD$  the transverse the upright (I 21),

but also

rect  $AG, GB$  sq  $GE$  the transverse the upright (I 21),

therefore also

rect  $BF, FA$  sq  $FD$  rect  $AG, GB$  sq  $GE$  (I 14)

And alternately

rect  $BF, FA$  rect  $AG, GB$  sq  $FD$  sq  $GE$

But

sq  $FD$  sq  $GE$  sq  $FC$  sq  $CG$  (Eucl vi 4),

therefore alternately

rect  $BF, FA$  sq  $FC$  rect  $AG, GB$  sq  $CG$

Therefore also *componendo* in the case of the ellipse, and *inversely* and *con-  
tertendo* in the case of the opposite sections (Eucl v Defs 14 13, 16),

sq  $1C$  sq  $CF$  sq  $BC$  sq  $CG$  (Eucl II 5, 6),

and alternately But

$$\text{sq } CB = \text{sq } AC,$$

therefore also

$$\text{sq } CG = \text{sq } CF$$

Therefore

$$CG = CF$$

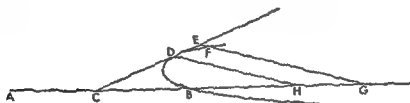
And the straight lines  $DF$  and  $GE$  are parallel, therefore also

$$DC = CE$$

### PROPOSITION 31

If on the transverse side of the figure of an hyperbola some point be taken cutting off from the vertex of the section not less than half of the transverse side of the figure,

let  $C$  be a point on the diameter  $AB$  taken cutting off from the vertex  $A$  not less than half of  $AB$ .



than half of  $AB$ , and let some straight line  $CD$  be drawn to meet the section

and first let

$$AC = CB$$

And since

$$sq\ EG + sq\ DH > sq\ FG + sq\ DH \text{ (Eucl v 8),}$$

but

$$sq\ EG + sq\ DH = sq\ CG + sq\ CH$$

because of  $EG$  &  $DH$  being parallel to  $CH$ , and

$$sq\ FG + sq\ DH = rect\ AG, GB + rect\ AH, HB$$

because of the section (i 21),

therefore

$$sq\ CG + sq\ CH > rect\ AG, GB + rect\ AH, HB$$

Alternately therefore

$$sq\ CG + rect\ AG, GB > sq\ CH + rect\ AH, HB$$

Therefore separando

$$sq\ CB + rect\ AG, GB > sq\ CB + rect\ AH, HB,$$

since it will also fall inside  $CD$

since it will also fall inside  $CD$

The rules governing operations on inequalities in proportions are not developed by Euclid in Book V of the *Elements*. But they can be deduced on Euclid's principles.



## PROPOSITION 32

If a straight line is drawn through the vertex of a section of a cone, parallel to an ordinate, then it touches the section, and another straight line will not fall into the space between the conic section and this straight line

Let there be a section of a cone, first the so called parabola whose diameter is the straight line  $AB$ , and from  $A$  let the straight line  $AC$  be drawn parallel to an ordinate

Now it has been shown that it falls outside the section (1 17)

Then I say that also another straight line will not fall into the space between the straight line  $AC$  and the section

For if possible, let it fall in, as the straight line  $AD$  and let some point  $D$  be taken on it at random and let the straight line  $DE$  be dropped ordinatewise, and let the straight line  $AF$  be the parameter of the ordinates And since

$\text{sq } DE \text{ sq } EA > \text{sq } GE \text{ sq } EA$  (Eucl v 8),  
and

$$\text{sq } GE = \text{rect } FA \text{ } AE \text{ (1 11),}$$

therefore also

$\text{sq } DE \text{ sq } EA > \text{rect } FA \text{ } AE \text{ sq } EA,$   
or

$$> FA \text{ } EA$$

Let it be contrived then that

$\text{sq } DE \text{ sq } EA \text{ } FA \text{ } HA$  (Eucl vi 20, 11),

and through the point  $H$  let the straight line  $HLK$  be drawn parallel to  $ED$   
Since then

$$\text{sq } DE \text{ sq } EA \text{ } FA \text{ } AH \text{ rect } FA \text{ } AH \text{ sq } AH,$$

and

$$\text{sq } DE \text{ sq } EA \text{ sq } KH \text{ sq } HA \text{ (Eucl vi 22),}$$

and

$$\text{sq } HL = \text{rect } FA \text{ } AH \text{ (1 11),}$$

therefore also

$$\text{sq } KH \text{ sq } HA \text{ sq } LH \text{ sq } HA$$

Therefore

$$KH = HL$$

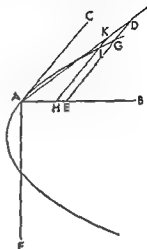
and this is absurd Therefore another straight line will not fall into the space between the straight line  $AC$  and the section

Next let the section be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line  $AB$  and whose upright side is the straight line  $AF$ , and let the straight line  $BF$  be joined and produced, and from the point  $A$  let the straight line  $AC$  be drawn parallel to an ordinate

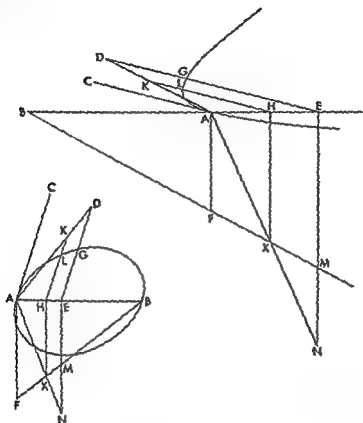
Now it has been shown that it falls outside the section (1 17)

Then I say that also another straight line will not fall into the space between the straight line  $AC$  and the section

For if possible, let it fall as the straight line  $AD$ , and let some point  $D$  be taken at random on it and from it let the straight line  $DE$  be dropped ordi-



natewise, and through  $E$  let the straight line  $EM$  be drawn parallel to the straight line  $AF$



And since

$$\text{sq } GE = \text{rect } AE, EM \text{ (I 12 13)}$$

let it be contrived that

$$\text{rect } AE, EN = \text{sq } DE,$$

and let the straight line joining  $AN$  cut the straight line  $FM$  at  $X$ , and through  $X$  let the straight line  $XH$  be drawn parallel to  $FA$ , and through  $H$ ,  $HLK$  parallel to  $AC$ . Since then

$$\text{sq } DE = \text{rect } AM, EN,$$

hence

$$NE \cdot FD = DE \cdot EA,$$

and therefore

$$NE \cdot EA = \text{sq } DE = \text{sq } EA \text{ (Eucl vi 20)}$$

But

$$NE \cdot EA = XH \cdot HA,$$

and

$$\text{sq } DE = \text{sq } EA = \text{sq } KH = \text{sq } HA$$

Therefore

$$XH \cdot HA = \text{sq } KH = \text{sq } HA,$$

therefore

$$XH \cdot HK = KH \cdot HA \text{ (Eucl vi 20).}$$

Therefore

$$\text{sq } KH = \text{rect } AH, HX,$$

but also

$$\text{sq } LH = \text{rect } AH, HX$$

because of the section (i 12, 13),

therefore

$$\text{sq } KH = \text{sq } HL,$$

and this is absurd. Therefore another straight line will not fall into the space between the straight line  $AC$  and the section.

### PROPOSITION 33

If in a parabola some point is taken, and from it an ordinate is dropped to the diameter, and, to the straight line cut off by it on the diameter from the vertex, a straight line in the same straight line from its extremity is made equal, then the straight line joined from the point thus resulting to the point taken will touch the section.

Let there be a parabola whose diameter is the straight line  $AB$ , and let the straight line  $CD$  be dropped ordinately, and let the straight line  $AE$  be made equal to the straight line  $ED$ , and let the straight line  $AC$  be joined.

I say that the straight line  $AC$  produced will fall outside the section.

For if possible, let it fall within, as the straight line  $CF$ , and let the straight line  $GB$  be dropped ordinately. And since

$$\text{sq } BG = \text{sq } CD > \text{sq } FB = \text{sq } CD,$$

but

$$\text{sq } FB = \text{sq } CD = \text{sq } BA = \text{sq } AD,$$

and

$$\text{sq } BG = \text{sq } CD = BE \cdot DE \text{ (i 20),}$$

therefore

$$BE \cdot DE > \text{sq } BA = \text{sq } AD$$

But

$$BE \cdot DE = 4 \text{ rect } BE, EA = 4 \text{ rect } DE, EA,$$

therefore also

$$4 \text{ rect } BE, EA = 4 \text{ rect } DE, EA > \text{sq } AB = \text{sq } AD$$

Therefore alternately

$$4 \text{ rect } BE, EA = \text{sq } AB > 4 \text{ rect } DE, EA = \text{sq } AD,$$

and this is absurd, for since

$$DE,$$

hence

But

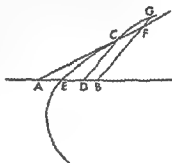
$$4 \cdot$$

for  $E$  is not the vertex

of  $A$

$AC$  does not fall

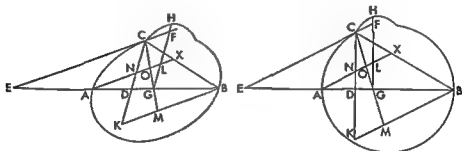
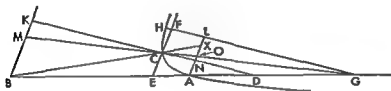
line



### PROPOSITION 34

*If on an hyperbola or ellipse or circumference of a circle some point is taken, and from it a straight line is dropped ordinatewise to the diameter, and whatever ratio the straight lines cut off by the ordinate from the ends of the figure's transverse side have to each other, that ratio have the segments of the transverse side to each other so that the segments from the vertex are corresponding, then the straight line joining the point taken on the transverse side and that taken on the section will touch the section.*

Let there be an hyperbola or ellipse or circumference of a circle whose diam-



eter is the straight line  $AB$ , and let some point  $C$  be taken on the section, and from  $C$  let the straight line  $CD$  be drawn ordinatewise, and let it be contrived that

$BD \quad DA \quad BE \quad EA.$

and let the straight line  $EC$  be joined

I say that the straight line  $CE$  touches the section

<sup>1</sup>This construction is easy. In the case of the hyperbola *componendo*

BD+DA DA BA EA.

and in the case of the ellipse *separando*

BD-DA DA BA EA

This proportion is the same as the harmonic proportion defined by Nicomachus in his *Introduction to Arithmetic*. For if

BD D4 BE EA

BD+D4 BD BA BE

$BA \quad BD \quad BE-EA \quad BE$

$$BD+DA \quad B4 \quad BA \quad BE-E4.$$

$$DA = BD - BA \quad EA = BA - BE$$

2BD-B1 BA BA 2BE-BA

And so  $BA$  is the harmonic mean between  $RD$  and  $RE$ .

1992, 1993, 1994, 1995, 1996, 1997, 1998, 1999, 2000, 2001, 2002, 2003, 2004, 2005, 2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030, 2031, 2032, 2033, 2034, 2035, 2036, 2037, 2038, 2039, 2040, 2041, 2042, 2043, 2044, 2045, 2046, 2047, 2048, 2049, 2050, 2051, 2052, 2053, 2054, 2055, 2056, 2057, 2058, 2059, 2060, 2061, 2062, 2063, 2064, 2065, 2066, 2067, 2068, 2069, 2070, 2071, 2072, 2073, 2074, 2075, 2076, 2077, 2078, 2079, 2080, 2081, 2082, 2083, 2084, 2085, 2086, 2087, 2088, 2089, 2090, 2091, 2092, 2093, 2094, 2095, 2096, 2097, 2098, 2099, 2100, 2101, 2102, 2103, 2104, 2105, 2106, 2107, 2108, 2109, 2110, 2111, 2112, 2113, 2114, 2115, 2116, 2117, 2118, 2119, 2120, 2121, 2122, 2123, 2124, 2125, 2126, 2127, 2128, 2129, 2130, 2131, 2132, 2133, 2134, 2135, 2136, 2137, 2138, 2139, 2140, 2141, 2142, 2143, 2144, 2145, 2146, 2147, 2148, 2149, 2150, 2151, 2152, 2153, 2154, 2155, 2156, 2157, 2158, 2159, 2160, 2161, 2162, 2163, 2164, 2165, 2166, 2167, 2168, 2169, 2170, 2171, 2172, 2173, 2174, 2175, 2176, 2177, 2178, 2179, 2180, 2181, 2182, 2183, 2184, 2185, 2186, 2187, 2188, 2189, 2190, 2191, 2192, 2193, 2194, 2195, 2196, 2197, 2198, 2199, 2200, 2201, 2202, 2203, 2204, 2205, 2206, 2207, 2208, 2209, 2210, 2211, 2212, 2213, 2214, 2215, 2216, 2217, 2218, 2219, 2220, 2221, 2222, 2223, 2224, 2225, 2226, 2227, 2228, 2229, 2230, 2231, 2232, 2233, 2234, 2235, 2236, 2237, 2238, 2239, 2240, 2241, 2242, 2243, 2244, 2245, 2246, 2247, 2248, 2249, 2250, 2251, 2252, 2253, 2254, 2255, 2256, 2257, 2258, 2259, 2260, 2261, 2262, 2263, 2264, 2265, 2266, 2267, 2268, 2269, 2270, 2271, 2272, 2273, 2274, 2275, 2276, 2277, 2278, 2279, 2280, 2281, 2282, 2283, 2284, 2285, 2286, 2287, 2288, 2289, 2290, 2291, 2292, 2293, 2294, 2295, 2296, 2297, 2298, 2299, 2300, 2301, 2302, 2303, 2304, 2305, 2306, 2307, 2308, 2309, 2310, 2311, 2312, 2313, 2314, 2315, 2316, 2317, 2318, 2319, 2320, 2321, 2322, 2323, 2324, 2325, 2326, 2327, 2328, 2329, 2330, 2331, 2332, 2333, 2334, 2335, 2336, 2337, 2338, 2339, 2340, 2341, 2342, 2343, 2344, 2345, 2346, 2347, 2348, 2349, 2350, 2351, 2352, 2353, 2354, 2355, 2356, 2357, 2358, 2359, 2360, 2361, 2362, 2363, 2364, 2365, 2366, 2367, 2368, 2369, 2370, 2371, 2372, 2373, 2374, 2375, 2376, 2377, 2378, 2379, 2380, 2381, 2382, 2383, 2384, 2385, 2386, 2387, 2388, 2389, 2390, 2391, 2392, 2393, 2394, 2395, 2396, 2397, 2398, 2399, 2400, 2401, 2402, 2403, 2404, 2405, 2406, 2407, 2408, 2409, 2410, 2411, 2412, 2413, 2414, 2415, 2416, 2417, 2418, 2419, 2420, 2421, 2422, 2423, 2424, 2425, 2426, 2427, 2428, 2429, 2430, 2431, 2432, 2433, 2434, 2435, 2436, 2437, 2438, 2439, 2440, 2441, 2442, 2443, 2444, 2445, 2446, 2447, 2448, 2449, 2450, 2451, 2452, 2453, 2454, 2455, 2456, 2457, 2458, 2459, 2460, 2461, 2462, 2463, 2464, 2465, 2466, 2467, 2468, 2469, 2470, 2471, 2472, 2473, 2474, 2475, 2476, 2477, 2478, 2479, 2480, 2481, 2482, 2483, 2484, 2485, 2486, 2487, 2488, 2489, 2490, 2491, 2492, 2493, 2494, 2495, 2496, 2497, 2498, 2499, 2500, 2501, 2502, 2503, 2504, 2505, 2506, 2507, 2508, 2509, 2510, 2511, 2512, 2513, 2514, 2515, 2516, 2517, 2518, 2519, 2520, 2521, 2522, 2523, 2524, 2525, 2526, 2527, 2528, 2529, 2530, 2531, 2532, 2533, 2534, 2535, 2536, 2537, 2538, 2539, 2540, 2541, 2542, 2543, 2544, 2545, 2546, 2547, 2548, 2549, 2550, 2551, 2552, 2553, 2554, 2555, 2556, 2557, 2558, 2559, 2560, 2561, 2562, 2563, 2564, 2565, 2566, 2567, 2568, 2569, 2570, 2571, 2572, 2573, 2574, 2575, 2576, 2577, 2578, 2579, 2580, 2581, 2582, 2583, 2584, 2585, 2586, 2587, 2588, 2589, 2590, 2591, 2592, 2593, 2594, 2595, 2596, 2597, 2598, 2599, 2600, 2601, 2602, 2603, 2604, 2605, 2606, 2607, 2608, 2609, 2610, 2611, 2612, 2613, 2614, 2615, 2616, 2617, 2618, 2619, 2620, 2621, 2622, 2623, 2624, 2625, 2626, 2627, 2628, 2629, 2630, 2631, 2632, 2633, 2634, 2635, 2636, 2637, 2638, 2639, 2640, 2641, 2642, 2643, 2644, 2645, 2646, 2647, 2648, 2649, 2650, 2651, 2652, 2653, 2654, 2655, 2656, 2657, 2658, 2659, 2660, 2661, 2662, 2663, 2664, 2665, 2666, 2667, 2668, 2669, 2670, 2671, 2672, 2673, 26

straight line  $EC$ , and let the straight lines  $DC$ ,  $BC$ , and  $GC$  be joined and produced to the points  $M$ ,  $X$ , and  $\Lambda$ . And since

$$BD \text{ DA} \quad BE \text{ EA}$$

but

$$BD \text{ DA} \quad BK \text{ AN},$$

and

$$BE \text{ AE} \quad BC \text{ CX} \quad BK \text{ XN} \text{ (Eucl vi 4),}$$

therefore

$$BK \text{ AN} \quad BK \text{ XN},$$

therefore

$$AN = NX$$

Therefore

$$\text{rect } AN, NX > \text{rect } AO \text{ OA} \text{ (Eucl vi 27 or 5)}$$

Therefore

$$NX \text{ XO} > OA \text{ AN}^1$$

But

$$NX \text{ XO} \quad KB \text{ BM} \text{ (Eucl vi 4),}$$

therefore

$$KB \text{ BM} > OA \text{ AN}$$

Therefore

$$\text{rect } KB \text{ AN} > \text{rect } BM, OA$$

And so

$$\text{rect } KB, AN \text{ sq } CE > \text{rect } BM \text{ OA sq } CE \text{ (Eucl v 8)}$$

But

$$\text{rect } KB \text{ AN sq } CE \quad \text{rect } BD \text{ DA sq } DE$$

through the similarity of the triangles  $BKD$   $ECD$  and  $NAD$ <sup>2</sup> and

<sup>1</sup>Eutocius commenting says - For since

$$\text{rect } AN \text{ NX} > \text{rect } AO \text{ OA}$$

let

$$\text{rect } AN, NX = \text{rect } AO \text{ XP}$$

where  $XP$  is some line such that

$$XP > XC$$

therefore

$$OA \text{ AN} \quad NX \text{ XP}$$

But

$$NX \text{ XO} > NX \text{ XP} \text{ (Eucl v 8)}$$

and therefore

$$NX \text{ XO} > OA \text{ AN}$$

Then the converse is also evident that if

$$NX \text{ XO} > OA \text{ AN}$$

then

$$\text{rect } AN \text{ NA} > \text{rect } AO \text{ OA}$$

For let it be that

$$OA \text{ AN} \quad NX \text{ XP}$$

where

$$XP > XO$$

therefore

$$\text{rect } XN \text{ NA} = \text{rect } AO \text{ XP}$$

and so

$$\text{rect } XN \text{ NA} > \text{rect } AO \text{ OA}$$

<sup>2</sup>Eutocius commenting, says - Since then because  $AN \text{ EC}$  and  $KB$  are parallel

$$AN \text{ EC} \quad AD \text{ DE}$$

and

$$EC \text{ KB} \quad ED \text{ DB}$$

therefore ex aequali

$$AN \text{ KB} \quad AD \text{ DB}$$

therefore also

$$\text{sq } AA \text{ rect } AN \text{ KB} \quad \text{sq } AD \text{ rect } AD \text{ DB}$$

But

$$\text{sq } EC \text{ sq } AN \quad \text{sq } ED \text{ sq } AD$$

therefore ex aequali

$$\text{sq } EC \text{ rect } AN \text{ KB} \quad \text{sq } ED \text{ rect } AD \text{ DB}$$

and inversely

$$\text{rect } KB \text{ AN} \text{ sq } EC \quad \text{rect } AD \text{ DB} \text{ sq } ED$$

A similar proof holds for the proportion following

therefore

$$\text{rect } BM, OA \text{ sq } CE \text{ rect } BG, GA \text{ sq } GE,$$

$$\text{rect } BD, DA \text{ sq } DE > \text{rect } BG, GA \text{ sq } GE.$$

Therefore alternately

$$\text{rect } BD, DA \text{ rect } BG, GA > \text{sq } DE \text{ sq } GE$$

But

$$\text{rect } BD, DA \text{ rect } AG, GB \text{ sq } CD \text{ sq } GH \text{ (I 21),}$$

and

$$\text{sq } DE \text{ sq } EG \text{ sq } CD \text{ sq } FG \text{ (Eucl vi 4),}$$

therefore also

$$\text{sq } CD \text{ sq } HG > \text{sq } CD \text{ sq } FG$$

Therefore

$$HG < FG \text{ (Eucl v 10),}$$

and this is impossible. Therefore the straight line  $EC$  does not cut the section, therefore it touches it

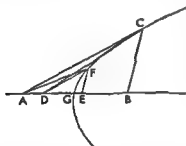
### PROPOSITION 35

*If a straight line touches a parabola, meeting the diameter outside the section, the straight line drawn from the point of contact ordinatewise to the diameter will cut off on the diameter beginning from the vertex of the section a straight line equal to the straight line between the vertex and the tangent, and no straight line will fall into the space between the tangent and the section*

Let there be a parabola whose diameter is the straight line  $AB$ , and let the straight line  $BC$  be erected ordinatewise, and let the straight line  $AC$  be tangent to the section

I say that the straight line  $AG$  is equal to the straight line  $GB$

For if possible, let it be unequal to it, and let the straight line  $GE$  be made equal to  $AG$ , and let the straight line  $EF$  be erected ordinatewise and let the straight line  $AF$  be joined. Therefore  $AF$  produced will meet the



unequal to the straight line  $GB$ , therefore it is equal

Then I say that no straight line will fall into the space between the straight line  $AC$  and the section

For if possible, let the straight line  $CD$  fall in between, and let  $GE$  be made equal to  $GD$ , and let the straight line  $EF$  be erected ordinatewise. Therefore

### PROPOSITION 36

*If some straight line, meeting the transverse side of the figure touches an hyperbola or ellipse or circumference of a circle, and a straight line is dropped from the*

of contact ordinatewise to the diameter then as the straight line cut off by the tangent from the end of the transverse side is to the straight line cut off by the tangent from the other end of that side, so will the straight line cut off by the ordinate from the end of the side be to the straight line cut off by the ordinate from the other end of the side in such a way that the corresponding straight lines are continuous and another straight line will not fall into the space between the tangent and the section of the cone

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line  $AB$ , and let the straight line  $CD$  be tangent and let the straight line  $CE$  be dropped ordinatewise

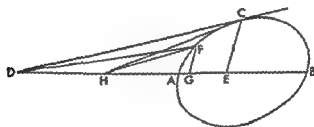
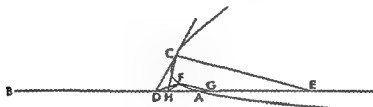
I say that

$$BE \text{ EA } BD \text{ DA}$$

For if it is not let it be

$$BD \text{ DA } BG \text{ GA},$$

and let the straight line  $GF$  be erected ordinatewise, therefore the straight line



joined from  $D$  to  $F$  will touch the section (1 34), therefore produced it will meet  $CD$ . Therefore two straight lines will have the same ends and this is impossible

I say that no straight line will fall between the section and the straight line  $CD$

For if possible, let it fall between as the straight line  $CH$ , and let it be contrived that

$$BH \text{ HA } BG \text{ GA},$$

and let the straight line  $GF$  be erected ordinatewise, therefore the straight line joined from  $H$  to  $F$ , when produced will meet  $HC$  (1 34). Therefore two straight lines will have the same ends and this is impossible. Therefore a straight line will not fall into the space between the section and the straight line  $CD$

## PROPOSITION 37

ratio to the square on the ordinate which the transverse has to the upright

center

I say that

$$\text{rect } DF \cdot FE = \text{sq } FB,$$

and

$$\text{rect } DE, EF \text{ sq } EC : \text{the transverse the upright}$$

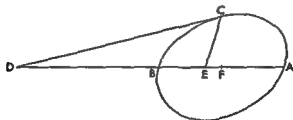
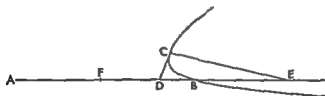
For since  $CD$  touches the section and  $CE$  has been dropped ordinatewise hence

$$AD \cdot DB = AE \cdot EB \quad (\text{I 36})$$

Therefore *componendo*

$$AD + DB \cdot DB = AE + EB \cdot EB$$

And let the halves of the antecedents be taken (Eucl I 15), in the case of the



hyperbola we shall say but

$$\text{half } (AE + EB) = FE,$$

and

$$\text{half } AB - FB,$$

therefore

$$FE \cdot EB = FB \cdot BD$$

Therefore *conuertendo*

$$FE \cdot FB = FB \cdot FD,$$



therefore

$$\text{rect } EF \cdot FD = \text{sq } FB$$

And since

$$FE \cdot EB \cdot FB \cdot BD \cdot AF \cdot BD,$$

alternately

$$AF \cdot FE \cdot DB \cdot BE$$

componendo

$$AE \cdot EF \cdot DE \cdot EB$$

and so

$$\text{rect } AE \cdot EB = \text{rect } FE \cdot ED$$

But

$$\text{rect } AE \cdot EB \cdot \text{sq } CE \quad \text{the transverse} \quad \text{the upright (I 21),}$$

therefore also

$$\text{rect } FE \cdot ED \cdot \text{sq } CE \quad \text{the transverse} \quad \text{the upright}$$

And in the case of the ellipse and of the circle we shall say but

$$\text{half } (AD + DB) = DF,$$

and

$$\text{half } AB = FB,$$

therefore

$$FD \cdot DB \cdot FB \cdot BE$$

Therefore componendo

$$DF \cdot FB \cdot BF \cdot FE$$

Therefore

$$\text{rect } DF \cdot FE = \text{sq } BF$$

But

$$\text{rect } DF \cdot FE = \text{rect } DE \cdot EF + \text{sq } FE \quad (\text{Eucl II 3}),$$

and

$$\text{sq } BF = \text{rect } AE \cdot EB + \text{sq } FE \quad (\text{Eucl II 5})$$

Let the common square on  $EF$  be subtracted therefore

$$\text{rect } DE \cdot EF = \text{rect } AE \cdot EB$$

Therefore

$$\text{rect } DE \cdot EF \cdot \text{sq } CE \quad \text{rect } AE \cdot EB \cdot \text{sq } CE$$

But

$$\text{rect } AE \cdot EB \cdot \text{sq } CE \quad \text{the transverse} \quad \text{the upright (I 21)}$$

Therefore

$$\text{rect } DE \cdot EF \cdot \text{sq } CE \quad \text{the transverse} \quad \text{the upright}$$

### PROPOSITION 38

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the second diameter and from the point of contact a straight line is dropped to the same diameter parallel to the other diameter then the straight line cut off by the dropped straight ( $\kappa\alpha\tau\alpha\gamma\mu\epsilon\tau\eta$ )<sup>1</sup> line from the center of the section with the straight

<sup>1</sup>When the word  $\kappa\alpha\tau\alpha\gamma\mu\epsilon\tau\eta$  is used in connection with the first diameter we translate it as ordinate but we have preferred to stick more closely to the original when it is referred to the second diameter. For although it is certainly an ordinate in the case of the ellipse yet in the case of the hyperbola it is only analogically an ordinate. This analogy however becomes stronger and stronger as the treatise moves on. It is therefore no accident that  $\kappa\alpha\tau\alpha\gamma\mu\epsilon\tau\eta$  is used in both cases. On the other hand in First Definitions 2-5 Apollonius definitely calls both cases ordinates as if announcing the culmination of an analogy to be worked out in the course of the treatise.

line cut off by the tangent from the center of the section will contain an area equal to the square on the half of the second diameter, and with the straight line between the dropped straight line and the tangent will contain an area having a ratio to the square on the dropped straight line which the upright side of the figure has to the transverse

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line  $AGB$ , and whose second diameter is the straight line  $CGD$ , and let the straight line  $ELI'$ , meeting  $CD$  at  $F$ , be a tangent to the section, and let the straight line  $HE$  be parallel to  $AB$

I say that

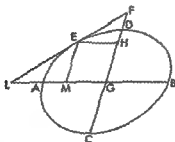
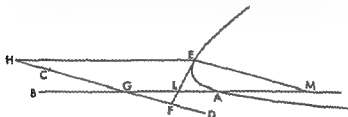
$$\text{rect } FG, GH = \text{sq } GC,$$

and

$$\text{rect } GH, HF = \text{sq } HE \quad \text{the upright} \quad \text{the transverse}$$

Let the straight line  $ME$  be drawn ordinatewise, therefore

$$\text{rect } GM, ML = \text{sq } ME \quad \text{the transverse} \quad \text{the upright (I. 37)}$$



But

$$\text{the transverse } BA \quad CD \quad CD \quad \text{the upright (see Def 11)}$$

and therefore

$$\text{the transverse} \quad \text{the upright} \quad \text{sq } BA \quad \text{sq } CD \quad (\text{Eucl VI 20}),$$

and as the quarters of them, that is

$$\text{the transverse} \quad \text{the upright} \quad \text{sq } GA \quad \text{sq } GC,$$

therefore also

$$\text{rect } GM, ML = \text{sq } ME \quad \text{sq } GA \quad \text{sq } GC$$

But

$$\text{rect } GM, ML = \text{sq } ME \text{ comp } GM \quad ME, LM \quad MF$$

or

$$\text{rect } GM, ML = \text{sq } ME \text{ comp } GM \quad GH, LM$$

Therefore inversely

$$\text{sq } CG \text{ sq } GA \text{ comp } EM : MG \text{ or } HG \text{ } GM, EM \text{ } ML \text{ or } FG \text{ } GL$$

Therefore

$$\text{sq } GC \text{ sq } GA \text{ comp } HG \text{ } GM, FG, GL,$$

which is the same as

$$\text{rect } FG, GH \text{ rect } MG, GL$$

Therefore

$$\text{rect } FG, GH \text{ rect } MG, GL \text{ sq } CG \cdot \text{sq } GA$$

And alternately therefore

$$\text{rect } FG, GH \text{ sq } CG \text{ rect } MG, GL \text{ sq } GA$$

But

$$\text{rect } MG, GL = \text{sq } GA \text{ (1 37),}$$

therefore also

$$\text{rect. } FG, GH = \text{sq } CG$$

Again since

$$\text{the upright the transverse sq } EM \text{ rect } GM \text{ } ML \text{ (1 37),}$$

and

$$\text{sq } EM \text{ rect } GM, ML \text{ comp } EM \text{ } GM, EM \text{ } ML$$

or

$$\text{sq } EM \text{ rect } GM, ML \text{ comp } HG \text{ } HE \text{ } FG \text{ } GL \text{ or } FH \text{ } HE$$

which is the same as

$$\text{rect } FH, HG \text{ sq } HE,$$

therefore

$$\text{rect } FH \text{ } HG \text{ sq } HE \text{ the upright the transverse}$$

With the same things supposed it remains to be shown that, as the straight line between the tangent and the end of the (second) diameter on the same side with the dropped straight line is to the straight line between the tangent and the second diameter, so is the straight line between the other end and the dropped straight line to the straight line between the first end and the dropped straight line

For since

$$\text{rect } FG \text{ } GH = \text{sq } GC = \text{rect } CG, GD \text{ (2 para above),}$$

for

$$CG = GD,$$

therefore

$$\text{rect } FG, GH = \text{rect } CG, GD,$$

therefore

$$FG \text{ } GD \text{ } CG \text{ } GH$$

And *convertendo*

$$GF \text{ } FD \text{ } GC \text{ } CH$$

And let the doubles of the antecedents be taken, but

$$2GF = CF + FD$$

because

$$CG = GD,$$

and

$$2GC = CD,$$

therefore

$$CF + FD \text{ } FD \text{ } DC \text{ } CH$$

And *separando*

*CF FD DH HC.*

and this was to be shown

Then it is clear from what has been said that the straight line  $EF$  touches the section either if

rect  $FG, GH = sq \text{ } GC$ .

or if

rect  $FH$   $HG$  sq  $GC'$

in the ratio we said, for it could be shown conversely

### PROPOSITION 39

straight line between the ordinate and the center of the section and the other is between the ordinate and the tangent then the ordinate will have to it the ratio compounded of the ratio of the other of the two straight lines to the ordinate and of the ratio of the upright side of the figure to the transverse

T + + L      t =      t = "      f =      1 m h      d =

dropped ordmatwise

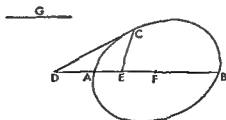
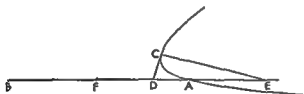
I say that

*CE FE* comp the upright the transverse *ED EC*.

And

*CE ED comp the upright the transverse FE EC*

For let

$$\text{rect } FE \text{ } ED = \text{rect } EC \text{ } G$$


And since

rect  $FE$   $ED$  sq  $CE$  the transverse the upright (1 37).

and

$$\text{rect } FE, ED = \text{rect } CE, G,$$

therefore

$$\text{rect } CE, G \text{ sq } CE \cdot G \text{ CE the transverse the upright}$$

And since

$$\text{rect } FE, ED = \text{rect } CE, G,$$

hence

$$FE \cdot EC = G \cdot ED$$

And since

$$CE \cdot ED \text{ comp } CE \cdot G, G \cdot ED,$$

but

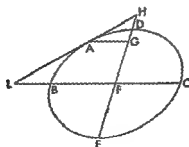
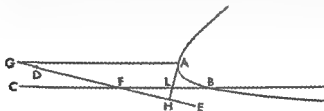
$$CE \cdot G = \text{the upright the transverse,}$$

therefore

$$CE \cdot ED \text{ comp the upright the transverse, } FE \cdot EC$$

## PROPOSITION 40

*If a straight line touching an hyperbola or ellipse or circumference of a circle meets the second diameter, and from the point of contact a straight line is dropped to the same diameter parallel to the other diameter, then whichever of the two straight lines is taken of which one is the straight line between the dropped straight line and the center of the section, and the other is between the dropped straight line and the*



*tangent, the dropped straight line will have to it the ratio compounded of the ratio of the transverse side to the upright and of the ratio of the other of the two straight lines to the dropped straight line*

Let there be an hyperbola or ellipse or circumference of a circle  $AB$  and its diameter the straight line  $BFC$  and its second diameter the straight line  $DFE$ , and let the straight line  $HLA$  be drawn tangent, and the straight line  $AG$  parallel to the straight line  $BC$

I say that

and  $AG \perp HG$  comp the transverse the upright  $FG \perp GA$ ,

Let  $AG$   $FG$  comp the transverse the upright  $HG$   $GA$

$$\text{rect } GA \wedge = \text{rect } HG \text{ } GF$$

And since

the upright the transverse rect  $HG$   $GF$  sq  $GA$  (1 38)

$$\text{rect } GA \wedge = \text{rect } HG \text{ } GF$$

therefore also

And since

 $AG \quad GF \text{ comp } AG \quad \wedge \quad \wedge \quad GF.$ 

but

**AG**  $\Delta$  the transverse the upright

and

*K GF HG GA*

because

$$\text{rect } HG \cdot GF = \text{rect } AG \cdot K$$

therefore

*AG GF* comp the transverse the upright *GH GA*

### PROPOSITION 41

If in an hyperbola or ellipse or circumference of a circle a straight line is dropped ordinatewise to the diameter and equiangular parallelogrammic figures are described both on the ordinate and on the radius and the ordinate side has to the remaining side of the figure the ratio compounded of the ratio of the radius to the remaining side of the figure and of the ratio of the ordinate to the remaining side of the figure.

the area and circumference of a circle the logarithm with the figure on the ordinate is equal to the figure on the radius

Le h l h m h e e n a = m f m f  
 eter  
 CD  
 equi

I

and in the case of the ellipse and circle

figure on  $ED+GD=AF$

For let it be contrived that

the upright    the transverse    *DC*    *CH*

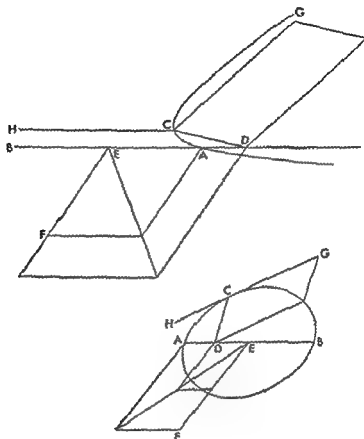
And since

*DC CH*    the upright    the transverse

but

*DC CH*    *sq DC*    *rect DC CH*

and

the upright the transverse sq  $DC$  rect  $BD, DA$  (1 21),

therefore

$$\text{rect } BD \ DA = \text{rect } DC \ CH$$

And since

 $DC \ CG \text{ comp } AE \ EF$  the upright the transverse

or

$$DC \ CG \text{ comp } AE \ EF \ DC \ CH,$$

and further

$$DC \ CG \text{ comp } DC \ CH \ CH \ CG$$

therefore

$$\text{ratio comp } AE \ EF \ DC \ CH = \text{ratio comp } DC \ CH \ CH \ CG$$

Let the common ratio  $DC \ CH$  be taken away, therefore  
 $AE \ EF \ CH \cdot CG.$

But

$$HC \ CG \text{ rect } HC, CD \text{ rect } GC, CD,$$

and

$$AE \ EF \text{ sq } AE \text{ rect } AE, EF,$$

therefore

$$\text{rect } HC, CD \text{ rect } GC, CD \text{ sq } AE \text{ rect } AE, EF.$$

And it has been shown that

$$\text{rect } HC, CD = \text{rect } BD, DA,$$

therefore

$$\text{rect } BD, DA \text{ rect } GC, CD \text{ sq } AE \text{ rect } AE, EF$$

Alternately

$$\text{rect } BD, DA \text{ sq } AE \text{ rect } GC, CD \text{ rect } AE, EF$$

And

$$\text{rect } GC, CD \text{ rect } AE, EF \text{ pll } DG \text{ pll } FA,$$

for they are equiangular and have to one another the ratio compounded of their sides  $GC$   $AE$  and  $CD$   $EF$  (Eucl vi 23), and therefore

$$\text{rect } BD, DA \text{ sq } EA \text{ pll } DG \text{ pll } FA$$

Moreover in the case of the hyperbola we are to say *componendo*

$$\text{rect } BD, DA + \text{sq } AE \text{ sq } AE \text{ pll } GD + \text{pll } AF \text{ pll } AF,$$

or

$$\text{sq } DE \text{ sq } EA \text{ pll } GD + \text{pll } AF \text{ pll } AF \text{ (Eucl II 6)}$$

$$\text{pll } GD + \text{pll } AF \text{ pll } AF \text{ figure on } ED \text{ pll } AF$$

Therefore

$$\text{figure on } ED = \text{pll } GD + \text{pll } AF,$$

the figure on  $ED$  being similar to the parallelogram  $AF$

And in the case of the ellipse and of the circumference of a circle we shall say since then

And

$$\text{sq } AC - \text{rect } BD, DA = \text{sq } DE \text{ (Eucl II 5),}$$

therefore

$$\text{sq } DE \text{ pll } AF - \text{pll } DG \text{ sq } AE \text{ pll } AF$$

But

$$\text{sq } DE \text{ pll } AF - \text{pll } DG = \text{figure on } DE + \text{pll } DG = \text{pll } AF \text{ (Eucl II 5)} \\ \text{the figure}$$

Therefore

$$\text{figure on } DE + \text{pll } DG = \text{pll } AF$$

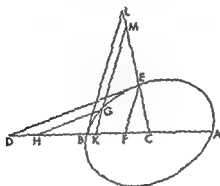
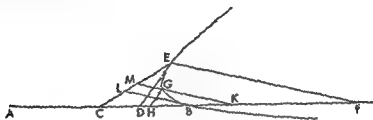
#### PROPOSITION 42





straight line  $BL$  be erected ordinatewise

I say that triangle  $KMC$  differs from triangle  $CLB$  by triangle  $GKH$



For since the straight line  $ED$  touches and the straight line  $EF$  has been dropped, hence

$EF \cdot FD \text{ comp } CF \cdot FE$ , the upright the transverse (1 39)

But

$$EF \cdot FD = GK \cdot KH,$$

and

$$CF \cdot FE = CB \cdot BL \text{ (Eucl vi 4),}$$

therefore

$$GK \cdot KH \text{ comp } BC \cdot BL, \text{ the upright the transverse}$$

#### PROPOSITION 44

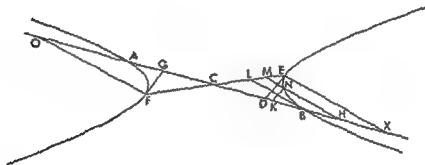
line drawn through the point of contact and the center, and, some point being taken at random on the section, let straight lines be dropped to the diameter, one of which

Let there be the  $\text{opp}$  straight line  $AB$ , and center the point  $C$ , and let the point  $F$  of those on the section  $FA$  let the straight line  $FG$  be drawn tangent to the section, and the

from  $N$  let the straight line  $NH$  be dropped ordinatewise, and let the straight line  $NK$  be drawn parallel to the straight line  $FG$

I say that

$$\text{trgl HKN} + \text{trgl CBL} = \text{trgl CMH}$$



For through  $E$  let the straight line  $ED$  be drawn tangent to the section  $BE$ , and let the straight line  $EX$  be drawn ordinatewise. Since then  $FA$  and  $BE$  are

diameter  $\square$  the straight line  $AB$ , and whose center is  $C$ , and the straight line  $DE$  is tangent to the section, and  $EX$  drawn ordinatewise, and  $BL$  is parallel to  $EA$  and  $N$  has been taken on the section as the point from which  $NH$  has

<sup>1</sup>Futourus commenting says 'For since  $AF$  is an hyperbola and  $BG$  a tangent and  $FO$  an ordinate

(r 37) likewise then also  
Therefore  $\text{rect } OC, CG = \text{sq } AC$  and  $\text{rect } AC, CD = \text{sq } CB$   
and alternately,  $\text{rect } OC, CG = \text{rect } AC, CD$  and  $\text{sq } AC = \text{sq } CB$   
But  $\text{sq } AC = \text{sq } CB$ ,  
therefore also  $\text{rect } OC, CG = \text{rect } AC, CD$   
And  $OC = CA$  (r 14 30),  
and therefore  $GC = CD$ ,  
and also  $FC = CE$  (r 30),  
therefore  $FC = FC, GC = CD$

And they contain equal angles at the point  $C$ , for they are vertical. And so also  $FG = ED$

and angle  $CFG = \text{angle } CFD$

And they are alternate, therefore the straight line  $FG$  is parallel to the straight line  $ED$ "

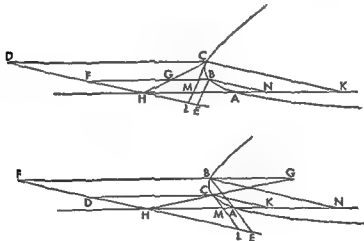
been dropped ordinatewise and  $KN$  has been drawn parallel to  $DE$ , therefore  
 $\text{trgl } NHK + \text{trgl } BCL = \text{trgl } HMC$ ,  
 for this has been shown in the forty-third theorem (1.43)

### PROPOSITION 45

and the center a straight line is produced, and, some point being taken at random on the section, two straight lines are drawn to the second diameter one of which is

and circle, together with the triangle cut off will be equal to the triangle whose base is the tangent and whose vertex is the center of the section

Let there be an hyperbola or ellipse or circumference of a circle  $ABC$ , whose



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1.

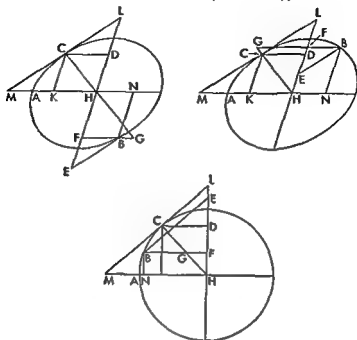
<sup>7</sup>+trgl LCH.

$$\text{trgl } BEF + \text{trgl } FGH = \text{trgl } CLH$$

For let the straight lines  $CK$  and  $BN$  be drawn parallel to  $DH$ . Since then the straight line  $CM$  is tangent and the straight line  $CK$  has been dropped ordinatewise, hence

$CA \wedge H \text{ comp } MK \wedge C$ , the upright the transverse (1 39).

$MK \quad KC \quad CD \quad DL$  (Eucl vi 4);



therefore

$CK \quad KH$  comp  $CD \quad DL$ , the upright the transverse  
And triangle  $CDL$  is the figure on  $AH$ , and triangle  $CKH$ , that is triangle  $CDH$ , is the figure on  $CK$ , that is on  $DH$ , therefore, in the case of the hyperbola,

$\text{trgl } CDL = \text{trgl } CKH + \text{trgl on } AH \text{ similar to trgl } CDL$ ,  
and, in the case of the ellipse and the circle,

$\text{trgl } CDH + \text{trgl } CDL = \text{trgl on } AH \text{ similar to trgl } CDL$ ,  
for this was also shown in the case of their doubles in the forty-first theorem (i 41)

Since then triangle  $CDL$  differs either from triangle  $CKH$  or from triangle  $CDH$  by the triangle on  $AH$  similar to triangle  $CDL$ , and it also differs by triangle  $CHL$  therefore

$\text{trgl } CHL = \text{trgl on } AH \text{ similar to trgl } CDL$

$BN$ , that is on  $FH$ , and by things already shown (i 41) triangle  $BFE$  differs from triangle  $GHE$  by the triangle on  $AH$  similar to  $CDL$ , and so also by triangle  $CHL$

<sup>1</sup>That is (Eucl vi 4)

and  $\frac{BF}{GF} = \frac{FE}{FH} = \frac{CD}{CD} = \frac{DL}{DH} = \frac{CK}{CK} = \frac{KH}{KH}$

Therefore these first ratios can be substituted in the central proportion of the theorem

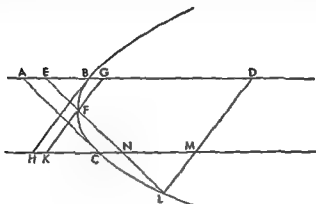
$CK \quad KH$  comp  $CD \quad DL$  the upright the transverse,

and so satisfy i 41

## PROPOSITION 46

If a straight line touching a parabola meets the diameter, the straight line drawn through the point of contact parallel to the diameter in the direction of the section bisects the straight lines drawn in the section parallel to the tangent

Let there be a parabola whose diameter is the straight line  $ABD$ , and let the



be drawn parallel to  $AC$

I say that

$$LN = NF$$

Let the straight lines  $BH$ ,  $KFG$ , and  $LMD$  be drawn ordinatewise. Since then by the things already shown in the forty-second theorem (I. 42)

$$\text{trgl } ELD = \text{pllg } BM,$$

and

$$\text{trgl } EFG = \text{pllg } BK,$$

therefore the remainders

$$\text{pllg } GM = \text{quadr } LFGD$$

Let the common pentagon  $MDGFN$  be subtracted, therefore the remainders

$$\text{trgl } KFN = \text{trgl } LVN$$

And  $AF$  is parallel to  $LV$ , therefore

$$FN = LN \text{ (Eucl VI 22 lemma)}$$

## PROPOSITION 47

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter and through the point of contact and the center a straight line is drawn in the direction of the section, it bisects the straight lines drawn in the section parallel to the tangent

let a point  $N$  be taken at random on the section, and through  $N$  let the straight line  $HN$  be drawn parallel.

I say that

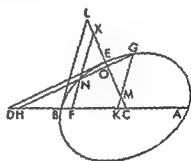
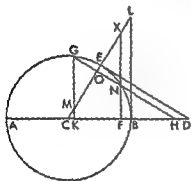
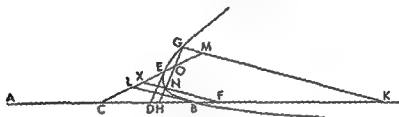
$$NO = OG.$$

For let the straight lines  $XNF$ ,  $BL$ , and  $GMK$  be dropped ordinatewise. Therefore by things already shown in the forty-third theorem (I. 43)

$$\text{trgl } HNF = \text{quadr } LBFX,$$

and

$$\text{trgl } GHK = \text{quadr } LBKM.$$



Therefore the remainders

$$\text{quadr } NGKF = \text{quadr } MKFX,$$

Let the common pentagon  $ONFKM$  be subtracted, therefore the remainders

$$\text{trgl } HNF = \text{trgl } GHK$$

∴

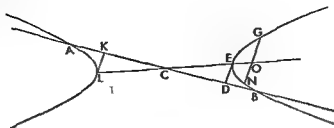
∴

### PROPOSITION 48

If a straight line touching one of the opposite sections meets the diameter, and through the point of contact and the center a straight line produced cuts the other section, then whatever line is drawn in the other section parallel to the tangent, will be bisected by the straight line produced

AB and  
straight  
on the

section  $B$ , and through  $N$  let the straight line  $NG$  be drawn parallel to the straight line  $LA$ .



I say that

$$NO = OG$$

For  
theref  
is an l  
joined and a point  $N$  has been taken on the section and through it  $NG$  has been  
drawn parallel to  $DE$  by a theorem already shown (I 47) for the hyperbola  
 $NO = OG$

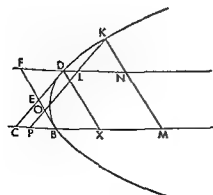
#### PROPOSITION 49

If a straight line touching a parabola meets the diameter, and through the point of contact a parallel to the diameter is drawn and from the vertex a straight line is drawn parallel to an ordinate and it is contrived that as the segment of the tangent between the diameter and the point of contact is to the segment of the

no 111

1

2 = 2 and the point of contact is to the segment of the



$ED$   $DF$  some straight line  
 $G$   $2CD$

and let some point  $K$  be taken on the section and let the straight line  $KLP$  be drawn through  $K$  parallel to  $CD$

I say that

$$\text{sq } KL = \text{rect } GDL$$

that is that, with the straight line  $DL$  as diameter, the straight line  $G$  is the upright side

For let the straight lines  $DX$  and  $KN$  be dropped ordinate-wise And since the straight line



$CD$  touches the section, and the straight line  $DX$  has been dropped ordinate-wise, then

$$CB = BX \text{ (I 35)}$$

But

$$BX = FD$$

and therefore

$$CB = FD$$

And so also

$$\text{trgl } ECB = \text{trgl } EFD$$

Let the common figure  $DEBMN$  be added, therefore

$$\begin{aligned} \text{quadr } DCMN &= \text{pllg } FM \\ &= \text{trgl } KPM \text{ (I 42)} \end{aligned}$$

Let the common quadrilateral  $LPMN$  be subtracted, therefore the remainders

$$\text{trgl } KLN = \text{pllg } LC$$

And

$$\text{angle } DLP = \text{angle } KLN,$$

therefore

$$\text{rect } KL, LN = 2 \text{ rect } LD, DC,^1$$

And since

$$ED \text{ DF } G \text{ } 2CD,$$

and

$$ED \text{ DF } KL \text{ LN},$$

therefore also

$$G \text{ } 2CD \text{ } KL \text{ LN}$$

But

$$KL \text{ LN } \text{sq } KL \text{ rect } KL \text{ LN},$$

and

$$G \cdot 2CD \cdot \text{rect } G, DL \text{ } 2 \text{ rect } LD \text{ } DC;$$

therefore

<sup>1</sup>Eutocius commenting says "For let the triangle  $KIN$  and the parallelogram  $DLPC$  be set out. And since

$$\text{trgl } ALN = \text{pllg } DP,$$

let the straight line  $NR$  be drawn through  $N$  parallel to  $LA$ , and through  $K$ ,  $KR$  parallel to  $LN$ , therefore  $LR$  is a parallelogram and

$$\text{pllg } LR = 2 \text{ trgl } ALN,$$

and so also

$$\text{pllg } LR = 2 \text{ pllg } DP$$

Then let the straight lines  $DC$  and  $LP$  be produced to  $S$  and  $T$  and let  $CS$  be made equal to  $DC$ , and  $PT$  to  $LP$  and let  $ST$  be joined, therefore

$$\text{pllg } DT = 2 \text{ pllg } DP,$$

and so

$$\text{pllg } LR = \text{pllg } LS$$

But it is also equiangular with it because of the angles at  $L$  being vertical, but in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional, therefore

$$KL \text{ } LT \text{ or } DS \text{ } DL \text{ } LN$$

and

$$\text{rect } AL, LN = \text{rect } LD, DS$$

And since

$$DS = 2DC$$

hence

$$\text{rect } KL, LN = 2 \text{ rect } LD, DC$$

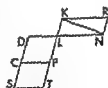
'And if  $DC$  is parallel to  $LP$  and  $CP$  is not parallel to  $LD$ , it is clear  $DCPL$  is a trapezoid, and so I say that

$$\text{rect } AL \text{ } LN = \text{rect } DL, CD + LR$$

For if  $LP$  is filled out as we have said before, and the straight lines  $DC$  and  $LP$  are produced, and  $CS$  is made equal to  $LP$ , and  $PT$  to  $DC$  and the straight line  $ST$  is joined, then

$$\text{pllg } DT = 2DP,$$

and the same demonstration will fit. And this will be useful in what follows (I 50) "



sq  $KL$  rect  $KL, LN \therefore$  rect  $G, DL$  2rect  $CD, DL$

And alternately, but

rect  $KL, LN = 2\text{rect } CD, DL,$

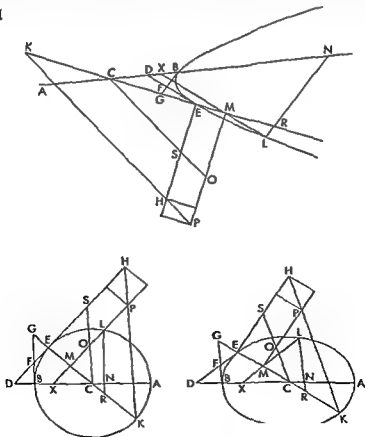
therefore also

sq  $KL = \text{rect } G, DL$

# PROPOSITION 50

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter, and a straight line is produced through the point of contact and the center, and from the vertex a straight line erected parallel to an ordinate meets the straight line drawn through the point of contact and the center, and if it is contrived that as the segment of the tangent between the point of contact and the straight line erected is to the segment of the straight line, drawn through the point of contact and the center, between the point of contact and the straight line erected, so some straight line is to the double of the tangent, then whatever straight line is drawn from the section to the straight line drawn through the point of contact and the center, parallel to the tangent, will equal in square = rectangular area applied to the straight line found, having as breadth the straight line cut off by it from the

## CASES I





and let the straight line  $HA$  be joined and produced and let some point  $L$  be taken on the section and through it let the straight line  $LM\lambda$  be drawn parallel to  $ED$  and the straight line  $LRN$  parallel to  $BG$  and the straight line  $MP$  parallel to  $EH$

I say that

$$\text{sq } LM = \text{rect } EM, MP$$

For let the straight line  $CSO$  be drawn through  $C$  parallel to  $AP$  And since  $EC = CA$ ,

and

Cases I

$$EC \quad AC \quad ES \quad SH$$

therefore also

$$ES = SH$$

Cases II

And since

$$FE \quad EG \quad HE \quad 2ED,$$

and

$$2ES = EH$$

therefore also

$$FE \quad EG \quad SE \quad ED$$

And

$$FE \quad EG \quad LM \quad MR$$

therefore

$$LM \quad MR \quad SE \quad ED$$

And since it was shown (I 43)

$$\text{trgl } RNC$$

and in the case of the ellipse and circle

$$\text{trgl } RNC + \text{trgl } LNX = \text{trgl } GBC \\ = \text{trgl } CDE$$

therefore in the case of the hyperbola with the common triangle  $ECD$  and the common quadrilateral  $NRM\lambda$  subtracted and in the case of the ellipse and circle with the common triangle  $M\lambda C$  subtracted<sup>2</sup>

That

$$\text{trgl } GBC = \text{trgl } CDE$$

is proved by Apollonius in the course of another proof of I 43 reported by Eutocius It is also proved in III 1 without the help of intervening propositions

<sup>2</sup>The position of point  $L$  furnishes different cases which at times as in the present the

Subtracting the first equals from the second identity we have

$$\text{trgl } LMR = \text{quadr } MEDX$$

```
trgl LMR=quadr MEDX
```

And  $MX$  is parallel to  $DE$ , and

angle  $LMR = \text{angle } EMX$ .

therefore

$$\text{rect } LM, MR = \text{rect } EM, ED + MX \text{ (1-49, note, para 2).}$$

And since

MC CE MX ED

and

MC CE MO ES.

therefore

MO ES MX ED

And componendo

MO+ES ES MX+ED ED.

alternately

MO+ES MX+ED ES ED

**But**

$$MO+ES \quad MX+ED \quad \text{rect} \quad MO+ES, EM \quad \text{rect} \quad MX+ED, EM,$$

and

$ES \quad ED \quad LM \quad MR \quad FE \quad EG$  (Eucl vi 4)

or

ES ED sq LV rect LV.MR.

therefore

rect  $MO+ES, ME$  rect  $MX+ED$   $CM$  sq  $LM$  rect  $LM, VR$

And alternately

rect *MO+ES,ME* sq *LM* rect *MY+ED EM* rect *LM,MR*

But

rect LM.MR=rect ME MX+ED (above).

therefore

sq  $LM \approx \text{rect } EM \text{ } MO + ES$

And

$SE = SH.$

and

$SH = OP.$

therefore

sq  $LM \approx \text{rect } EM, MP$

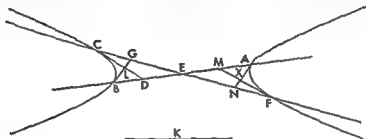
### PROPOSITION 51

If a straight line touching either of the opposite sections meets the diameter, and through the point of contact and the center some straight line is produced to the other section, and from the vertex a straight line is erected parallel to an ordinate and meets the straight line drawn through the point of contact and the center, and if it is contrived that, as the segment of the tangent between the straight line erected



and the point of contact is to the segment of the straight line, drawn through the point of contact and the center, between the point of contact and the straight line erected, so is some straight line to the double of the tangent, then whatever straight line in the other of the sections is drawn to the straight line through the point of contact and the center, parallel to the tangent, will equal in square the rectangle applied to the straight line found, having as breadth the straight line cut off by it from the point of contact and exceeding by a figure similar to the rectangle contained by the straight line between the opposite sections and the straight line found.

Let there be opposite sections whose diameter is the straight line  $AB$  and



center  $E$ , and let the straight line  $CD$  be drawn tangent to the section  $B$  and the straight line  $MF$  be drawn tangent to the section  $F$ .

Now it is evident that the straight lines in the section  $BC$ , parallel to  $CD$  and drawn to  $EC$  produced are equal in square to the areas applied to  $K$ , having as breadths the straight line cut off by them from the point of contact, and exceeding by a figure similar to the rectangle  $CF, K$ , for

$$GC = 2CE$$

I say then that in section  $FA$  the same thing will come about

For let the straight line  $MF$  be drawn through  $F$  tangent to the section  $AF$ , and let the straight line  $AXN$  be erected ordinatewise. And since  $BC$  and  $AF$  are opposite sections and  $CD$  and  $MF$  are tangents to them therefore  $CD$  is equal and parallel to  $MF$  (1. 44. note). But also

$$CE = EF,$$

therefore also

$$ED = EV$$

And since

$$LC \cdot CG = K \cdot 2CD \text{ or } 2MF,$$

therefore also

$$XF \cdot FN = K \cdot 2MF$$

Since then  $AF$  is an hyperbola whose diameter is  $AB$  and tangent  $MF$ , and  $AN$  has been drawn ordinatewise and

$$XF \cdot FN = K \cdot 2FM,$$

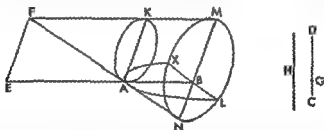
hence

(I 46), but in the hyperbola and ellipse and opposite sections each of the straight lines drawn through the center is a diameter (I 47-48), and that in the parabola the straight lines dropped to each of the diameters parallel to the tangents will equal in square the rectangles applied to it (I 49), but in the hyperbola and opposite sections they will equal in square the areas applied to it and exceeding by the same figure (I 50-51), but in the ellipse the areas applied to it and defective by the same figure (I 50), and that all the things which have been already proved about the sections as following when the principal diameters are used,<sup>1</sup> will also, those very same things, follow when the other diameters are taken

### PROPOSITION 52 (PROBLEM)

Given a straight line in a plane bounded at one point, to find in the plane the section of a cone called parabola, whose diameter is the given straight line, and whose vertex is the end of the straight line, and where whatever straight line is dropped from the section to the diameter at a given angle, will equal in square the rectangle contained by the straight line cut off by it from the vertex of the section and by some other given straight line

Let there be the straight line  $AB$  given in position and bounded at the point  $A$ , and another straight line  $CD$  given in magnitude, and first let the given



angle be a right angle it is required then to find a parabola in the plane of reference whose diameter is the straight line  $AB$  and whose vertex is the point  $A$ , and whose upright side is the straight line  $CD$ , and where the straight lines dropped ordinatewise will be dropped at a right angle that is so that  $AB$  is the axis (First Def 1 7)

Let  $AB$  be produced to  $E$  and let  $CG$  be taken as the fourth part of  $CD$ , and let

$$EA > CG,$$

and let

$$CD : H :: H : EA$$

Therefore

$$CD : EA :: sq H : sq EA,$$

and

$$CD < 4EA,$$

therefore also

$$sq H < 4 sq EA$$

<sup>1</sup>The principal diameter (*ἀκέρως ἀρχὴς*) is that whose being is established in I 7 porism.

Therefore

$$H < 2EA,$$

and the two straight lines  $EA$  are greater than  $H$ . It is therefore possible for a triangle to be constructed from  $H$  and two straight lines  $EA$ . Then let the triangle  $EAF$  be constructed on  $EA$  at right angles to the plane of reference so that

$$EA = AF,$$

and

$$H = FE,$$

and let the straight line  $AK$  be drawn parallel to  $FE$ , and  $FK$  to  $EA$  and let a cone be conceived whose vertex is the point  $F$  and whose base is the circle about diameter  $KA$ , at right angles to the plane through  $AFX$ . Then the cone will be a right cone (First Def 13), for

$$AF = FK$$

And let the cone be cut by a plane parallel to the circle  $KA$ , and let it make

circle. Since then circle  $MNX$  is at right angles to triangle  $MFN$ , and the plane of reference also at right angles to triangle  $MFN$ , therefore the straight line  $LX$ ,

point  $F$ , has been cut by a plane at right angles to the triangle  $MFN$  and makes as a section circle  $MNX$ , and since it has also been cut by another plane, the

$$CD = H = EA,$$

and

$$EA = AF = FK$$

and

$$H = EF = AK$$

therefore

$$CD = AK = AF$$

And therefore

$$CD = AF = AK = AF \text{ or rect } AF, FK$$

Therefore  $CD$  is the upright side of the section, for this has been shown in the eleventh theorem (11)

#### PROPOSITION 53 (PROBLEM)

With the same things supposed, let the given angle not be right, and let the angle  $HAE$  be made equal to it and let



$AH = \text{half } CD,$

and from  $H$  let the straight line  $HE$  be drawn perpendicular to  $AE$ , and through  $E$  let the straight line  $EL$  be drawn parallel to  $BH$ , and from  $A$  let the straight line  $AL$  be drawn perpendicular to  $EL$ , and let  $EL$  be bisected at  $K$ , and from  $K$  let the straight line  $KM$  be drawn perpendicular to  $EL$  and produced to  $F$  and  $G$ , and let rect  $LK, KM = \text{sq } AL$ . And given the two straight lines  $LK$  and  $KM, KL$  in position and bounded at  $K$ , and  $KM$  in magnitude, and let a parabola be described with a right angle whose diameter is the straight line  $KL$ , and whose vertex is the point  $K$ , and whose upright side is the straight line  $KM$ , as has been shown before (1 52), and it will pass through the point  $A$  because

$\text{sq } AL = \text{rect } LK, KM$  (1 11),

and the straight line  $EA$  will touch the section because

$EK = KL$  (1 33)

And  $HA$  is parallel to  $EAL$ , therefore  $HAB$  is the diameter of the section, and the straight lines dropped to it parallel to  $AE$  will be bisected by  $AB$  (1 46) And they will be dropped at angle  $HAE$ . And since

angle  $AEH = \text{angle } AGF,$

and angle at  $A$  is common, therefore triangle  $AHE$  is similar to triangle  $AGF$ . Therefore

$$HA \cdot EA = FA \cdot AG,$$

therefore

$$2AH \cdot 2AE = FA \cdot AG$$

But

$$CD = 2AH,$$

therefore

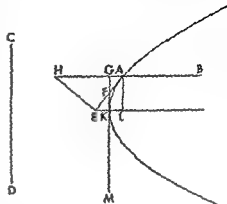
$$FA \cdot AG = CD \cdot 2AE$$

Then by things already shown in the forty ninth theorem (1 49) the straight line  $CD$  is the upright side

#### PROPOSITION 54

Given two bounded straight lines perpendicular to each other one of them being produced on the side of the right angle, to find on the straight line produced the section of a cone called hyperbola in the same plane with the straight lines, so that the straight line produced is a diameter of the section and the point at the angle is the vertex, and where whatever straight line is dropped from the section to the diameter, making an angle equal to the given angle will equal in square the rectangle applied to the other straight line having as breadth the straight line cut off by the dropped straight line beginning with the vertex and exceeding by a figure similar and similarly situated to that contained by the original straight lines

Let there be the two bounded straight lines  $AB$  and  $BC$  perpendicular to





$EK$  be drawn perpendicular from  $E$  to the straight line  $AB$  and let it be produced to  $L$ , therefore the straight line  $EL$  is a diameter (Eucl III 1) If then

$$AB \ BC \ EK \ KL$$

we use point  $L$ , but if not let it be contrived that

$$AB \ BC \ EK \ KM$$

with

$$KM < KL \text{ (Eucl V 8),}$$

and through  $M$  let  $MF$  be drawn parallel to  $AB$ , and let  $AF$ ,  $EF$ , and  $FB$  be joined, and through  $B$  let  $BX$  be drawn parallel to  $FE$  Since then

$$\text{angle } AFE = \text{angle } EFB,$$

but

$$\text{angle } AFE = \text{angle } AXB,$$

and

$$\text{angle } EFB = \text{angle } XBF,$$

therefore also

$$\text{angle } XBF = \text{angle } FXB,$$

therefore also

$$FB = FX$$

Let a cone be conceived whose vertex is the point  $F$  and whose base is the circle about diameter  $BX$  at right angles to triangle  $BFX$  Then the cone will be a right cone for

$$FB = FX$$

Then let the straight lines  $BF$ ,  $FX$  and  $MF$  be produced, and let the cone be cut by a plane parallel to the circle  $BX$ , then the section will be a circle (I 4) Let it be the circle  $GPR$ , and so  $GH$  will be the diameter of the circle (I 4 end) And let the straight line  $PDR$  be the common section of circle  $GH$  and of the plane of reference then  $PDR$  will be perpendicular to both of the straight lines  $GH$  and  $DB$ , for both of the circles  $XB$  and  $HG$  are perpendicular to triangle  $BFX$  and therefore  $PDR$  is perpendicular to the plane of triangle  $BFX$  touching it and in the same plane

$$AB < BC$$

it will be above  $D$

And now let it be below as  $G$  and with center  $G$  and radius  $GF$  let a circle be described then it will have to pass either within or without the points  $A$  and  $B$  And if it should pass

and

$$DH = DK$$

Then likewise also

$$KD \ DB \ FD \ DL$$

And therefore

$$FD \ DM \ ED \ DL$$

And alternately,

$$ED \ DF \ AB \ BC \ LD \ DM$$

'And likewise if the circle described on  $FE$  cuts  $AB$ , the same thing could be shown.'

line  $GDH$ , and the common section of the plane of reference and of triangle  $GFH$  that is the straight line  $DB$ , produced in the direction of  $B$ , meets the straight line  $GF$  at  $A$ , therefore by things already shown before (1 12) the

and

$$EA \cdot KM \cdot EN \cdot NF \text{ rect } EN \cdot NF \text{ sq } NF,$$

therefore

$$AB \cdot BC \text{ rect } EN \cdot NF \text{ sq } NF$$

And

$$\text{rect } EN, NF = \text{rect } AN \cdot NB,$$

therefore

$$AB \cdot CB \text{ rect } AN \cdot NB \text{ sq } NF$$

But

$$\text{rect } AN, NB \text{ sq } NF \text{ comp } AN \cdot NF \cdot BN \cdot NF,$$

but

$$AN \cdot NF \cdot AD \cdot DG \cdot FO \cdot OG,$$

and

$$BN \cdot NF \cdot FO \cdot OH,$$

therefore

$$AB \cdot BC \text{ comp } FO \cdot OG, FO \cdot OH,$$

that is

$$\text{sq } FO \text{ rect } OG \cdot OH$$

Therefore

$$AB \cdot BC \text{ sq } FO \text{ rect } OG \cdot OH$$

And the straight line  $FO$  is parallel to the straight line  $AD$ , therefore the straight line  $AB$  is the transverse side and  $BC$  the upright side, for these things have been shown in the twelfth theorem (1 12)

#### PROPOSITION 55 (PROBLEM)

Then let the given angle not be a right angle and let there be the two given

$HAB$

Let the straight line  $AB$  be bisected at  $D$  and let the semicircle  $AFD$  be described on  $AD$  and let some straight line  $FG$  parallel to  $AH$ , be drawn to the semicircle making

$$\text{sq } FG \text{ rect } DG \cdot GA \cdot AC \cdot AB^1$$

<sup>1</sup>Cutoc us, comment ng gives th a construction Let there be the semicircle  $ABC$  on the





of which the greater is the straight line  $AB$ , then it is required to describe in

$BA, AC$

And first let the given angle be a right angle, and let a plane be erected from  $AB$  at right angles to the plane of reference, and in it, on  $AB$ , let the sector of a circle  $ADB$  be described, and its midpoint be  $D$ , and let the straight lines  $DA$  and  $DB$  be joined, and let the straight line  $AX$  be made equal to  $AC$ , and through  $X$  let the straight line  $XO$  be drawn parallel to  $DB$ , and through  $O$  let  $OF$  be drawn parallel to  $AB$ , and let  $DF$  be joined and let it meet  $AB$  produced at  $E$ , then we will have

$$AB \ AC \ AB \ AX \ DA \ AO \ DE \ EF$$

And let the straight lines  $AF$  and  $FB$  be joined and produced, and let some point  $G$  be taken at random on  $FA$ , and through it let the straight line  $GL$  be drawn parallel to  $DE$  and let it meet  $AB$  produced at  $K$ , then let  $FO$  be produced and let it meet  $GK$  at  $L$ . Since then

$$\begin{aligned} \text{arc } AD &= \text{arc } DB, \\ \text{angle } ABD &= \text{angle } DFB \text{ (Eucl. III 27)} \end{aligned}$$

And since

$$\text{angle } EFA = \text{angle } FDA + \text{angle } FAD,$$

but

$$\text{angle } FAD = \text{angle } FBD,$$

and

$$\text{angle } FDA = \text{angle } FBA,$$

therefore also

$$\text{angle } EFA = \text{angle } DBA = \text{angle } DFB$$

And also  $DE$  is parallel to  $LG$ ; therefore

$$\text{angle } EFA = \text{angle } FGH,$$

and

$$\text{angle } DFB = \text{angle } FHG$$

And so also

$$\text{angle } FGH = \text{angle } FHG,$$

and

$$FG = FH$$

Then let circle  $GHN$  be described about  $HG$  at right angles to triangle  $HGF$ , let a cone be conceived whose base is the circle  $GHN$ , and whose vertex is the point  $F$ , then the cone will be a right cone because

$$FG = FH$$

And since the circle  $GHN$  is at right angles to plane  $HGF$ , and the plane of reference is also at right angles to the plane through  $GH$  and  $HF$ , therefore

point  $F$ , has been cut by a plane through the axis and makes as a section triangle  $GHF$ , and has been cut also by another plane through  $AK$  and  $KM$ ,

will be dropped at a right angle (I 13), for they are parallel to  $AM$ . And since  
 $DE \cdot EF = \text{rect } DE \cdot EF$  or  $\text{rect } BE \cdot EA = \text{sq } EF$ ,

and

$$\text{rect } BE \cdot EA = \text{sq } EF \text{ comp } BF \cdot EF, AE \cdot EF,$$

but

$$BE \cdot EF = BK \cdot KH,$$

and

$$AE \cdot EF = AK \cdot KG = FL \cdot LG,$$

therefore

$$BA \cdot AC \text{ comp } FL \cdot LG, FL \cdot LH \text{ (see above),}$$

which is the same as

$$\text{sq } FL = \text{rect } GL, LH,$$

therefore

$$BA \cdot AC = \text{sq } FL = \text{rect } GL, LH$$

And whenever this is so, the straight line  $AC$  is the upright side of the figure, as has been shown in the thirteenth theorem (I 13)

#### PROPOSITION 57 (PROBLEM)

With the same things supposed let the straight line  $AB$  be less than  $AC$ , and let it be required to describe an ellipse about diameter  $AB$  so that  $AC$  is the upright

Let  $AB$  be bisected at  $D$ , and from  $D$  let the straight line  $EDF$  be drawn perpendicular to  $AB$ , and let

$$\text{sq } FE = \text{rect } BA \cdot AC$$

so that

$$FD = DE,$$

and let  $FG$  be drawn parallel to  $AB$ , and let it be contrived that

$$AC \cdot AB = EF \cdot FG,$$

therefore also

$$EF > FG$$

And since

$$\text{rect } CA \cdot AB = \text{sq } EF,$$

hence

$$CA \cdot AB = \text{sq } FE = \text{sq } AB + \text{sq } DF = \text{sq } DA$$

But

$$CA \cdot AB = EF \cdot FG,$$

therefore

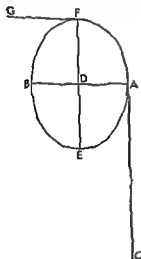
$$EF \cdot FG = \text{sq } FD + \text{sq } DA$$

But

$$\text{sq } FD = \text{rect } FD \cdot DE,$$

therefore

$$EF \cdot FG = \text{rect } ED \cdot DF + \text{sq } AD$$



Then  
and w  
upright



rect  $FD, DE : sq DA :: EF : FG$  (I. 21).

And

$$AD = DB;$$

then it will also pass through  $B$ . Then an ellipse has been described about  $AB$ .  
And since

$$CA \cdot AB :: sq FD \cdot sq DA,$$

and

$$sq DA = rect AD, DB,$$

therefore

$$CA \cdot AB = sq DF \cdot rect AD, DB.$$

And so the straight line  $AC$  is an upright side (I. 21)

### PROPOSITION 58 (PROBLEM)

But then let the given angle not be a right angle, and let the angle  $BAD$  be equal to it, and let the straight line  $AB$  be bisected at  $E$ , and let the semicircle  $AFE$  be described on  $AE$ , and in it let the straight line  $FG$  be drawn parallel to  $AD$  making

$$sq FG : rect AG, GE :: CA \cdot AB,$$

and let the straight lines  $AF$  and  $EF$  be joined and produced, and let  
 $DE : EH :: EH : EF$ ,

and let

$$LX = LN,$$

while like

componendo  $HG : GF :: LN : NM$ ,

inversely  $FG : GH :: NM : NL$ ,

and  $FG : GE :: VN : NX$ ,

separando  $FG : FE :: NM : MX$ .

And since  $AL = LX$ ,

and the straight line  $LK$  is common and at right angles, therefore also

$$AN = KX$$

And also  $KO = KP$ ,

therefore  $AP = PO$ .

And also

But also  $KM : KR :: MX : PR$ ,

and therefore  $NM : RO :: UX : PR$ ,

and alternately  $NM : MX :: RO : RP$ .

But  $AM : MX :: GF : FE :: DE : EF$ ,

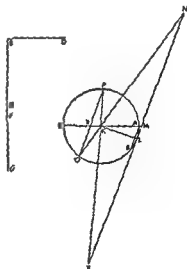
and  $OR : RP :: sq OR : rect OR, RP$ ;

and therefore

$$DE : EF :: sq OR : rect OR, RP$$

And  $rect OR, RP = rect AR, RC$  (Eucl III 35)

Therefore  $DE : EF :: sq OR : rect AR, RC$ "



and let

$EK = EH.$

and let it be contrived that

rect  $HF, FL = \text{sq } AF,$

and 1.12. 1.13. 1.14. 1.15. 1.16. 1.17. 1.18. 1.19. 1.20. 1.21. 1.22. 1.23. 1.24. 1.25. 1.26. 1.27. 1.28. 1.29. 1.30. 1.31. 1.32. 1.33. 1.34. 1.35. 1.36. 1.37. 1.38. 1.39. 1.40. 1.41. 1.42. 1.43. 1.44. 1.45. 1.46. 1.47. 1.48. 1.49. 1.50. 1.51. 1.52. 1.53. 1.54. 1.55. 1.56. 1.57. 1.58. 1.59. 1.60. 1.61. 1.62. 1.63. 1.64. 1.65. 1.66. 1.67. 1.68. 1.69. 1.70. 1.71. 1.72. 1.73. 1.74. 1.75. 1.76. 1.77. 1.78. 1.79. 1.80. 1.81. 1.82. 1.83. 1.84. 1.85. 1.86. 1.87. 1.88. 1.89. 1.90. 1.91. 1.92. 1.93. 1.94. 1.95. 1.96. 1.97. 1.98. 1.99. 2.00. 2.01. 2.02. 2.03. 2.04. 2.05. 2.06. 2.07. 2.08. 2.09. 2.10. 2.11. 2.12. 2.13. 2.14. 2.15. 2.16. 2.17. 2.18. 2.19. 2.20. 2.21. 2.22. 2.23. 2.24. 2.25. 2.26. 2.27. 2.28. 2.29. 2.30. 2.31. 2.32. 2.33. 2.34. 2.35. 2.36. 2.37. 2.38. 2.39. 2.40. 2.41. 2.42. 2.43. 2.44. 2.45. 2.46. 2.47. 2.48. 2.49. 2.50. 2.51. 2.52. 2.53. 2.54. 2.55. 2.56. 2.57. 2.58. 2.59. 2.60. 2.61. 2.62. 2.63. 2.64. 2.65. 2.66. 2.67. 2.68. 2.69. 2.70. 2.71. 2.72. 2.73. 2.74. 2.75. 2.76. 2.77. 2.78. 2.79. 2.80. 2.81. 2.82. 2.83. 2.84. 2.85. 2.86. 2.87. 2.88. 2.89. 2.90. 2.91. 2.92. 2.93. 2.94. 2.95. 2.96. 2.97. 2.98. 2.99. 3.00. 3.01. 3.02. 3.03. 3.04. 3.05. 3.06. 3.07. 3.08. 3.09. 3.10. 3.11. 3.12. 3.13. 3.14. 3.15. 3.16. 3.17. 3.18. 3.19. 3.20. 3.21. 3.22. 3.23. 3.24. 3.25. 3.26. 3.27. 3.28. 3.29. 3.30. 3.31. 3.32. 3.33. 3.34. 3.35. 3.36. 3.37. 3.38. 3.39. 3.40. 3.41. 3.42. 3.43. 3.44. 3.45. 3.46. 3.47. 3.48. 3.49. 3.50. 3.51. 3.52. 3.53. 3.54. 3.55. 3.56. 3.57. 3.58. 3.59. 3.60. 3.61. 3.62. 3.63. 3.64. 3.65. 3.66. 3.67. 3.68. 3.69. 3.70. 3.71. 3.72. 3.73. 3.74. 3.75. 3.76. 3.77. 3.78. 3.79. 3.80. 3.81. 3.82. 3.83. 3.84. 3.85. 3.86. 3.87. 3.88. 3.89. 3.90. 3.91. 3.92. 3.93. 3.94. 3.95. 3.96. 3.97. 3.98. 3.99. 4.00. 4.01. 4.02. 4.03. 4.04. 4.05. 4.06. 4.07. 4.08. 4.09. 4.10. 4.11. 4.12. 4.13. 4.14. 4.15. 4.16. 4.17. 4.18. 4.19. 4.20. 4.21. 4.22. 4.23. 4.24. 4.25. 4.26. 4.27. 4.28. 4.29. 4.30. 4.31. 4.32. 4.33. 4.34. 4.35. 4.36. 4.37. 4.38. 4.39. 4.40. 4.41. 4.42. 4.43. 4.44. 4.45. 4.46. 4.47. 4.48. 4.49. 4.50. 4.51. 4.52. 4.53. 4.54. 4.55. 4.56. 4.57. 4.58. 4.59. 4.60. 4.61. 4.62. 4.63. 4.64. 4.65. 4.66. 4.67. 4.68. 4.69. 4.70. 4.71. 4.72. 4.73. 4.74. 4.75. 4.76. 4.77. 4.78. 4.79. 4.80. 4.81. 4.82. 4.83. 4.84. 4.85. 4.86. 4.87. 4.88. 4.89. 4.90. 4.91. 4.92. 4.93. 4.94. 4.95. 4.96. 4.97. 4.98. 4.99. 5.00. 5.01. 5.02. 5.03. 5.04. 5.05. 5.06. 5.07. 5.08. 5.09. 5.10. 5.11. 5.12. 5.13. 5.14. 5.15. 5.16. 5.17. 5.18. 5.19. 5.20. 5.21. 5.22. 5.23. 5.24. 5.25. 5.26. 5.27. 5.28. 5.29. 5.30. 5.31. 5.32. 5.33. 5.34. 5.35. 5.36. 5.37. 5.38. 5.39. 5.40. 5.41. 5.42. 5.43. 5.44. 5.45. 5.46. 5.47. 5.48. 5.49. 5.50. 5.51. 5.52. 5.53. 5.54. 5.55. 5.56. 5.57. 5.58. 5.59. 5.60. 5.61. 5.62. 5.63. 5.64. 5.65. 5.66. 5.67. 5.68. 5.69. 5.70. 5.71. 5.72. 5.73. 5.74. 5.75. 5.76. 5.77. 5.78. 5.79. 5.80. 5.81. 5.82. 5.83. 5.84. 5.85. 5.86. 5.87. 5.88. 5.89. 5.90. 5.91. 5.92. 5.93. 5.94. 5.95. 5.96. 5.97. 5.98. 5.99. 6.00. 6.01. 6.02. 6.03. 6.04. 6.05. 6.06. 6.07. 6.08. 6.09. 6.10. 6.11. 6.12. 6.13. 6.14. 6.15. 6.16. 6.17. 6.18. 6.19. 6.20. 6.21. 6.22. 6.23. 6.24. 6.25. 6.26. 6.27. 6.28. 6.29. 6.30. 6.31. 6.32. 6.33. 6.34. 6.35. 6.36. 6.37. 6.38. 6.39. 6.40. 6.41. 6.42. 6.43. 6.44. 6.45. 6.46. 6.47. 6.48. 6.49. 6.50. 6.51. 6.52. 6.53. 6.54. 6.55. 6.56. 6.57. 6.58. 6.59. 6.60. 6.61. 6.62. 6.63. 6.64. 6.65. 6.66. 6.67. 6.68. 6.69. 6.70. 6.71. 6.72. 6.73. 6.74. 6.75. 6.76. 6.77. 6.78. 6.79. 6.80. 6.81. 6.82. 6.83. 6.84. 6.85. 6.86. 6.87. 6.88. 6.89. 6.90. 6.91. 6.92. 6.93. 6.94. 6.95. 6.96. 6.97. 6.98. 6.99. 7.00. 7.01. 7.02. 7.03. 7.04. 7.05. 7.06. 7.07. 7.08. 7.09. 7.10. 7.11. 7.12. 7.13. 7.14. 7.15. 7.16. 7.17. 7.18. 7.19. 7.20. 7.21. 7.22. 7.23. 7.24. 7.25. 7.26. 7.27. 7.28. 7.29. 7.30. 7.31. 7.32. 7.33. 7.34. 7.35. 7.36. 7.37. 7.38. 7.39. 7.40. 7.41. 7.42. 7.43. 7.44. 7.45. 7.46. 7.47. 7.48. 7.49. 7.50. 7.51. 7.52. 7.53. 7.54. 7.55. 7.56. 7.57. 7.58. 7.59. 7.60. 7.61. 7.62. 7.63. 7.64. 7.65. 7.66. 7.67. 7.68. 7.69. 7.70. 7.71. 7.72. 7.73. 7.74. 7.75. 7.76. 7.77. 7.78. 7.79. 7.80. 7.81. 7.82. 7.83. 7.84. 7.85. 7.86. 7.87. 7.88. 7.89. 7.90. 7.91. 7.92. 7.93.

*AF*L, for the angle at *F* is right And with the two given bounded straight lines *KH*, and *HM* perpendicular to each other, let an ellipse be described whose transverse diameter is *KH*, and the upright side of whose figure is *HM*, and where the ordinates to *HK* will be dropped at right angles (1 56-57), then the section will pass through *A* because sq *FA*=rect *HF*, *FL* (1 13) And since

**HE-EK.**

and

$AE = EB,$

the section -  
line  $AEB$  t

And since

$CA \parallel AB$  sq  $FG$  rect  $AG, GE$ .

that

CA AB comp CA 2AD 2AD AB or DA AE.

and

$$\text{sq } FG = \text{rect } AG, GE \text{ comp } FG \quad GE, FG \quad GA.$$

therefore

$$\text{ratio comp } CA \quad 2AD \quad DA \quad AE = \text{ratio comp } FG \quad GE \quad FG \quad GA$$

But

$DA \quad AE \quad FG \quad GE,$

and the common ratio being taken away we will have

CA 2AD FG GA

or

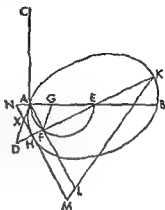
CA 2AD XA AN

And whenever this is so, the straight line  $AC$  is the upright side of the figure (r. 59)

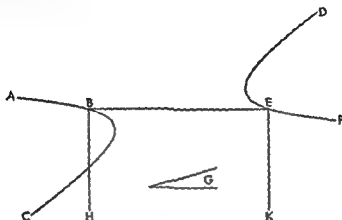
**PROPOSITION 59 (PROBLEM)**

Given two bounded straight lines perpendicular to each other, to find opposite sections whose diameter is one of the given straight lines and whose vertex is the ends of the straight line and where the straight lines dropped in each of the sections at a given angle will equal in square the rectangles applied to the other of the straight lines and exceeding by a figure similar to the rectangle contained by the given straight lines

Let there be the two given bounded straight lines  $BE$  and  $BH$ , perpendicular to each other, and let the given angle be  $G$ , then it is required to describe



opposite sections about one of the straight lines  $BE$  and  $BH$ , so that the ordinates are dropped at an angle  $G$



must be done (1.55) Then let the straight line  $EK$  be drawn through  $E$  perpendicular to  $BE$  and equal to  $BH$ , and let another hyperbola  $DEF$  be likewise described whose diameter is  $BE$  and the upright side of whose figure is  $EK$ , and where the ordinates from the section will be dropped at a same angle  $G$  Then it is evident that  $B$  and  $E$  are opposite sections, and there is one diameter for them, and their uprights are equal

#### PROPOSITION 60 (PROBLEM)

*Given two straight lines bisecting each other, to describe about each of them opposite sections, so that the straight lines are their conjugate diameters and the diameter of one pair of opposite sections is equal in square to the figure of the other pair, and likewise the diameter of the second pair of opposite sections is equal in square to the figure of the first pair*

equal in square to the figure about  $AC$ , and  $AC$  is equal in square to the figure about  $DE$

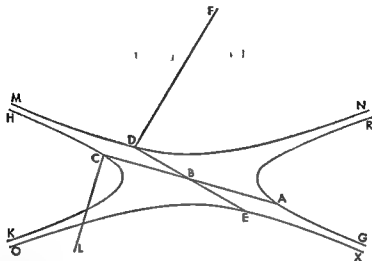
Let

$$\text{rect } AC \cdot CL = \text{sq } DE,$$

and let  $LC$  be perpendicular to  $CA$  And given two straight lines  $AC$  and  $CL$  perpendicular to each other, let the opposite sections  $RAG$  and  $HCK$  be described whose transverse diameter will be  $CA$  and whose upright side will be

Then again let

$$\text{rect } DE, DF = \text{sq } AC,$$



and let  $DF$  be perpendicular to  $DE$ . And given two straight lines  $ED$  and  $DF$  lying perpend. be described, whose figure

$AC$ , and this it was required to do

And let such sections be called conjugate

## BOOK TWO

APOLLONIUS to EUDEMUS,

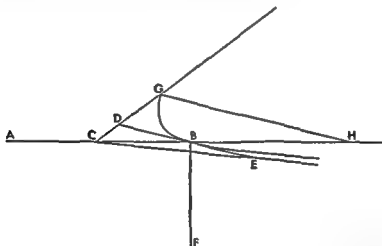
If you are well, well and good, and I too fare pretty well

I have sent you my son Apollonius bringing you the second book of the conics as arranged by us. Go through it then carefully and acquaint those with

### PROPOSITION 1

*the ends thus taken on the tangent will not meet the section*

Let there be an hyperbola whose diameter is the straight line  $AB$  and center  $C$ , and upright the straight line  $BF$ , and let the straight line  $DE$  touch the



section at  $B$ , and let the squares on  $BD$  and  $BE$  each be equal to the fourth of

at line  
then

but

$$\text{sq } CB = \text{fourth sq } AB,$$

and

$$\text{sq } BD = \text{fourth rect } AB, BF,$$

therefore

$$AB \cdot BF \text{ sq } CB \text{ sq } DB \text{ sq } CH \text{ sq } HG$$

And also

$$AB \cdot BF \text{ rect } AH, HB \text{ sq } HG \text{ (I 21),}$$

therefore

$$\text{sq } CH \text{ sq } HG \text{ rect } AH, HB \text{ sq } HG$$

Therefore

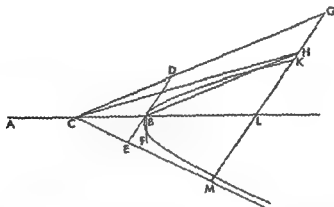
$$\text{rect } AH, HB = \text{sq } CH,$$

and this is absurd (Eucl II 6). Therefore the straight line  $CD$  will not meet the section. Then likewise we could show that neither does  $CE$ , therefore the straight lines  $CD$  and  $CE$  are asymptotes (*ἀσύμπτωτοι*)<sup>1</sup> to the section.

### PROPOSITION 2

With the same things it is to be shown that a straight line cutting the angle contained by the straight lines  $DC$  and  $CE$  is not another asymptote.

For if possible, let  $CH$  be it, and let the straight line  $BH$  be drawn through  $B$  parallel to  $CD$  and let it meet  $CH$  at  $H$ , and let  $DG$  be made equal to  $BH$  and



let  $GH$  be joined and produced to the points  $A$ ,  $L$  and  $M$ . Since then  $BH$  and  $DG$  are equal and parallel,  $DB$  and  $HG$  are also equal and parallel. And since  $AB$  is bisected at  $C$  and a straight line  $BL$  is added to it,

$$\text{rect } AL \cdot LB + \text{sq } CB = \text{sq } CL \text{ (Eucl II 6)}$$

Likewise then, since  $GM$  is parallel to  $DE$  and

$$DB = BE,$$

<sup>1</sup>The word *ἀσύμπτωτοι* means literally 'not capable of meeting' and is used in a general way in Euclid to refer to any non-meant lines or planes. In Apollonius it is also used in this way as for instance in II 14 *porism*, where it refers to any straight lines not meeting the hyperbola. The special case where in English the lines are spoken of as asymptotes is the one defined here. Book II proposition 14 *porism* further declares their peculiar property and significance.

therefore also

$$GL = LM$$

And since

$$GH = DB,$$

therefore

$$GK > DB$$

And also

$$KM > BE,$$

since also

$$LM > BE,$$

therefore

$$\text{rect } MK, AG > \text{rect } DB, BE \\ > \text{sq } DB$$

Since then

$$AB \cdot BF \text{ sq } CB \text{ sq } BD \text{ (ii 1),}$$

but

$$AB \cdot BF \text{ rect } AL, LB \text{ sq } LK \text{ (i 21),}$$

and

$$\text{sq } CB \text{ sq } BD \text{ sq } CL \text{ sq } LG,$$

therefore also

$$\text{sq } CL \text{ sq } LG \text{ rect } AL, LB \text{ sq } LK$$

Since then

$$\begin{array}{l} \text{whole sq } LC \text{ whole sq } LG \\ \text{part subtr rect } AL, LB \text{ part subtr sq } LK, \end{array}$$

therefore also

$$\text{sq } LC \text{ sq } LG \text{ remainder sq } CB \text{ remainder rect } MK, KG,$$

that is

$$\text{sq } CB \text{ rect } MK, KG \text{ sq } CB \text{ sq } DB$$

Therefore

$$\text{sq } DB = \text{rect } MK, KG,$$

and this is absurd, for it has been shown to be greater than it. Therefore the straight line  $CH$  is not an asymptote to the section

### PROPOSITION 3

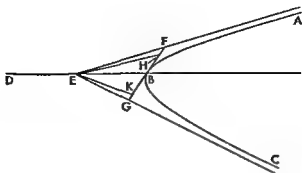
*If a straight line touches an hyperbola, it will meet both of the asymptotes and it will be bisected at the point of contact, and the square on each of its segments will be equal to the fourth of the figure resulting on the diameter drawn through the point of contact*

Let there be the hyperbola  $ABC$ , and its center  $E$ , and asymptotes  $FE$  and  $EG$ , and let some straight line  $HK$  touch it at  $B$

I say that the straight line  $HK$  produced will meet the straight lines  $FE$  and  $EG$

figure on  $BD$ , and let  $FI$  and  $EA$  be joined. Therefore they are asymptotes (ii 1), and this is absurd (ii 2), for  $FE$  and  $EG$  are supposed asymptotes. Therefore  $KH$  produced will meet the asymptotes  $EF$  and  $EG$  at  $F$  and  $G$

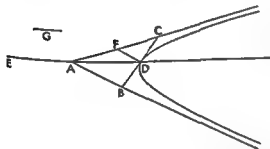
I say then also that the squares on  $BF$  and  $BG$  will each be equal to the fourth of the figure on  $BD$



For let it not be, but if possible, let the squares on  $BH$  and  $BK$  each be equal to the fourth of the figure. Therefore  $HE$  and  $EK$  are asymptotes (II 1), and this is absurd (II 2). Therefore the squares on  $FB$  and  $BG$  will each be equal to the fourth of the figure on  $BD$ .

#### PROPOSITION 4 (PROBLEM)

Given two straight lines containing an angle and a point within the angle to describe through the point the section of a cone called hyperbola so that the given



Let the straight line  $AD$  be joined and produced to  $E$ , and let  $AE$  be made equal to  $DA$  and let the straight line  $DF$  be drawn through  $D$  parallel to  $AB$  and let  $FC$  be made equal to  $AF$ , and let  $CD$  be joined and produced to  $B$ , and let it be contrived that

$$\text{rect } DE, G = \text{sq } CB,$$

and with  $AD$  extended let an hyperbola be described about it through  $D$  so that the

ordinates equal in square the areas applied to  $G$  and exceeding by a figure similar to rectangle  $DE, G$ . Since then  $DF$  is parallel to  $BA$ , and

$$CF = FA,$$

therefore

$$CD = DB,$$

and so

$$\text{sq } CB = 4 \text{ sq } CD$$

And

$$\text{sq } CB = \text{rect } DE, G,$$



therefore the squares on  $CD$  and  $DB$  are each equal to the fourth part of the figure  $DE, G$ . Therefore the straight lines  $AB$  and  $AC$  are asymptotes to the hyperbola described.

### PROPOSITION 5

*If the diameter of a parabola or hyperbola bisects some straight line, the tangent to the section at the end of the diameter will be parallel to the bisected straight line.*

Let there be the parabola or hyperbola  $ABC$  whose diameter is the straight line  $DBE$ , and let the straight line  $FBG$  touch the section, and let some straight line  $AEC$  be drawn in the section making  $AE$  equal to  $EC$ .

I say that  $AC$  is parallel to  $FG$ .

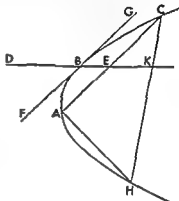
For if not, let the straight line  $CH$  be drawn through  $C$  parallel to  $FG$  and let  $HA$  be joined. Since then  $ABC$  is a parabola or hyperbola whose diameter is  $DE$ , and tangent  $FG$ , and  $CH$  is parallel to it, therefore

$$CK = KH \quad (1 \text{ 46, 47})$$

But also

$$CE = EA$$

Therefore  $AH$  is parallel to  $KE$ , and this is absurd, for produced it meets  $BD$  (1 22).



### PROPOSITION 6

*If the diameter of an ellipse or circumference of a circle bisects some straight line not through the center, the tangent to the section at the end of the diameter will be parallel to the bisected straight line.*

Let there be an ellipse or circumference of a circle whose diameter is the straight line  $AB$ , and let  $AB$  bisect  $CD$ , a straight line not through the center, at the point  $E$ .

I say that the tangent to the section at  $A$  is parallel to  $CD$ .

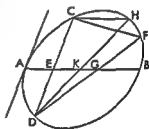
For let it not be, but if possible, let  $DF$  be parallel to the tangent at  $A$ , therefore

$$DG = FG$$

But also

$$DE = EC,$$

therefore  $CF$  is parallel to  $GE$ , and this is absurd.



then

$$DK = KH,$$

and also

$$DE = EC,$$

therefore  $CH$  is parallel to  $AB$ . But also  $CF$ , and this is absurd. Therefore the tangent at  $A$  is parallel to  $CD$ .

be  
ncc

## PROPOSITION 7

If a straight line touches a section of a cone or circumference of a circle, and a parallel to it is drawn in the section and bisected, the straight line joined from the point of contact to the midpoint will be a diameter of the section

Let there be a section of a cone or circumference of a circle  $ABC$ , and  $FG$  tangent to it, and  $AC$  parallel to  $FG$  and bisected at  $E$ , and let  $BE$  be joined

I say that  $BE$  is a diameter of the section

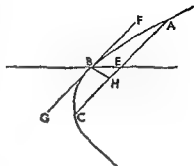
For let it not be, but, if possible, let  $BH$  be a diameter of the section Therefore

$$AH = HC \text{ (First Def 1 4),}$$

and this is absurd, for

$$AE = EC$$

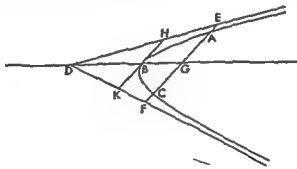
Therefore  $BH$  will not be a diameter of the section Then likewise we could show that there is no other than  $BE$



## PROPOSITION 8

If a straight line meets an hyperbola in two points produced both ways it will meet the asymptotes, and the straight lines cut off on it by the section from the asymptotes will be equal

Let there be the hyperbola  $ABC$ , and the asymptotes  $ED$  and  $DF$ , and let some straight line  $AC$  meet  $ABC$



I say that produced both ways it will meet the asymptotes

$DE$  and  $DF$

Let it meet them at  $E$  and  $F$  and

$$HB = BK \text{ (II 3),}$$

therefore also

$$FG = GE$$

And so also

$$CF = AE$$

## PROPOSITION 9

If a straight line meeting the asymptotes is bisected by the hyperbola it will touch the section in one point only

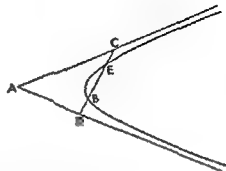
For let the straight line  $CD$  meeting the asymptotes  $CA, AD$  be bisected by the hyperbola at the point  $E$

I say that it touches the hyperbola at no other point

For if possible, let it touch it at  $B$

Therefore  $CE = BD$  (II 8)

and this is absurd for  $CE$  is supposed equal to  $ED$  Therefore it will not touch the section at another point



## PROPOSITION 10

If some straight line cutting the section meet both of the asymptotes the rectangle contained by the straight lines cut off between the asymptotes and the section is equal to the fourth of the figure resulting on the diameter bisecting the straight lines drawn parallel to the drawn straight line

and let  $BM$  be drawn from  $B$  perpendicular to  $HEB$  therefore  $BH$  is a diameter (II 7) and  $BM$  the upright side

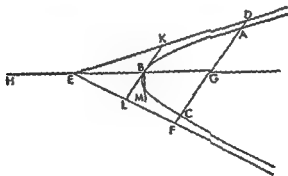
I say that

rect  $DA AF =$  fourth rect  $HB BM$ ,

then likewise also

rect  $DC CF =$  fourth rect  $HB BM$

For let  $KL$  be drawn through  $B$  tangent to the section therefore it is



parallel to  $DF$  (II 6) And since it has been shown

$HB \cdot BM = EB \cdot BA = EG \cdot GD$  (II 1, 3),

and

$HB \cdot BM = \text{rect } HG \cdot GB = GA^2$  (I 21)

therefore

$$\text{sq } EG : \text{sq } GD :: \text{rect } HG, GB : \text{sq } GA.$$

Since then

$$\begin{aligned} & \text{whole sq } EG : \text{whole sq } GD :: \\ & \text{part subtr rect } HG, GB . \text{ part subtr sq } AG, \end{aligned}$$

therefore also

$$\text{remainder sq } EB : \text{remainder rect } DA, AF :: \text{sq } EG : \text{sq } GD,$$

or

$$\text{remainder sq } EB - \text{remainder rect } DA, AF .. \text{sq } EB : \text{sq } BK$$

Therefore

$$\text{rect } FA, AD = \text{sq } BK$$

Then likewise it could be shown also that

$$\text{rect } DC, CF = \text{sq } BL,$$

therefore also

$$\text{rect } FA, AD = \text{rect } DC, CF.$$

### PROPOSITION 11

If some straight line cut each of the straight lines containing the angle adjacent to the angle containing the hyperbola, it will meet the section in one point only, and the rectangle contained by the straight lines cut off between the containing straight lines and the section will be equal to the fourth part of the square on the diameter drawn parallel to the cutting straight line.

Let there be an hyperbola whose asymptotes are  $CA, AD$ , and let  $DA$  be

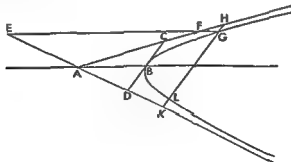
in one point only (I 26)

Let it meet it at  $G$

I say then also that

$$\text{rect } EG, GF = \text{sq } AB$$

For let the straight line  $HGLK$  be drawn ordinatewise through  $G$ , therefore



the tangent through  $B$  is parallel to  $GH$  (II 5) Let it be  $CD$  Since then

$$CB = BD \text{ (II 3),}$$

therefore

$$\text{sq } CB \text{ or rect. } CB, BD = \text{sq } BA \text{ comp } CB . BA, DB : BA$$

But

$$CB \ BA \ HG : GF,$$

and

$$DB \ BA \ GK \ GE,$$

therefore

$$\text{sq } CB \ \text{sq } B \ 1 \ \text{comp } HG \ GF, KG \ GE$$

But also

$$\text{rect } AG, GH \ \text{rect } EG, GF \ \text{comp } HG \ GF, KG \ GE,$$

therefore

$$\text{rect } AG, GH \ \text{rect } EG, GF \ \text{sq } CB \ \text{sq } BA$$

Alternately

$$\text{rect } KG, GH \ \text{sq } CB \ \text{rect } EG, GF \ \text{sq } BA$$

But it was shown

$$\text{rect } AG, GH = \text{sq } CB \ (\text{II } 10),$$

therefore also

$$\text{rect } EG, GF = \text{sq } AB$$

### PROPOSITION 12

If two straight lines at chance angles are drawn to the asymptotes from some point of those on the section, and parallels are drawn to them from some point of those on the section, then the rectangle contained by the parallels will be equal to that contained by those straight lines to which they were drawn parallel

Let there be an hyperbola whose asymptotes are  $AB$  and  $BC$ , and let some point  $D$  be taken on the section, and from it let  $DE$  and  $DF$  be dropped to  $AB$  and  $BC$ , and let some other point on the section  $G$  be taken, and through  $G$  let  $GH$  and  $GK$  be drawn parallel to  $ED$  and  $DF$

I say that

$$\text{rect } ED, DF = \text{rect } HG, GK$$

For let  $DG$  be joined and produced to  $A$  and  $C$ . Since then

$$\text{rect } AD \ DC = \text{rect } AG, GC \ (\text{II } 8),$$

therefore

$$AG \ AD \ DC \ CG$$

But

$$AG \ AD \ GH \ ED,$$

and

$$DC \ CG \ DF \ GK,$$

therefore

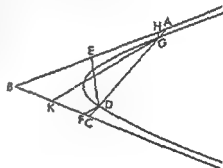
$$GH \ DE \ DF \ GK$$

Therefore

$$\text{rect } ED \ DF = \text{rect } HG, GK$$

### PROPOSITION 13

If in the place bounded by the asymptotes and the section some straight line is drawn parallel to one of the asymptotes, it will meet the section in one point only



Let there be an hyperbola whose asymptotes are  $CA$  and  $AB$ , and let some point  $E$  be taken, and through it let  $EF$  be drawn parallel to  $AB$

I say that it will meet the section

For if possible, let it not meet it, and let some point  $G$  on the section be taken, and through  $G$  let  $GC$  and  $GH$  be drawn parallel to  $CA$  and  $AB$ , and let

$\text{rect } CG, GH = \text{rect } AE, EF$ ,  
and let  $AF$  be joined and produced, then it will meet the section (I 2) Let it meet it at  $K$ , and through  $K$  parallel to  $CA$  and  $AB$  let  $KL$  and  $KD$  be drawn, therefore

$$\text{rect } CG, GH = \text{rect } LK, KD \text{ (II 12)}$$

And it is supposed that also

$$\text{rect } CG, GH = \text{rect } AE, EF,$$

therefore

$$\text{rect } LK, KD \text{ or } \text{rect } KL, LA = \text{rect } AE, EF,$$

and thus is impossible, for both

$$KL > EF$$

and

$$LA > AE$$

Therefore  $EF$  will meet the section

Let it meet it at  $M$

I say then that it will not meet it at any other point

For if possible, let it also meet it at  $N$ , and through  $M$  and  $N$  let  $MX$  and  $NB$  be drawn parallel to  $CA$  Therefore

$$\text{rect } EM, MX = \text{rect } EN, NB \text{ (II 12)},$$

and thus is impossible Therefore it will not meet the section in another point

#### PROPOSITION 14

*The asymptotes and the section if produced indefinitely draw nearer to each other and they reach a distance less than any given distance*

Let there be an hyperbola whose asymptotes are  $AB$  and  $AC$ , and a given distance  $K$

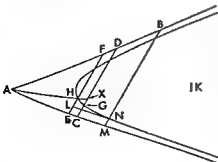
I say that  $AB$  and  $AC$  and the section if produced draw nearer to each other and will reach a distance less than  $K$

For let  $EHF$  and  $CGD$  be drawn parallel to the tangent, and let  $AH$  be joined and produced to  $X$  Since then  $\text{rect } CG, GD = \text{rect } FH, HE$  (II 10), therefore

$$DG \cdot FH = HE \cdot CG$$

But

$$DG > FH \text{ (I 8, 26),}$$



therefore also

$$HE > CG$$

Then likewise we could show that the succeeding straight lines are less

Then let the distance  $EL$  be taken less than  $K$ , and through  $L$  let  $LN$  be drawn parallel to  $AC$ , therefore it will meet the section (11 13) Let it meet it at  $N$ , and through  $N$  let  $MNB$  be drawn parallel to  $EF$  Therefore

$$MN = EL$$

and so

$$MN < K$$

#### PROBLEM

Then from this it is evident that the straight lines  $AB$  and  $AC$  are nearer than all the asymptotes to the section, and the angle contained by  $BA, AC$  is clearly less than that contained by other asymptotes to the section.

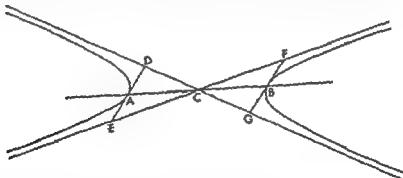
#### PROPOSITION 15

*The asymptotes of opposite sections are common*

Let there be opposite sections whose diameter is  $AB$  and center  $C$

I say that the asymptotes of the sections  $A$  and  $B$  are common

Let the straight lines  $DAE$  and  $FBG$  be drawn tangent to the sections through the points  $A$  and  $B$ , they are therefore parallel (1 44, note) Then let



each of the straight lines  $DA$ ,  $AE$ ,  $FB$ , and  $BG$  be cut off equal in square to the fourth of the figure applied to  $AB$ , therefore

$$DA = AE = FB = BG$$

Then let  $CD$ ,  $CE$ ,  $CF$ , and  $CG$  be joined. Then it is evident that  $DC$  is in a straight line with  $CG$  and  $CE$  with  $CF$  because of the parallels. Since then it is a hyperbola whose diameter is  $AB$  and tangent  $DE$ , and  $DA$  and  $AE$  are each equal in square to the fourth of the figure applied to  $AB$ , therefore  $DC$  and  $CE$  are asymptotes (11 1) For the same reasons then  $FC$  and  $CG$  are also asymptotes to section  $B$ . Therefore the asymptotes of opposite sections are common.

#### PROPOSITION 16

*If in opposite sections some straight line containing the angle adjacent to the vertex*

*is drawn, it will meet*

each of the opposite sections in one point only, and the straight lines cut off on it by the sections from the asymptotes will be equal

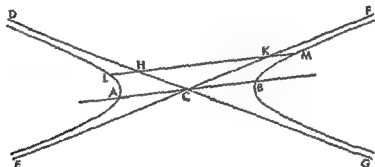
For let there be the opposite sections  $A$  and  $B$  whose center is  $C$  and asymptotes  $DCG$  and  $ECF$ , and let some straight line  $HK$  be drawn through cutting each of the straight lines  $DC$  and  $CF$

I say that produced it will meet each of the sections in one point only

For since  $DC$  and  $CE$  are asymptotes of section  $A$ , and some straight line  $HK$  has been drawn across cutting both of the straight lines containing the adjacent angle  $DCF$ , therefore  $HK$  produced will meet the section (II 11). Then likewise also  $B$

Let it meet them at  $L$  and  $M$

Let the straight line  $ACB$  be drawn through  $C$  parallel to  $LM$ , therefore



$$\text{rect } KL, LH = \text{sq } AC \text{ (II 11)}$$

and

$$\text{rect } HM, MK = \text{sq } CB \text{ (II 11)}$$

And so also

$$\text{rect } KL, LH = \text{rect } HM, MK,$$

and

$$LH = HM$$

#### PROPOSITION 17

The asymptotes of conjugate opposite sections are common

Let there be conjugate opposite sections whose conjugate diameters are  $AB$  and  $CD$ , and whose center is  $E$





I say that their asymptotes are common

For let the straight lines  $FAG$ ,  $GDH$ ,  $HBK$ , and  $KCF$  be drawn through the points  $A$ ,  $B$ ,  $C$ , and  $D$  touching the sections, therefore  $FGHK$  is a parallelogram (I 44, note) Then let  $FEH$  and  $KLG$  be joined, therefore they are straight lines (II 15) and diagonals of the parallelogram, and they are all bisected at the point  $E$  And since the figure on  $AB$  is equal to the square on  $CD$  (I 60), and

$$CE = ED,$$

therefore each of the squares on  $FA$ ,  $AG$ ,  $KB$ , and  $BH$  is equal to a fourth of the figure on  $AB$  Therefore the straight lines  $FEH$  and  $KEG$  are asymptotes of the sections  $A$  and  $B$  (II 1) Then likewise we could show that the same straight lines are also asymptotes of the sections  $C$  and  $D$  Therefore the asymptotes of conjugate opposite sections are common

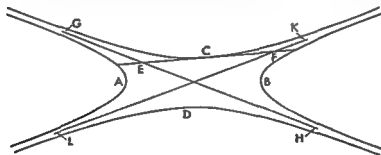
### PROPOSITION 18

*If a straight line meeting one of the conjugate opposite sections, when produced both ways, falls outside the section, it will meet both of the adjacent sections in one point only*

Let there be the conjugate opposite sections  $A$ ,  $B$ ,  $C$ , and  $D$ , and let some straight line  $EF$  meet the section  $C$  and produced both ways fall outside the section

I say that it will meet both of the sections  $A$  and  $B$  in one point only

For let  $GH$  and  $KL$  be asymptotes of the sections Therefore  $EF$  meets both



$GH$  and  $KL$  (II 3) Then it is evident that it will also meet the sections  $A$  and  $B$  in one point only (II 16)

### PROPOSITION 19

*If some straight line is drawn touching some one of the conjugate opposite sections at random, it will meet the adjacent sections and will be bisected at the point of contact*

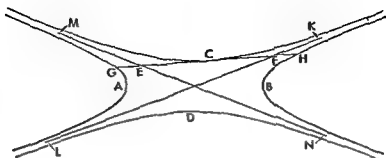
Let some

ed at  $C$

It is evident now that it will meet sections  $A$  and  $B$  (II 18), let it meet them at  $G$  and  $H$

I say that

$$CG=CH.$$



For let the asymptotes of the sections  $KL$  and  $MN$  be drawn. Therefore  
 $EG=FH$  (II 16),  
 and

$$CE=CF \text{ (II 3),}$$

and

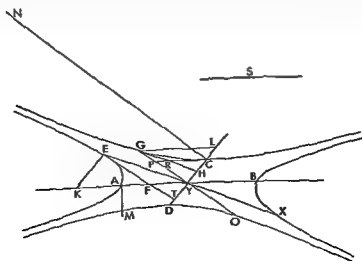
$$CG=CH$$

#### PROPOSITION 20

If a straight line touches one of the conjugate opposite sections, and two straight

drawn through the point of contact and the center, and those through the points of contact and the center will be conjugate diameters of the opposite sections

Let there be conjugate opposite sections whose conjugate diameters are the



straight lines  $AB$  and  $CD$ , and center  $Y$ , and let  $EF$  be drawn touching the section  $A$ , and produced let it meet  $CY$  at  $T$ , and let  $EY$  be joined and produced to  $\Lambda$ , and through  $Y$  let  $YG$  be drawn parallel to  $EF$ , and through  $G$  let  $HG$  be drawn touching the section

I say that  $HG$  is parallel to  $YE$ , and  $GO$  and  $EX$  are conjugate diameters

For let the straight lines  $KE$ ,  $GL$ , and  $CRP$  be drawn ordinatewise, and let  $AM$  and  $CN$  be the parameters Since then

$$BA \cdot AM = NC \cdot CD \quad (\text{I } 60),$$

but

$$BA \cdot AM = \text{rect } YK, KF = \text{sq } KE \quad (\text{I } 37),$$

and

$$NC \cdot CD = \text{sq } GL = \text{rect } YL, LH \quad (\text{I } 37),$$

therefore also

$$\text{rect } YK, KF = \text{sq } EK = \text{sq } GL = \text{rect } YL, LH$$

But

$$\text{rect } YK, KF = \text{sq } EK \text{ comp } YK = KE, FK = KE,$$

and

$$\text{sq } GL = \text{rect } YL, LH \text{ comp } GL = LY, GL = LH,$$

therefore

$$\text{ratio comp } YK = KE, FK = KE = \text{ratio comp } GL = LY, GL = LH,$$

and of these

$$FK = KE = GL = LY,$$

for each of the straight lines  $EK$ ,  $KF$ , and  $FE$  is parallel to each of the straight lines  $YL$ ,  $LG$ , and  $GY$  respectively Therefore as remainder

$$YK = KE = GL = LH$$

Also the sides about the equal angles at  $K$  and  $L$  are proportional, therefore triangle  $EKF$  is similar to triangle  $GHL$  and will have equal the angles the corresponding sides subtend Therefore

$$\text{angle } EYK = \text{angle } LGH$$

But also

$$\text{angle } KYG = \text{angle } LGY,$$

and therefore

$$\text{angle } EYG = \text{angle } HGY$$

Therefore  $EY$  is parallel to  $GH$

Then let it be contrived that

$$PG = GR = HQ = S,$$

therefore  $S$  is a half of the parameter of the ordinates to the diameter  $GO$  in sections  $C$  and  $D$  (I 51) And since  $CD$  is the second diameter of the sections  $A$  and  $B$  and  $CT$  meets it, therefore

$$\text{rect } TY, EK = \text{sq } CY,$$

for if we draw from  $C$  a parallel to  $KY$ , the rectangle contained by  $TY$  and the straight line cut off by the parallel will be equal to the square on  $CY$  (I 38)

And therefore

$$TY \cdot EK = \text{sq } TY = \text{sq } TC \quad (\text{Eucl vi } 20)$$

But

$$TY \cdot EK = TF \cdot FE$$

or

$$TY \cdot EK = \text{trgl } TYF = \text{trgl } EYF \quad (\text{Eucl vi } 1),$$

and

sq  $TY$  sq  $CY$  trgl  $YTF$  trgl  $YCP$  (Eucl vi 19)

Therefore sq  $TY$  sq  $CY$  trgl  $YTF$  trgl  $GHY$  (iii 1)

Therefore trgl  $TYF$  trgl  $EFY$  trgl  $TFY$  trgl  $YGH$

$$\text{trgl } GHY = \text{trgl } YEF$$

But they also have

$$\text{angle } HGY = \text{angle } YEF,$$

for  $EY$  || parallel to  $GH$ , and  $EF$  to  $GY$ . Therefore the sides about the equal angles are reciprocally proportional (Eucl vi 15). Therefore

$$GH : EY = EF : GY,$$

therefore

$$\text{rect } HG, GY = \text{rect } YE, EF$$

And since

$$S : HG = RG : GP,$$

and

$$RG : GP = YE : EF,$$

for they are parallel, therefore also

$$S : HG = YE : EF$$

But, with  $YG$  taken as common height,

$$S : HG = \text{rect } S, YG : \text{rect } HG, GY,$$

and

$$YE : EF = \text{sq } YE : \text{rect } YE, EF$$

And therefore

$$\text{rect } S, YG : \text{rect } HG, GY = \text{sq } YE : \text{rect } YE, EF$$

Alternately

$$\text{rect } S, GY : \text{sq } EY = \text{rect } HG, GY : \text{rect } FE, EY$$

But

$$\text{rect } HG, GY = \text{rect } YE, EF \text{ (above),}$$

therefore also

$$\text{rect } S, GY = \text{sq } EY$$

And rectangle  $S, GY$  is a fourth of the figure on  $GO$ , for

$$GY = \text{half } GO,$$

and  $S$  is the parameter, and

$$\text{sq } EY = \text{fourth sq } EX,$$

for

$$EY = YX$$

Therefore the square on  $EY$  is equal to the figure on  $GO$ . Then likewise we could show also that  $GO$  is equal in square to the figure on  $EX$ . Therefore  $EY$  and  $GO$  are conjugate diameters of the opposite sections  $A, B, C$ , and  $D$ .

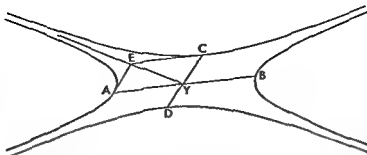
#### PROPOSITION 21

*The same things being supposed it is to be shown that the point of meeting of the tangents is on one of the asymptotes*

Let

the straight lines  $AC$  and  $BD$  be drawn tangent

For since the square on  $CY$  is equal to the fourth of the figure on  $AB$  (I 60), and



$$\text{sq } AE = \text{sq } CY \text{ (I 17),}$$

therefore also the square on  $AE$  is equal to the fourth part of the figure on  $AB$ . Let  $EY$  be joined, therefore  $EY$  is an asymptote (I 1), therefore the point  $E$  is on the asymptote

### PROPOSITION 22

*If in conjugate opposite sections a radius is drawn to any one of the sections, and a parallel is drawn to it meeting one of the adjacent sections and meeting the asymptotes, then the rectangle contained by the segments produced between the section and the asymptotes on the straight line drawn is equal to the square on the radius*

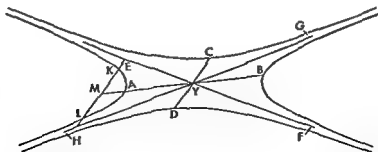
Let there be the conjugate opposite sections  $A, B, C$  and  $D$ , and let there be the asymptotes of the sections  $YEF$  and  $YGH$ , and from the center  $Y$  let some straight line  $YCD$  be drawn across, and let  $HE$  be drawn parallel to it cutting both the adjacent section and the asymptotes

I say that

$$\text{rect } EK, KH = \text{sq } CY$$

Let  $KL$  be bisected at  $M$ , and let  $MY$  be joined and produced, therefore  $AB$

(I 1st Def 1 6) Therefore the square on  $CY$  is equal to the fourth of the figure



on  $AB$  (I 60) And the rectangle  $HK \cdot KE$  is equal to the fourth part of the figure on  $AB$  (I 10), therefore also

$$\text{rect } HK \cdot KE = \text{sq } CY$$

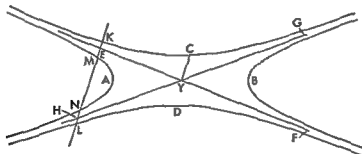
## PROPOSITION 23

If in conjugate opposite sections some radius  $AM$  drawn to any one of the sections, and a parallel is drawn to it meeting the three adjacent sections, then the rectangle contained by the segments produced between the three sections on the straight line drawn is twice the square on the radius.

Let  $C$ , and  $D$ , and let the straight line  $CY$  be drawn to meet any one of the sections and let  $KL$  be drawn parallel to  $CY$  cutting the three adjacent sections. I say that

$$\text{rect } KM, ML = 2 \text{ sq } CY$$

Let the asymptotes to the sections,  $EF$  and  $GH$ , be drawn, therefore



$$\text{Eq } CY = \text{rect } HM, ME \text{ (II 22)} = \text{rect } HK, KE \text{ (II 11)}$$

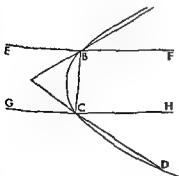
And

$$\text{rect } HM, ME + \text{rect } HK, KE = \text{rect } LM, MK$$

because of the straight lines on the ends being equal (II 8, 16) Therefore also  
 $\text{rect } LM, MK = 2 \text{ sq } CY$

## PROPOSITION 24

If two straight lines meet a parabola each at two points, and if a point of meeting of one of them is the point of meeting of the other



$ABCD$ , and let a point of meeting of neither of them be contained by the points of meeting of the other

I say that the straight lines produced will meet each other

Let the diameters of the section  $EBF$  and  $GCH$ , be drawn through the points  $B$  and  $C$ , therefore they are parallel (I 51, end) and each one cuts the section in one point only (I 26) Then let  $BC$  be joined, therefore

$$\text{angle } EBC + \text{angle } BCG = 2 \text{ rt angles,}$$

and  $DC$  and  $BA$  produced make angles less than two right angles. Therefore they will meet each other outside the section (I 10, Eucl Post 5)

### PROPOSITION 25

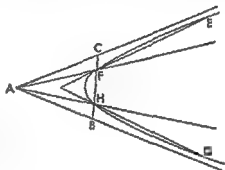
*If two straight lines meet an hyperbola each at two points and if a point of meeting of neither of them is contained by the points of meeting of the other, then the straight lines will meet each other outside the section, but within the angle containing the section*

Let there be an hyperbola, whose asymptotes are  $AB$  and  $AC$ , and let the two straight lines  $EF$  and  $GH$  cut the section, and let a point of meeting of neither of them be contained by the points of meeting of the other

I say that the straight lines  $EF$  and  $GH$  produced will meet outside the section, but within the angle  $CAB$

For let the straight lines  $AF$  and  $AH$  be joined and produced, and let  $FH$  be joined. And since the straight lines  $EF$  and  $GH$  produced cut the angles  $AFH$  and  $AHF$ , and the said angles are less than two right angles (Eucl 1 17), the straight lines  $EF$  and  $GH$  produced will meet each other outside the section, but within the angle  $BAC$

Then we could likewise show it, even if the straight lines  $EF$  and  $GH$  are tangents to the sections

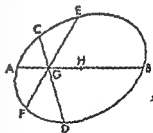


### PROPOSITION 26

*If in an ellipse or circumference of a circle two straight lines not through the center cut each other, then they do not bisect each other*

For if possible, in the ellipse or circumference of a circle let the two straight lines  $CD$  and  $EF$  not through the center bisect each other at  $G$ , and let the point  $H$  be the center of the section, and let  $GH$  be joined and produced to  $A$  and  $B$

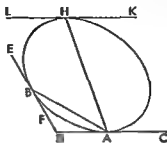
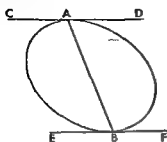
Since then the straight line  $AB$  is a diameter bisecting  $EF$ , therefore the tangent at  $A$  is parallel to  $EF$  (II 6). We could then likewise show that it is also parallel to  $CD$ . And so also  $EF \parallel CD$ . And this is impossible. Therefore  $CD$  and  $EF$  do not bisect each other



### PROPOSITION 27

*If two straight lines touch an ellipse or circumference of a circle, and if the straight line joining the points of contact is through the center of the section, the tangents will be parallel, but if not, they will meet on the same side of the center.*

Let there be the ellipse or circumference of a circle  $AB$ , and let the straight lines  $CAD$  and  $EBF$  touch it, and  $AB$  be joined, and first let it be through the center



I say that  $CD$  is parallel to  $EF$

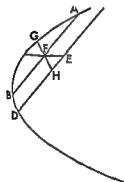
For since  $AB$  is a diameter of the section, and  $CD$  touches it at  $A$ , therefore  $CD$  is parallel to the ordinates to  $AB$  (I 17) Then for the same reasons  $BF$  is also parallel to the same ordinates Therefore  $CD$  is also parallel to  $EF$

the center as  $AB$

### PROPOSITION 28

If in  
para

joined and produced



and this is impossible, for it is supposed  
 $CE = ED$

Therefore  $GH$  is not a diameter Then likewise we could show that there is no other except  $EF$  Therefore  $EF$  will be a diameter of the section

### PROPOSITION 29

straight lines  $AB$  and  $AC$ , meeting at  $A$ , be drawn tangent, and let  $BC$  be joined and bisected at  $D$ , and let  $AD$  be joined



I say that it is a diameter of the section

For if possible, let  $DE$  be a diameter, and let  $EC$  be joined, then it will cut the section (1 35, 36) Let it cut it at  $F$ , and through  $F$  let  $FKG$  be drawn parallel to  $CDB$  Since then

$$CD = DB$$

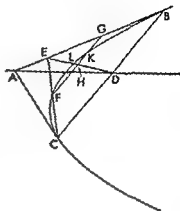
also

$$FH = HG$$

And since the tangent at  $L$  is parallel to  $BC$  (11 5, 6), and  $FG$  is also parallel to  $BC$ , therefore also  $FG$  is parallel to the tangent at  $L$  Therefore

$$FH = HK \text{ (1 46, 47),}$$

and this is impossible Therefore  $DE$  is not a diameter Then likewise we could show that there is no other except  $AD$



### PROPOSITION 30

If two straight lines tangent to a section of a cone or to a circumference of a circle meet the diameter drawn from the point of meeting will bisect the straight line joining the points of contact

Let there be the section of a cone or circumference of a circle  $BC$ , and let two tangents  $BA$  and  $AC$  be drawn to it meeting at  $A$  and let  $BC$  be joined and let  $AD$  be drawn through  $A$  as a diameter of the section

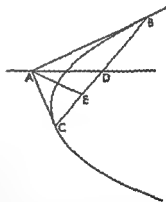
I say that

$$DB = DC$$

For let it not be, but if possible, let

$$BE = EC,$$

and let  $AE$  be joined, therefore  $AE$  is a diameter of the section (11 29) But  $AD$  is also a diameter, and this is absurd For if the section is an ellipse, the point  $A$  at which the diameters meet each other, will be a center outside the section, and this is impossible, and if the section is a parabola, the diameters meet each other (1 51, end), and if it is an hyperbola and the straight lines  $BA$  and  $AC$  meet the section without containing one another's points of meeting then the center is within the angle containing the hyperbola (11 25), but it is also on it, for it has been supposed a center since  $DA$  and  $AE$  are diameters (1 51 end), and this is absurd Therefore  $BE$  is not equal to  $EC$



### PROPOSITION 31

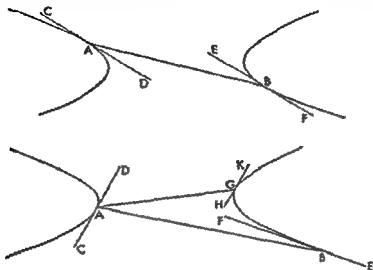
If two straight lines touch each of the opposite sections then if the straight line joining the points of contact falls through the center, the tangents will be parallel, but if not, they will meet on the same side as the center

Let there be the opposite sections  $A$  and  $B$ , and let the straight lines  $CAD$  and  $EBF$  be tangent to them at  $A$  and  $B$ , and let the straight line joined from

$A$  to  $B$  fall first through the center of the sections.

I say that  $CD$  is parallel to  $EF$

For since they are opposite sections of which  $AB$  is a diameter, and  $CD$  touches one of them at  $A$ , therefore the straight line drawn through  $B$  parallel

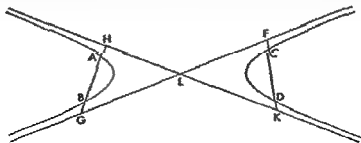


to  $CD$  touches the section (1 44, note) But  $EF$  also touches it, therefore  $CD$  is parallel to  $EF$

Then let the straight line from  $A$  to  $B$  not be through the center of the sections, and let  $AG$  be drawn as a diameter of the sections, and let  $HK$  be drawn tangent to the section, therefore  $HK$  is parallel to  $CD$ , and since the straight lines  $EF$  and  $HK$  touch an hyperbola, therefore they will meet (11 25, end) And  $HK$  is parallel to  $CD$ , therefore also the straight lines  $CD$  and  $EF$  produced will meet. And it is evident they are on the same side as the center

### PROPOSITION 32

If straight lines meet each of the opposite sections, in one point when touching or in two points when cutting, and, when produced, the straight lines meet, then their point of meeting will be in the angle adjacent to the angle containing the section.



Let there be opposite sections and the straight lines  $AB$  and  $CD$

touching the opposite sections in one point or cutting them in two points, and let them meet when produced

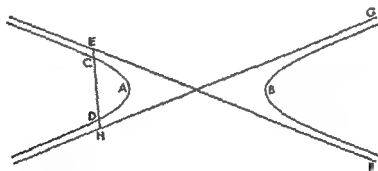
I say that their point of meeting will be in the angle adjacent to the angle containing the section

Let  $FG$  and  $HK$  be asymptotes to the sections, therefore  $AB$  produced will meet the asymptotes (II 8). Let it meet them at  $H$  and  $G$ . And since  $FK$  and  $HG$  are supposed as meeting, it is evident that either they will meet in the place under angle  $HLF$  or in that under angle  $KLK$ . And likewise also, if they touch (II 3).

### PROPOSITION 33

If a straight line meeting one of the opposite sections, when produced both ways, falls outside the section, it will not meet the other section, but will fall through the three places of which one is that contained by the angle containing the section, and two are those contained by the angle adjacent to the angle containing the section.

Let there be the opposite sections  $A$  and  $B$ , and let some straight line  $CD$



cut  $A$ , and, when produced both ways, let it fall outside the section

I say that the straight line  $CD$  does not meet the section  $B$ .

For let  $EF$  and  $GH$  be drawn as asymptotes to the sections, therefore  $CD$  produced will meet the asymptotes (II 8). And it only meets them in the points  $E$  and  $H$ . And so it will not meet the section  $B$ .

And it is evident that it will fall through the three places. For if some straight line meets both of the opposite sections, it will meet neither of the opposite sections in two points. For if it meets it in two points, by what has just been proved it will not meet the other section.

### PROPOSITION 34

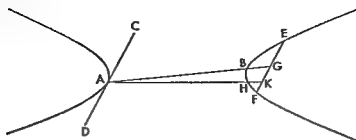
If some straight line touch one of the opposite sections and a parallel to it be drawn in the other section, then the straight line drawn from the point of contact to the midpoint of the parallel will be a diameter of the opposite sections.

Let there be the opposite sections  $A$  and  $B$ , and let some straight line  $CD$  touch one of them  $A$  at  $A$ , and let  $EF$  be drawn parallel to  $CD$  in the other section, and let it be bisected at  $G$ , and let  $AG$  be joined.

I say that  $AG$  is a diameter of the opposite sections.

For if possible, let  $AHA$  be. Therefore the tangent at  $H$  is parallel to  $CD$ .

(II.31) But  $CD$  is also parallel to  $EF$ , and therefore the tangent at  $H$  is parallel to  $EF$ . Therefore



$$EK = KT \text{ (I. 47),}$$

and this is impossible, for

$$EG = GF$$

Therefore  $AH$  is not a diameter of the opposite sections

Therefore  $AB$  is

### PROPOSITION 35

*If a diameter in one of the opposite sections bisects some straight line, the straight line touching the other section at the end of the diameter will be parallel to the bisected straight line*

Let there be the opposite sections  $A$  and  $B$ , and let their diameter  $AB$  bisect the straight line  $CD$  in section  $B$  at  $E$



I say that the tangent to the section at  $A$  is parallel to  $CD$

For if possible, let  $DF$  be parallel to the tangent to the section at  $A$ , therefore

$$DG = GF \text{ (I. 48)}$$

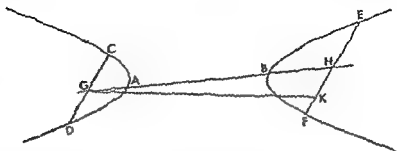
But also

$$DE = EC$$

### PROPOSITION 36

*If parallel straight lines are drawn one in each of the opposite sections, then the straight line joining their midpoints will be a diameter of the opposite sections*

Let there be the opposite sections  $A$  and  $B$ , and let the straight lines  $CD$  and  $EF$  be drawn, one in each of them, and let them be parallel, and let them both



be bisected at points  $G$  and  $H$ , and let  $GH$  be joined

I say that  $GH$  is a diameter of the opposite sections

For if not, let  $GK$  be one. Therefore the tangent to  $A$  is parallel to  $CD$  (II 5), and so also to  $EF$ . Therefore

$$EK = KF \text{ (I 48),}$$

and this is impossible, since also

$$EH = HF$$

Therefore  $GK$  is not a diameter of the opposite sections. Therefore  $GH$  is

### PROPOSITION 37

If a straight line not through the center cuts the opposite sections, then the straight line joined from its midpoint to the center is a so-called upright diameter of the opposite sections, and the straight line drawn from the center parallel to the bisected straight line is a transverse diameter conjugate to it

Let there be the opposite sections  $A$  and  $B$  and let some straight line  $CD$  not through the center cut the sections  $A$  and  $B$  and let it be bisected at  $E$ , and let  $Y$  be the center of the sections, and let  $YE$  be joined, and through  $Y$  let  $AB$  be drawn parallel to  $CD$

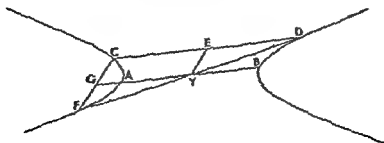
I say that the straight lines  $AB$  and  $YE$  are conjugate diameters of the sections

For let  $DY$  be joined and produced to  $F$ , and let  $CF$  be joined. Therefore

$$DY = YF \text{ (I 30)}$$

But also

$$DE = EC,$$



therefore  $EY$  is parallel to  $FC$  Let  $BA$  be produced to  $G$  And since

$$DY = YF,$$

therefore also

$EY = FG,$

and so also

$CG=FG$

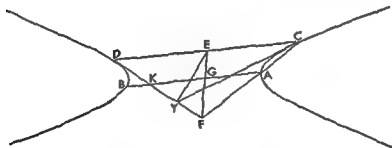
Therefore the tangent at  $A$  is parallel to  $CF$  (II 5), and so also to  $EY$ . Therefore  $EY$  and  $AB$  are conjugate diameters (I 16).

### PROPOSITION 38

If two straight lines meeting touch opposite sections, the straight line joined from the point of meeting to the midpoint of the straight line joining the points of contact will be a so-called upright diameter of the opposite sections, and the straight line drawn through the center parallel to the straight line joining the points of contact is a transverse diameter conjugate to it.

lgrwp

se diameter conjugate to it  
 $\Gamma$  be a point taken at random;  
 and let  $CF$  be joined, therefore


$$AG \models \psi$$

And since

$$CE = ED.$$

and is on triangle  $CFD$ , therefore also

$AG = GK$

And so also

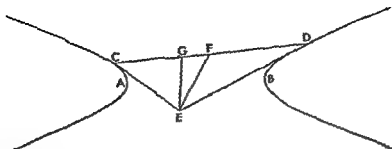
$$GK=GB,$$

and this is impossible. Therefore  $EF$  will not be a diameter

### PROPOSITION 39

*If two straight lines meeting touch opposite sections, the straight line drawn through the center and the point of meeting of the tangents bisects the straight line joining the points of contact*

Let there be the opposite sections  $A$  and  $B$ , and let two straight lines  $CE$  and



$ED$  be drawn touching  $A$  and  $B$ , and let  $CD$  be joined, and let  $EF$  be drawn as a diameter

I say that

$$CF = FD$$

For if not, let  $CD$  be bisected at  $G$ , and let  $GE$  be joined, therefore  $GE$  is a diameter (II 38). But  $EF$  is also, therefore  $E$  is the center (I 31, end). Therefore the point of meeting of the tangents is at the center of the sections, and this is absurd (II 32). Therefore  $CF$  is not unequal to  $FD$ . Therefore equal.

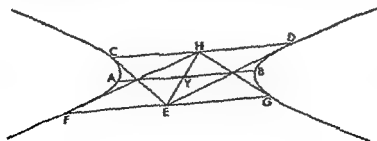
#### PROPOSITION 40

If two straight lines touching opposite sections meet, and through the point of meeting a straight line is drawn, parallel to the straight line joining the points of contact, and meeting the sections, then the straight lines drawn from the points of meeting to the midpoint of the straight line joining the points of contact touch the sections.

Let there be the opposite sections  $A$  and  $B$ , and let two straight lines  $CE$  and  $ED$  be drawn touching  $A$  and  $B$ , and let  $CD$  be joined, and through  $E$  let  $FEG$  be drawn parallel to  $CD$ , and let  $CD$  be bisected at  $H$ , and let  $FH$  and  $HG$  be joined.

I say that  $FH$  and  $HG$  touch the sections.

Let  $EH$  be joined, therefore  $EH$  is an upright diameter, and the straight line drawn through the center parallel to  $CD$  a transverse diameter conjugate to it (II 38). And let the center  $Y$  be taken and let  $AYB$  be drawn parallel to  $CD$ , therefore  $HE$  and  $AB$  are conjugate diameters. And  $CH$  has been drawn



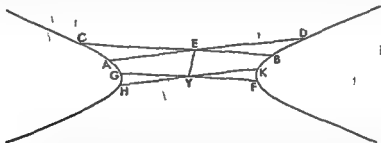
ordinatewise to the second diameter, and  $CE$  has been drawn touching the section and meeting the second diameter. Therefore the rectangle  $EY, YH$  is

equal to the square on the half of the second diameter (I 38), that is to the fourth part of the figure on  $AB$  (Second Def 1 10) And since  $FE$  has been drawn ordinatewise and  $FH$  joined, therefore  $FH$  touches the section  $A$  (I 38) Likewise then also  $GH$  touches section  $B$  Therefore  $FH$  and  $HG$  touch sections  $A$  and  $B$

PROPOSITION 41

*If in opposite sections two straight lines not through the center cut each other, then they do not bisect each other*

For if possible, let them bisect each other, and let  $Y$  be the center of the sections, and let  $EY$  be joined, therefore  $EY$  is a diameter (II 37) Let  $YF$  be drawn through  $Y$  parallel to  $BC$ , therefore  $EF$  is a diameter and conjugate to

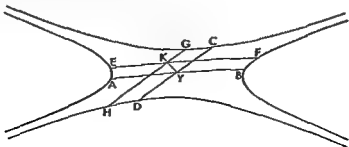


$EY$  (II 37) Therefore the tangent at  $F$  is parallel to  $EY$  (First Def 1 6) Then for the same reasons, with  $HK$  drawn parallel to  $AD$ , the tangent at  $H$  is

PROPOSITION 42

*If in conjugate opposite sections two straight lines not through the center cut each other, they do not bisect each other*

Let there be the conjugate opposite sections  $A, B, C$  and  $D$ , and in  $A, B, C$



and  $D$  let the two straight lines not through the center,  $EF$  and  $GH$ , cut each other at  $K$



I say that they do not bisect each other

For if possible, let them bisect each other, and let the center of the sections be  $Y$ , and let  $AB$  be drawn parallel to  $EF$  and  $CD$  to  $HG$ , and let  $KI$  be joined, therefore  $KY$  and  $AB$  are conjugate diameters (II 37) Likewise  $YK$  and  $CD$  are also conjugate diameters. And so also the tangent at  $A$  is parallel to the tangent at  $C$ , and this is impossible, for it meets it since the tangent at  $C$  cuts the sections  $A$  and  $B$  (II 19), and the tangent at  $A$  sections  $C$  and  $D$ , it is evident also that their point of meeting is in the place under angle  $AYC$  (II 21). Therefore the straight lines  $EF$  and  $GH$  not being through the center do not bisect each other

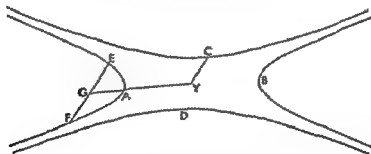
### PROPOSITION 43

If a straight line cuts one of the conjugate opposite sections in two points, and through the center one straight line is drawn to the midpoint of the cutting straight line and another straight line is drawn parallel to the cutting straight line, they will be conjugate diameters of the opposite sections

Let there be the conjugate opposite sections  $A$ ,  $B$ ,  $C$  and  $D$ , and let some straight line cut section  $A$  at the two points  $E$  and  $F$ , and let  $FE$  be bisected at  $G$ , and let  $Y$  be center, and let  $YG$  be joined, and let  $CY$  be drawn parallel to  $EF$

I say that  $AY$  and  $YC$  are conjugate diameters

For since  $AY$  is a diameter and bisects  $EF$ , the tangent at  $A$  is parallel to



$EF$  (II 5) and so also to  $CY$ . Since then they are opposite sections and a tangent has been drawn to one of them,  $A$  at  $A$  and from the center  $Y$  one straight line  $YA$  is joined to the point of contact and another  $CY$  has been drawn parallel to the tangent therefore  $YA$  and  $CY$  are conjugate diameters, for this has been shown before (II 20)

### PROPOSITION 44 (PROBLEM)

Given a section of a cone to find a diameter

Let there be the given conic section on which are the points  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$ . Then it is required to find a diameter

Let it have been done and let it be  $CH$ . Then with  $DF$  and  $EH$  drawn ordinately and produced

$$DF = FB$$

and

$$EH = HA \text{ (First Def 1 4)}$$

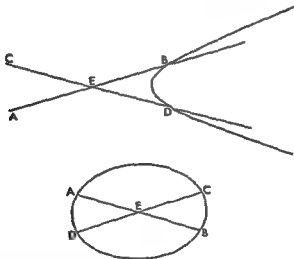
If then we fix the straight lines  $BD$  and  $EA$  in position to be parallel, the points  $H$  and  $F$  will be given. And so  $HFC$  will be given in position.

Then it will be constructed (*συμβεῖησεται*) thus let there be the given conic section on which are the points  $A, B, C, D$  and  $E$ , and let the straight lines  $BD$  and  $AE$  be drawn parallel and be bisected at  $F$  and  $H$ . And the straight line  $FH$  joined will be a diameter of the section (First Def 1 4). And in the same way we could also find an indefinite number of diameters.

### PROPOSITION 45 (PROBLEM)

*Given an ellipse or hyperbola, to find the center*

And this is evident, for if two diameters of the section  $AB$  and  $CD$ , are



drawn through (II 44) the point at which they cut each other will be the center of the section, as indicated below.

### PROPOSITION 46 (PROBLEM)

*Given a section of a cone, to find the axis*

Let the given section of a cone first be a parabola on which are the points  $F, C$  and  $E$ . Then it is required to find its axis.

the straight lines drawn perpendicular to it (First Def 1 7). And the perpendiculars to  $CD$  are also perpendiculars to  $AB$ , and so  $CD$  bisects the perpendiculars

ulars to  $AB$ . If then I fix  $EF$ , the perpendicular to  $AB$ , it will be given in position, and therefore

$$ED = DF,$$

therefore the point  $D$  is given. Therefore through the given point  $D$ ,  $CD$  has been drawn parallel in position to  $AB$ , therefore  $CD$  is given in position.

Then it will be constructed thus: let there be the given parabola on which are the points  $F$ ,  $E$  and  $A$ , and let  $AB$ , a diameter of it, be drawn (I 44), and let  $BE$  be drawn perpendicular to it and let it be produced to  $F$ . If then

$$EB = BF$$

it is evident that  $AB$  is the axis (First Def 1 7), but if not, let  $EF$  be bisected by  $D$ , and let  $CD$  be drawn parallel to  $AB$ . Then it is evident that  $CD$  is the axis of the section, for being parallel to a diameter, that is being a diameter (I 51, end), it bisects  $EF$  at right angles. Therefore  $CD$  has been found as the axis of the given parabola (First Def 1 7).

And it is evident that the parabola has only one axis. For if there is another, as  $AB$ , it will be parallel to  $CD$  (I 51, end). And it cuts  $EF$ , and so it also bisects it (First Def 1 4).

Therefore

$$BE = BF,$$

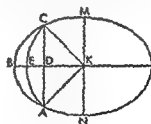
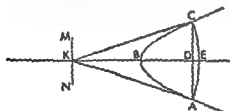
and this is absurd.

### PROPOSITION 47 (PROBLEM)

*Given an hyperbola or ellipse to find the axis*

Let there be the hyperbola or ellipse  $ABC$ , then it is required to find its axis.

Let it have been found and let it be  $AD$ , and  $A$  the center of the section, therefore  $AD$  bisects the ordinates to itself and at right angles (First Def 1 7).



Let the perpendicular  $CDA$  be drawn and let  $KA$  and  $AC$  be joined. Since then

$$CD = DA,$$

therefore

$$CA = KA$$

If then we fix the given point  $C$ ,  $CA$  will be given. And so the circle described with center  $K$  and radius  $KC$  will also pass through  $A$  and will be given in position. And the section  $ABC$  is also given in position, therefore the point  $A$

is given. But the point  $C$  is also given, therefore  $CA$  is given in position. Also  $CD = DA$ , therefore the point  $D$  is given. But also  $A$  is given, therefore  $DA$  is given in position.

Then it will be constructed thus: let there be the given hyperbola or ellipse  $ABC$  the section. Then let  $AD$  be drawn perpendicular to the tangent at  $A$ , and let  $KD$  be drawn through to  $B$ .

Since then

$$AD = DC$$

and  $DK$  is common, therefore the two straight lines  $CD$  and  $DA$  are equal to the two straight lines  $AD$  and  $DA$ , and

$$\text{base } AD = \text{base } DC$$

Therefore  $KBD$  bisects  $ADC$  at right angles. Therefore  $AD$  is an axis (First Def 1.7).

Let  $AKN$  be drawn through  $K$  parallel to  $CA$ , therefore  $AKN$  is the axis of the section conjugate to  $BK$  (First Def 1.8).

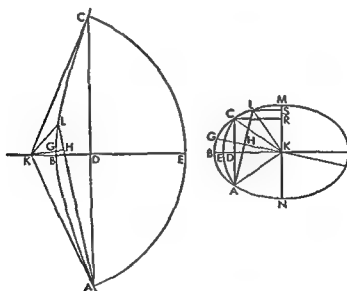
#### PROPOSITION 48 (PROBLEM)

Then to show that there are no other

$\angle G$  Then in the same way as

before, with  $AH$  drawn perpendicular,

$$AH = HL \text{ (First Def 1.4),}$$



and so also

$$AK = AL$$

But also

$$AK = KC,$$

therefore

$$KL = KC,$$

and this is absurd

Now that the circle  $AEC$  does not hit the section also in another point between the points  $A$ ,  $B$  and  $C$ , is evident in the case of the hyperbola, and in the case of the ellipse let the perpendiculars  $CR$  and  $LS$  be drawn Since then

$$KC = KL,$$

for they are radii, also

$$\text{sq } KC = \text{sq } KL$$

But

$$\text{sq } CR + \text{sq } RK = \text{sq } CK,$$

and

$$\text{sq } KS + \text{sq } SL = \text{sq } LK,$$

therefore

$$\text{sq } CR + \text{sq } RK = \text{sq } KS + \text{sq } SL$$

Therefore

$$\begin{aligned} &\text{difference between sq } CR \text{ and sq } SL = \\ &\text{difference between sq } KS \text{ and sq } RK \end{aligned}$$

Again since

$$\text{rect } MR, RN + \text{sq } RK = \text{sq } KM,$$

and also

$$\text{rect } MS, SN + \text{sq } SK = \text{sq } KM \text{ (Eucl II 5),}$$

therefore

$$\text{rect } MR, RN + \text{sq } RK = \text{rect } MS, SN + \text{sq } SK$$

Therefore

$$\begin{aligned} &\text{difference between sq } SK \text{ and sq } KR = \\ &\text{difference between rect } MR, RN \text{ and rect } MS, SN \end{aligned}$$

And it was shown that

$$\begin{aligned} &\text{difference between sq } SK \text{ and sq } KR = \\ &\text{difference between sq } CR \text{ and sq } SL, \end{aligned}$$

therefore

$$\begin{aligned} &\text{difference between sq } CR \text{ and sq } SL = \\ &\text{difference between rect } MR, RN \text{ and rect } MS, SN \end{aligned}$$

And since  $CR$  and  $LS$  are ordinates

$$\text{sq } CR \text{ rect } MR, RN \quad \text{sq } SL \text{ rect } MS, SN \text{ (I 21)}$$

But the same difference was also shown for both therefore

$$\text{sq } CR = \text{rect } MR, RN,$$

and

$$\text{sq } SL = \text{rect } MS, SN \text{ (Eucl V 16, 17 9)}$$

Therefore the line  $LCW$  is a circle, and this is absurd, for it is supposed an ellipse

#### PROPOSITION 49 (PROBLEM)

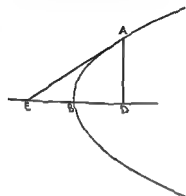
Given a section of a cone and a point not within the section to draw from the point a straight line touching the section in one point

Let the given section of a cone first be a parabola whose axis is  $BD$  Then it is

required to draw a straight line as prescribed from the given point which is not within the section

Then the given point is either on the line or on the axis or somewhere else outside

Now let it be on the line, and let it be  $A$ , and let it have been done, and let it be  $AE$ , and let  $AD$  be drawn perpendicular,



position

Then it will be constructed thus let  $AD$  be drawn perpendicular from  $A$ , and let  $BE$  be made equal to  $BD$ , and let  $AE$  be joined. Then it is evident that it touches the section (I 33)

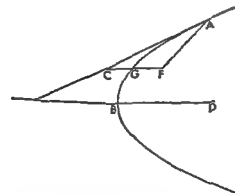
Again let the given point  $E$  be on the axis, and let it have been done, and let  $AE$  be drawn tangent, and let  $AD$  be drawn perpendicular, therefore

$$BE = BD \text{ (I 35)}$$

And  $BE$  is given, therefore also  $BD$  is given. And the point  $B$  is given, therefore

let  $DA$  be drawn perpendicular to  $ED$ , and let  $AE$  be joined. Then it is evident that  $AE$  touches (I 33)

And it is evident also that, even



17)

Then let  $C$  be the given point, and let it have been done, and let  $CA$  be it, and through  $C$  let  $CF$  be drawn parallel to the axis that is to  $BD$ , therefore  $CF$  is given in position. And from  $A$  let  $FA$  be drawn ordinatewise to  $CF$ , then

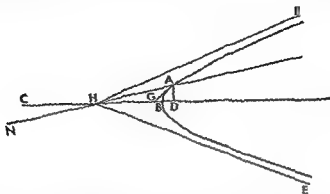
$$CG = FG \text{ (I 35)}$$

And the point  $G$  is given, therefore  $F$  is also given. And  $FA$  has been

erected ordinatewise, that is, parallel to the tangent at  $G$  (I 32), therefore  $FA$  is given in position. Therefore  $A$  is also given, but also  $C$ . Therefore  $CA$  is given in position.

le  
(1  
(I 33)

Again let it be an hyperbola whose axis is  $DBC$  and center  $H$ , and asymptotes  $HE$  and  $HF$ . Then the given point will be given either on the section or



on the axis or within angle  $EHF$  or in the adjacent place or on one of the asymptotes containing the section or in the place between the straight lines containing the angle vertical to angle  $EHF$ .

Let the point  $A$  first be on the section and let it have been done, and let  $AG$  be tangent and let  $AD$  be drawn perpendicular and let  $BC$  be the transverse side of the figure, then

$$CD : DB :: CG : GB \quad (1.36)$$

And the ratio of  $CD$  to  $DB$  is given, for both the straight lines are given, therefore also the ratio of  $CG$  to  $GB$  is given. And  $BC$  is given, therefore point  $G$  is

and let  $AG$  be joined. Then it is evident that  $AG$  touches the section (1.34).

Then again let the given point  $G$  be on the axis and let it have been done, and let  $AG$  be drawn tangent, and let  $AD$  be drawn perpendicular. Then for the same reasons

$$CG : GB :: CD : DB \quad (1.36)$$

And  $BC$  is given, therefore the point  $D$  is given. And  $DA$  is perpendicular, therefore  $DA$  is given. Therefore the point

Then it will be

and let it be contrived that

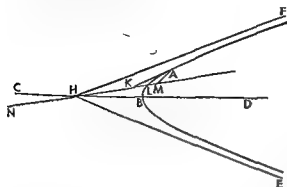
$$CG : GB :: CD : DB$$

and let  $DA$  be drawn perpendicular and let  $AG$  be joined. Then it is evident that  $AG$  does the problem (1.34) and that from  $G$  another tangent to the section could be drawn on the other side.

With the same things supposed let the given point  $A$  be in the place inside angle  $EHF$  and let it be required to draw a tangent to the section from  $A$ . Let it have been done and let it be  $KA$  and let  $KA$  be joined and produced and let  $HN$  be made equal to  $LI$  therefore they are all given. Then also  $LN$  will be given. Then let  $AM$  be drawn ordinatewise to  $VN$  then also

$$VA : AI :: VM : MI$$

And the ratio of  $NK$  to  $KL$  is given, therefore also the ratio of  $NM$  to  $ML$  is given. And the point  $L$  is given, therefore also  $M$  is given. And  $MA$  has been erected parallel to the tangent at  $L$ , therefore  $MA$  is given in position. And also the section  $ALB$  is given in position, therefore the point  $A$  is given. But  $K$  is also given, therefore  $AK$  is given.



Then it will be constructed thus: let the other things be supposed the same and the given point  $K$ , and  $AH$  be joined and produced, and

let  $HN$  be made equal to  $HL$ , and let it be contrived that

$$NK : KL = NM : ML$$

and let  $MA$  be drawn parallel to the tangent at  $L$  (above), and let  $KA$  be joined, therefore  $KA$  touches the section (I 34).

And it is evident that a tangent to the section could also be drawn to the other side.

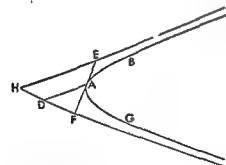
With the same things supposed let the given point  $F$  be on one of the asymptotes containing the section, and let it be required to draw from  $F$  a tangent to the section. And let it have been done, and let it be  $FAE$ , and through  $A$  let  $AD$  be drawn parallel to  $EH$ , then

$$DH = DF,$$

since also

$$FA = AE \text{ (II 3)}$$

And  $FH$  is given, therefore also point  $D$  is given. And through the given point  $D$ ,  $DA$  has been drawn parallel in position to  $EH$ , therefore  $DA$  is given in position. And the section is also given in position, therefore the point  $A$  is given. But  $F$  is also given, therefore the straight line  $FAE$  is given in position.



Then it will be constructed thus: let there be the section  $AB$ , and asymptotes  $EH$  and  $HF$ , and the given point  $F$  on one of the asymptotes containing the section, and let  $FH$  be bisected at  $D$ , and through  $D$

let  $DA$  be drawn parallel to  $HE$ , and let  $FA$  be joined. And since

$$FD = DH,$$

therefore also

$$FA = AE$$

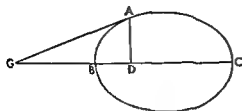
And so by things shown before, the straight line  $FAE$  touches the section (II 9).

With the same things supposed, let the given point be in the place under the





Then it will be constructed thus let  $AD$  be drawn perpendicular, and let  $CG \cdot GB = CD \cdot DE$ ,

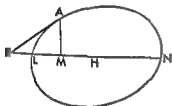


and let  $AG$  be joined. Then it is evident that  $AG$  touches, as also in the case of the hyperbola (I 34)

Then again let the given point be  $K$ , and let it be required to draw a tangent. Let it have been done, and let it be  $KA$ , and let the straight line  $KLH$  be joined to the center  $H$  and produced to  $N$ , then

it will be given in position. And if  $AM$  is drawn ordinately, then

$NK \cdot KL = NM \cdot ML$  (I 36)



And the ratio of  $NK \cdot KL$  is given, therefore the ratio of  $NM$  to  $LM$  is also given. Therefore the point  $M$  is given. And  $MA$  has been erected ordinately, for it is parallel to the tangent at  $L$ , therefore  $MA$  is given in position. Therefore the point  $A$  is given. But also  $K$ , therefore  $KA$  is given in position.

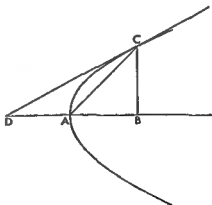
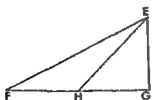
And the construction (*συμβεβηκός*) is the same as for the preceding.

#### PROPOSITION 50 (PROBLEM)

Given the section of a cone, to draw a tangent which will make with the axis, on the same side as the section, an angle equal to a given acute angle.

Let the section of a cone first be a parabola whose axis is  $AB$ , then it is required to draw a tangent to the section which will make with the axis  $AB$ , on the same side as the section, an angle equal to the given acute angle.

Let it have been done, and let it be  $CD$ , therefore angle  $BDC$  is given. Let  $BC$



is also given And it is in position with respect to  $BA$  and the given point  $A$ ; therefore  $CA$  is given in position And the section is also given in position, therefore the point  $C$  is given And  $CD$  touches, therefore  $CD$  is given in position

(r 33)

I say then that

$$\text{angle } CDB = \text{angle } EFG$$

For since

$$FG : GH :: DB : BA$$

and

$$HG : GE :: AB : BC,$$

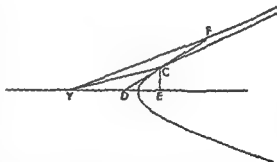
therefore ex aequali

$$FG : GE :: DB : BC,$$

And the angles at  $G$  and  $B$  are right angles therefore

$$\text{angle at } F = \text{angle at } D$$

the ratio of the square on  $CE$  to the square on  $ED$  is given, for each of the rectangles  $CD, DE$  and  $DE, EC$  is given Therefore the ratio of rectangle  $YE, ED$  to the square on  $ED$  is given, and so also the ratio of  $YE$  to  $ED$  is given And the angle at  $E$  is given therefore the angle at  $Y$  is also given Then some straight line  $CY$  has been drawn across in position with respect to the straight line  $YE$  and to the given point  $Y$  at a given angle, therefore  $CY$  is given in position And the section



is also given in position, therefore the point  $C$  is given And  $CD$  has been drawn across as tangent, therefore  $CD$  is given in position

Let the asymptote to the section  $YF$  be drawn, therefore  $CD$  produced will meet the asymptote (II 3) Let it meet it at  $F$  Therefore

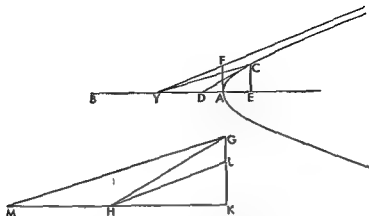
$$\text{angle } FDE > \text{angle } FYD$$

Therefore, for the construction the given acute angle will have to be greater

a hyperbola  
HG greater

angle  $KHL = \text{angle } A Y F$ ,

and let  $AF$  be drawn from  $A$  perpendicular to  $AB$ , and let some point  $G$  be



taken on  $GH$ , and let  $GK$  be drawn from it perpendicular to  $HK$ . Since then

angle  $FYA = \text{angle } LHK$ ,

and also the angles at  $A$  and  $K$  are right, therefore

$$YA : AF :: HK : KL$$

$$HK \cdot KL > HK : KG;$$

therefore also

$$YA \cdot AF > HK : KG.$$

And so also

$$\text{sq } YA \cdot \text{sq } AF > \text{sq } HK \cdot \text{sq } KG.$$

But

$$\text{sq } YA \cdot \text{sq } AF :: \text{transverse} \cdot \text{upright (II 1);}$$

therefore also

$$\text{transverse} \cdot \text{upright} > \text{sq } HK : \text{sq } KG$$

If then we shall contrive that

$$\text{sq } YA \cdot \text{sq } AF \text{ some other } \text{sq } KG,$$

it will be greater than the square on  $HK$ . Let it be the rectangle  $MK, KH$ ; and

let  $GM$  be joined. Since then

$$\text{sq } MK > \text{rect } MK, KH,$$

therefore

$$\begin{aligned} \text{sq } MK \cdot \text{sq } KG &> \text{rect } MK, KH \cdot \text{sq } KG \\ &> \text{sq } YA \cdot \text{sq } AF \end{aligned}$$

And if we shall contrive that

$$\text{sq } MK \cdot \text{sq } KG : \text{sq } YA : \text{some other,}$$

it will be

joined from

..

Let angle  $AYC$  be made equal to angle  $GMA$ ; therefore  $YC$  will cut the sec-

$$\text{sq } YE :: \text{sq } EC :: \text{sq } MA : \text{sq } KG.$$

But also

transverse  $\cdot$  upright rect  $YE, ED$  sq  $EC$  (I 37),

and

transverse upright rect  $MK, KH$  sq  $KG$

And inversely

sq  $CE$  rect  $YE, ED$  sq  $GK$  rect  $MK, KH$ ,

therefore *ex aequali*

sq  $YE$  rect  $YE, ED$  sq  $MK$  rect  $MK, KH$

And therefore

$YE \cdot ED = MK \cdot KH$

But also we had

$CE \cdot EY = GK \cdot KM$ ,

therefore *ex aequali*

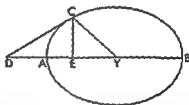
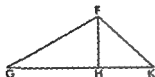
$CE \cdot ED = GK \cdot KH$

And the angles at  $E$  and  $K$  are right angles, therefore

angle at  $D =$  angle  $GKH$

Let the section be an ellipse whose axis is  $AB$ . Then it is required to draw a tangent to the section which with the axis will contain, on the same side as the section, an angle equal to the given acute angle

Let it have been done, and let it be  $CD$ . Therefore angle  $CDA$  is given. Let  $CE$  be drawn perpendicular, therefore the ratio of the square on  $DE$  to the square on  $EC$  is given. Let  $Y$  be the center of the section, and let  $CY$  be joined. Then the ratio of the square on  $CE$  to the rectangle  $DE, EY$  is given, for it is



Then the problem will be constructed thus: let there be the given acute angle  $FGH$ , and let some point  $F$  be taken on  $FG$ , and let  $FH$  be drawn perpendicular, and let it be contrived that

upright transverse sq  $FH$  rect  $GH, HK$ ,

and let  $KF$  be joined, and let  $Y$  be the center of the section, and let angle  $AYC$  be constructed equal to angle  $GKF$ , and let  $CD$  be drawn tangent to the section (II 49)

I say that  $CD$  does the problem that =

angle  $CDE =$  angle  $FGH$

For since

$$YE \cdot EC = AH \cdot FH$$

therefore also

$$sq \text{ } YE = sq \text{ } EC = sq \text{ } AH = sq \text{ } FH$$

But also

$$sq \text{ } EC \text{ rect } DE \cdot EY = sq \text{ } FH \text{ rect } AH \cdot HG$$

for each is the same ratio as that of the upright to the transverse (1 37 and above) And *ex aequali* therefore

$$sq \text{ } YE \text{ rect } DE \cdot EY = sq \text{ } AH \text{ rect } AH \cdot HG$$

And therefore

$$YE \cdot ED = AH \cdot HG$$

But also

$$YE \cdot EC = AH \cdot FH$$

*ex aequali* therefore

$$DE \cdot EC = HG \cdot FH$$

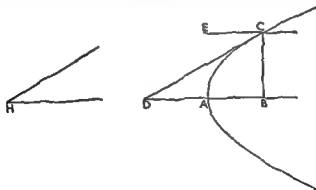
And the sides about the right angles are proportional therefore  
angle  $CDE = \text{angle } FGH$

Therefore  $CD$  does the problem

#### PROPOSITION 51 (PROBLEM)

Given a sect on of a cone to draw a tangent which with the diameter drawn through the point of contact will contain an angle equal to a given acute angle

Let the given section of a cone first be a parabola whose axis is  $AB$  and the given angle  $H$  then it is required to draw a tangent to the parabola which with



the diameter from the point of contact will contain an angle equal to the angle at  $H$

Let it have been done and let  $CD$  be drawn a tangent making with the diameter  $EC$  drawn through the point of contact angle  $ECD$  equal to angle  $H$  and let  $CD$  meet the axis at  $D$  (1 24) Since then  $AD$  is parallel to  $EC$  (1 51 end)

$$\text{angle } ADC = \text{angle } ECD$$

But angle  $ECD$  is given for it is equal to angle  $H$  therefore angle  $ADC$  is also given

Then it will be constructed thus let there be a parabola whose axis is  $AB$

and

therefore also

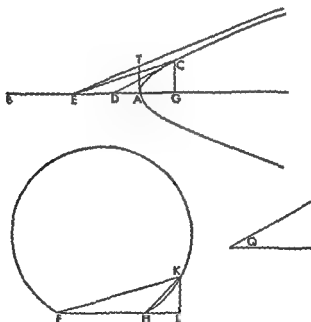
$$\text{angle } H = \text{angle } ADC,$$

$$\text{angle } ADC = \text{angle } ECD,$$

$$\text{angle } H = \text{angle } ECD$$

ratio of the transverse to the upright is given, and so also the ratio of rectangle  $EG, GD$  to the square on  $CG$  (1 37) Then let some given straight line  $\Gamma H$  be

$KL$  be drawn perpendicular making



rect  $FL$   $IH$  sq  $LK$  transverse upright,  
and let  $FA$  and  $KH$  be joined Since then

$$\text{angle } FKH = \text{angle } LCD,$$

but also

$$\text{rect } EG \text{ } GD \text{ sq } GC \text{ transverse upright,}$$

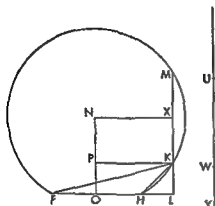
and

$$\text{rect } FL \text{ } LH \text{ sq } LK \text{ transverse upright,}$$

therefore triangle  $AFI$  is similar to triangle  $ECG$ , and triangle  $FHA$  to triangle  $ECD$ <sup>1</sup> And so

$$\text{angle } HFA = \text{angle } CED$$

Then it will be constructed thus let there be the given hyperbola  $AC$  and



a segment of a circle greater than

drawn perpendicular to  $FH$  and let  $NO$  be cut at  $P$  in the ratio of  $UW$  to  $WY$  and through  $P$  let  $PK$  be drawn paral

drawn perpendicular to it therefore it is parallel to  $FH$  And therefore

$$NP : PO \text{ or } UW : WY :: XA : AL$$

And doubling the antecedents

$$ZW : WY :: MK : KL$$

componendo

$$ZY : YW :: ML : LK$$

But

$$ML : LK :: \text{rect } ML : LA :: \text{sq } LA$$

therefore

$$ZY : YW :: \text{rect } ML : LA :: \text{sq } LK :: \text{rect } FL : LH :: \text{sq } LK \text{ (Eucl III 36)}$$

<sup>1</sup>Pappus in lemma IX to this book Let triangle  $ABC$  be similar to triangle  $DEF$  and triangle  $AGB$  to  $DEH$  the result is

$$\text{rect } BC : CG :: \text{sq } CA :: \text{rect } EF : FH :: \text{sq } DF$$



For since because of similarity

and

therefore

But also

therefore

But also

therefore also compounded ratios the same with compounded. Therefore

$$\text{rect } BC : CG :: \text{sq } CA :: \text{rect } EF : FH :: \text{sq } FD$$



But

$ZI \parallel H$  transverse upright,

therefore also

rect  $FL \cdot LH$  sq  $LK$  transverse upright

Then let  $AT$  be drawn from  $A$  perpendicular to  $AB$ . Since then  
sq  $EA$  sq  $AT$  transverse upright (II 1)

and also

transverse upright rect  $FL \cdot LH$  sq  $LK$

and

sq  $FL$  sq  $LK$  > rect  $FL \cdot LH$  sq  $LK$

therefore also

sq  $FL$  sq  $LK$  > sq  $EA$  sq  $AT$

And the angles at  $A$  and  $L$  are right angles therefore  
angle  $F$  < angle  $E$

Therefore also

rect  $FL \cdot LH$  sq  $LK$  rect  $EG \cdot GD$  sq  $CG$

therefore triangle  $AFL$  is similar to triangle  $ECG$  and triangle  $AHL$  to triangle  $CGD$  and triangle  $AFL$  to triangle  $CGD$ . And so  
angle  $ECD$  = angle  $FAL$  = angle  $Q$

And if the ratio of the transverse to the upright is equal to equal  $AL$  touches the circle  $FAL$  (Eucl III 37) and the straight line joined from the center to  $A$  will be parallel to  $FL$  and itself will do the problem

### PROPOSITION 52

If a straight line touches an ellipse making an angle with the diameter drawn through the point of contact it is not less than the angle adjacent to the one contained by the straight lines deflected at the middle of the section

Let there be an ellipse whose axes are  $AB$  and  $CD$  and center  $E$  and let  $AB$  be the major axis and let the straight line  $GFL$  touch the section and let  $AC \cdot CB$  and  $FE$  be joined and let  $BC$  be produced to  $L$

I say that angle  $LFE$  is not less than angle  $LCI$

For  $FE$  is either parallel to  $LB$  or not

Let it first be parallel and

$AE = EB$

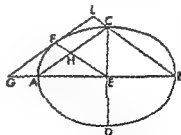
therefore also

$FI = IC$

And  $FE$  is a diameter therefore the tangent at  $F$  is parallel to  $AC$  (II 6). But also  $FE$  is parallel to  $LB$  therefore  $FHCL$  is a parallelogram and therefore  
angle  $LFH$  = angle  $ICL$

And since  $AE$  and  $FB$  are each greater than  $FC$  angle  $ACB$  is obtuse therefore angle  $ICA$  is acute. And so also angle  $LFE$ . And therefore angle  $GFE$  is obtuse

Then let  $EF$  not be parallel to  $LB$ , and let  $FA$  be drawn perpendicular,



therefore  $LBE$  is not equal to angle  $FEA$  But  
 rt angle at  $E$  = rt angle at  $K$ ,  
 therefore it is not true that

$$sq\ BE\ sq\ EC\ sq\ EK\ sq\ KF$$

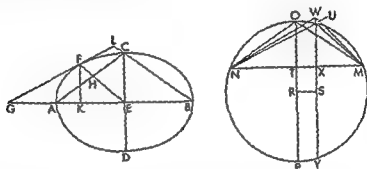
But  
 $sq\ BE\ sq\ EC\ rect\ AE, EB\ sq\ EC$  transverse upright (I 21)  
 and

$$transverse\ upright\ rect\ GK, KE\ sq\ KF\ (I\ 37)$$

Therefore it is not true that

$$rect\ GA, KE\ sq\ KF\ sq\ KE\ sq\ KF$$

Therefore  $GK$  is not equal to  $KE$  Let there be laid out a segment of a circle



$MUN$  admitting an angle equal to angle  $ACB$  (Eucl III 33), and angle  $ACB$  is obtuse, therefore  $MUN$  is a segment less than a semicircle (Eucl III 31) Then let it be contrived that

$$GK\ KE\ NX\ XV$$

$$RO > SU$$

and  $AB$  and  $MN$   
 angles at  $E$  and  
 similar Therefore

$$sq\ TN\ sq\ TO\ sq\ BE\ sq\ EC$$

And since

$$TR > SY$$

and

$$RO > SU,$$

therefore

$$RO\ TR > SU\ SY,$$

and *convertendo*

$$RO\ OT < SU\ UX$$

And, doubling the antecedents therefore

$$PO\ TO < YU\ UX$$

And *separando*

$$PT\ TO < YX\ UX$$



And  $KE$  is a diameter, therefore the tangent to the section at  $K$ , that is  $HKG$ , is parallel to  $CA$  (ii G) And also  $EK$  is parallel to  $GB$ , therefore  $HFCG$  is a parallelogram, and therefore

$$\text{angle } GKF = \text{angle } GCF$$

And angle  $GCT$  is equal to the given angle, that is  $U$ , therefore also

angle  $GHE = \text{angle } U$

Then let

angle  $U > \text{angle } ACG$ ,

then inversely

angle  $Y < \text{angle } ACB$

lore

$\angle MNP < \angle ACB$

*But*

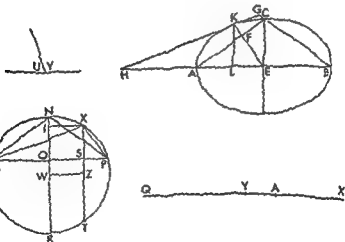
angle  $MNO = \text{half angle } MNP,$

અણલ

angle  $ACE = \text{half angle } ACB,$

therefore

Adx



And so also

$$sq \ 4E \ sq \ EC > sq \ MO \ sq \ NO$$

But

$$\text{sq } AE = \text{rect } AE, EB,$$

and

sq  $MO \cong \text{rect } MO \cong \text{rect } NO, OR$  (Eucl III 35),

therefore

rect 1E EB sq EC or transverse upright (x 21) > EO ov

Then let it be that

transverse upright  $QA' A'X'$ ,  
and let  $QX'$  be bisected at  $Y'$  Since then  
transverse upright  $> RO ON$ ,  
also

$$QA' A'X' > RO ON$$

And *componendo*

$$QX' X'A' > RN NO$$

Let the center of the circle be  $W$ , and so also

$$Y'X' X'A' > WN NO$$

And *separando*

$$A'Y' A'X' > WO ON$$

Then let it be contrived that

$$A'Y' A'X' WO \text{ less than } ON$$

such as  $IO$ , and let  $IX$  and  $XT$  and  $WZ$  be drawn parallel

Therefore

$$A'Y' A'X' WO OI ZS SX,$$

and *componendo*

$$Y'X' X'A' ZX XS$$

And doubling the antecedents

$$QX X'A' TX XS$$

And *separando*

$$QA' A'X' \text{ or transverse upright } TS SX$$

Then let  $MX$  and  $XP$  be joined, and let angle  $AEL$  be constructed on straight line  $AE$  at point  $E$  equal to angle  $M'PV$  and then at  $Z$  let  $KH$  be drawn touching the section (II

and

$$\text{rt angle at } S = \text{rt angle at } L,$$

therefore triangle  $YSP$  is equiangular with triangle  $KEL$

And

$$\text{transverse upright } TS SX \text{ rect } TS, SX$$

$$\text{sq } SX \text{ rect } M\lambda, SP \text{ sq } SX,$$

therefore triangle  $ALE$  is similar to triangle  $SXP$ , and triangle  $MXP$  to triangle  $AHE$ , and therefore

$$\text{angle } M\lambda P = \text{angle } HKE$$

But

$$\text{angle } MXP = \text{angle } MNP = \text{angle } Y,$$

therefore also

$$\text{angle } HKE = \text{angle } Y$$

And therefore

$$\text{adjacent angle } GAE = \text{adjacent angle } U$$

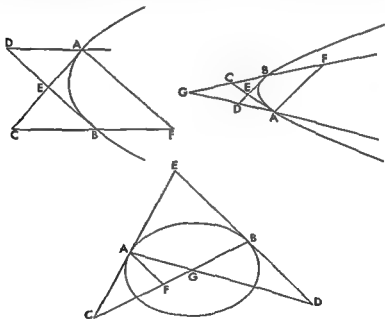
Therefore  $GH$  has been drawn across tangent to the section and making with the diameter  $AE$ , drawn through the point of contact angle  $GAE$  equal to the given angle  $U$ , and this it was required to do

## BOOK THREE

### PROPOSITION 1

*If straight lines, touching a section of a cone or circumference of a circle, meet, and diameters are drawn through the points of contact meeting the tangents, the resulting vertically related triangles will be equal*

Let there be the section of a cone or circumference of a circle  $AB$ , and let  $AC$  and  $BD$ , meeting at  $E$ , touch  $AB$ , and let the diameters of the section  $CB$  and



$DA$  be drawn through  $A$  and  $B$ , meeting the tangents at  $C$  and  $D$

I say that

$$\text{trgl } ADE = \text{trgl } EBC$$

For let  $AF$  be drawn from  $A$  parallel to  $BD$ , therefore it has been dropped ordinatewise (1 32) Then in the case of the parabola

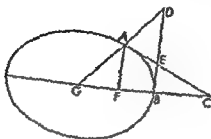
$$\text{pllg } ADBF = \text{trgl } ACF \text{ (1 42),}$$

and, with the common area  $AEBF$  subtracted,

$$\text{trgl } ADE = \text{trgl } CBE$$

And in the case of the others let the diameters meet at center  $G$  Since then

$AF$  has been dropped ordinatewise, and  $AC$  to the  
rect  $FG$ ,  $GC$  = sq  $BG$  (1.5)



Therefore

therefore also

$$FG \cdot GB = BG \cdot GC,$$

But

$$FG \cdot GC = \text{sq } FG = \text{sq } GB \text{ (Euc 7.1)}$$

and

$$\text{sq } FG = \text{sq } GB \quad \text{trgl } AGF : \text{trgl } DOB \text{ (E. 1)}$$

therefore also

$$FG \cdot GC = \text{trgl } AGF = \text{trgl } AGC,$$

Therefore

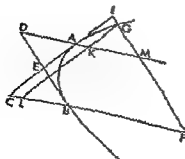
$$\text{trgl } AGF = \text{trgl } AGC = \text{trgl } AGF = \text{trgl } AGC.$$

Let the common area  $DOBE$  be subtracted, therefore as  
 $\text{trgl } AGC = \text{trgl } DOB$   
 $\text{trgl } AFD = \text{trgl } CEB$

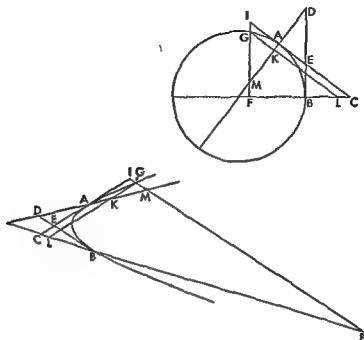
### PROPOSITION 2

With the same things supposed if some point is taken on the  
ence of a circle and through it parallels to the tangents are  
diameters then the quadrilateral produced on one of the tang  
diameters will be equal to the triangle produced on the same to  
diameter

For let there be a section of a cone or circumference of a



$AEC$  and  $BED$  be tangents, and  $AD$  and  $BC$  diameters, and let some point  $G$



be taken on the section, and  $GKL$  and  $GMF$  be drawn parallel to the tangents

I say that

$$\text{trgl } AIM = \text{quadr } CLGI$$

For triangle  $GKM$  has been shown equal to quadrilateral  $AL$  (1 42, 43), let the common quadrilateral  $IA$  be added or subtracted, and

$$\text{trgl } AIM = \text{quadr } CG^1$$

### PROPOSITION 3

With the same things supposed, if two points are taken on the section or circumfer-

I say that

$$\text{quadr } LG = \text{quadr } MH,$$

<sup>1</sup>Eutocius commenting gives the proof for another and important case "It must be remarked that, if the point  $G$  is taken between  $A$  and  $B$  so that the parallels are, for instance,



and

$$\text{quadr } LN = \text{quadr } RN$$

For since it has already been shown that

$$\text{trgl } RPA = \text{quadr } CG \text{ (III 2),}$$

and

$$\text{trgl } AMI = \text{quadr } CF \text{ (III 2),}$$

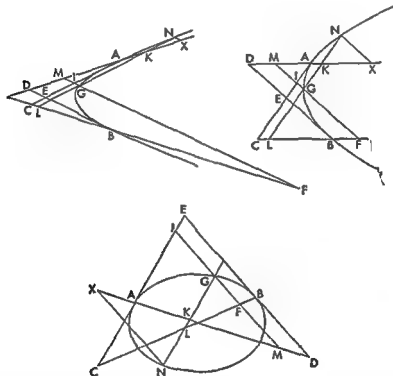
and

$$\text{trgl } RPA = \text{trgl } AMI + \text{quadr } PM,$$

therefore also

$$\text{quadr } CG = \text{quadr } CF + \text{quadr } PM,$$

*MIGI* and *LGA* one must draw *IA* to the section at *N* for instance and through *N* draw



infer-  
the  
the

*AA* parallel to *BD* for by what was said in the forty ninth and fiftieth theorems of the first book (1 49 50) and in the notes to them

$$\text{trgl } AA\lambda = \text{quadr } AC$$

But triangle *AAλ* is similar to triangle *AMG* because *MG* is parallel to *Nλ*, but it is also equal to it because *AC* is a tangent and *GV* is parallel to it and *MY* is a diameter and *GK = AA*

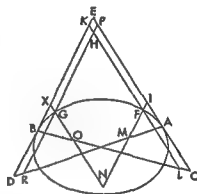
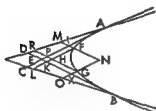
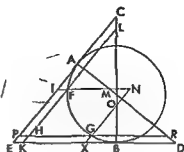
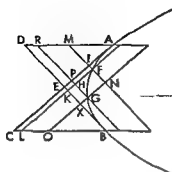
Since then  $\text{trgl } AA\lambda = \text{quadr } AC = \text{trgl } AMG$   
with the common quadrilateral *AG* subtracted as remainders

$$\text{trgl } AIM = \text{quadr } CC$$

It will be noticed that, just as in the second note to 1 50 the quadrilateral *CG* is to be considered as the difference between the triangles *CIF* and *GFI*

and so

$$\text{quadr } CG = \text{quadr } CH + \text{quadr } RF$$



be taken

If the common quadrilateral  $CH$  be subtracted therefore as remainders  
 $\text{quadr } LG = \text{quadr } HM$   
 For the and therefore as wholes  
 $\text{quadr } LN = \text{quadr } RN$

#### PROPOSITION 4

duced to  $F$  and  $G$

I say that

$$\text{trgl } AGD = \text{trgl } BDF$$

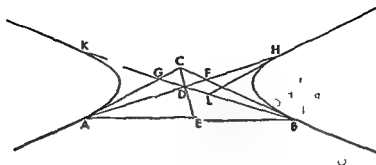
and

$$\text{trgl } ACF = \text{trgl } BCG$$

For let  $HL$  be drawn through  $H$  tangent to the section therefore it is parallel to  $AG$  (1 44 note) And since

$$AD = DH \text{ (1 30)}$$

$\text{trgl } AGD = \text{trgl } DHL$  (Eucl vi 19)



But

$$\text{trgl } DHL = \text{trgl } BDF \text{ (III 1);}$$

therefore also

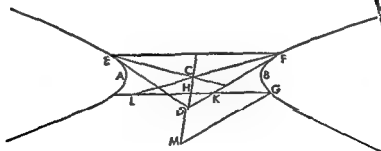
$$\text{trgl } AGD = \text{trgl } BDF$$

And so also

$$\text{trgl } ACF = \text{trgl } BCG.$$

#### PROPOSITION 5

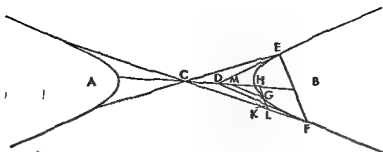
*If two straight lines touching opposite sections meet, and some point is taken on either of the sections, and from it two straight lines are drawn, the one the tangent, the other parallel to the line joining the points of intersection of the triangle produced by them on the diameter drawn through the point of meeting of the tangents, the triangle cut off at the point of meeting of the tangents differs from the triangle cut off at the point of meeting of the tangents by the triangle cut off at the point of meeting of the tangents.*



duced, and let  $FC$  and  $EC$  be joined and produced, and let some point  $C$  be taken on the section, and through it let  $HGKL$  be drawn parallel to  $EF$ ,  $GM$  parallel to  $DF$ .

I say that triangle  $GHM$  differs from triangle  $KHD$  by triangle  $KLF$ .  
For since  $CD$  has been shown to be a diameter of the opposite sections

39, 38), and  $EF$  to be an ordinate to it (II 38, First Def 1 5), and  $GH$  has been



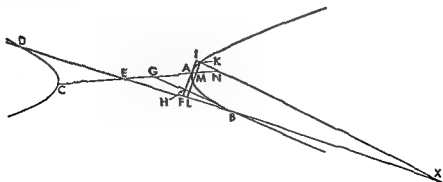
$$\text{trgl } KLF = \text{quadr } MGKD$$

## PROPOSITION 6

be taken  
I say  
the cor. tangents  
that

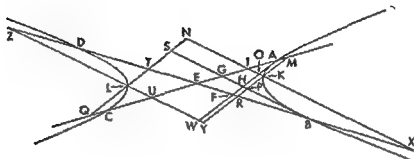
$$\text{quadr } KF = \text{trgl } AIN$$

the d there since  $AB$  and  $CD$  are opposite sections, and  $AF$ , meeting  $BD$ , touches



section  $AB$  and  $KL$  has been drawn parallel to  $AF$ , therefore

## PROPOSITION 7

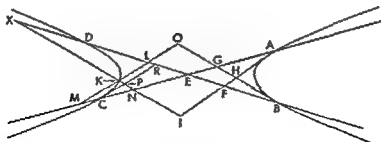


For since

$$\text{trgl. } AOI = \text{quadr. } RO \text{ (III. 2),}$$

And another and important area has the same value as the area of the triangle  $AOI$ .

(and  $IOU$ ), therefore



$$\text{whole trgl. } MEL = \text{trgl. } MKN + \text{trgl. } AEF$$

With common triangle  $KVN$  subtracted, as remainders

$$\text{trgl. } AEF = \text{quadr. } KLEN$$

Let the common quadrilateral  $FENI$  be added, therefore, in whole,

$$\text{trgl. } AIN = \text{quadr. } KLEI$$

And likewise also

$$\text{trgl. } BOL = \text{quadr. } KAGO."$$

let the quadrilateral  $EO$  be added to both, therefore  
whole  $\text{trgl } AEF = \text{quadr } KE$

But also

$$\text{trgl } BGE = \text{quadr } LE \text{ (III 5, note),}$$

and

$$\text{trgl } AEF = \text{trgl } BGE \text{ (III 1),}$$

therefore

$$\text{quadr } LE = \text{quadr } IKRE$$

Let the common quadrilateral  $NE$  be added, therefore as wholes  
whole  $\text{quadr } TK = \text{quadr } IL$ ,

and also

$$\text{quadr } KU = \text{quadr } RL$$

### PROPOSITION 8

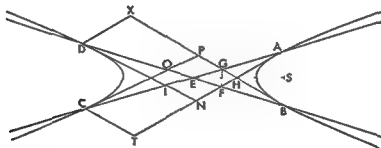
With the same things supposed, instead of  $K$  and  $L$  let there be taken the points  $C$  and  $D$  at which the diameters hit the sections, and through them let the parallels to the tangents be drawn

I say that

$$\text{quadr } DG = \text{quadr } FC$$

and

$$\text{quadr } XI = \text{quadr } OT$$



For since it was shown

$$\text{trgl } AGH = \text{trgl } HBF \text{ (III 1),}$$

and the straight line from  $A$  to  $B$  is parallel to the straight line from  $G$  to  $F$ ,<sup>1</sup>  
therefore

$$AE \quad EG \quad BE \quad EF,$$

and *convertendo*

$$EA \quad AG \quad EB \quad BF$$

And also

$$CA \quad AE \quad DB \quad BE,$$

<sup>1</sup>For the point  $H$  falls within the angle  $AEB$  (II 25) and the straight line drawn from  $H$  to the midpoint of  $AB$  that is  $S$  is a diameter (II 29) and must therefore pass through  $E$  (I 51 end). An analogous series of propositions is found for the opposite sections II. 32 38 39

Then, since  
therefore

$$\text{trgl } GHA = \text{trgl } FHB$$

$$\text{trgl } GFB = \text{trgl } GFA$$

Their bases are the same, therefore their heights are equal (Eucl. VI 1)

for each is double the other, therefore *ex aequali*

$$CA \cdot AG = DB \cdot BF$$

And the triangles are similar because of the parallels; therefore

$$\text{trgl } CTA \sim \text{trgl } AHG \sim \text{trgl } XBD \sim \text{trgl } HBF \text{ (Eucl vi 19)}$$

And alternately, but

$$\text{trgl } AHG = \text{trgl } HBF \text{ (iii 1);}$$

therefore

$$\text{trgl } CTA = \text{trgl } XBD$$

As parts of these it was shown

$$\text{trgl } AHG = \text{trgl } HBF,$$

therefore also as remainders

$$\text{quadr } DH = \text{quadr } CH.$$

And so also

$$\text{quadr } DG = \text{quadr } CF$$

And since  $CO$  is parallel to  $AF$ ,

$$\text{trgl } COE = \text{trgl } AEF.$$

And likewise also

$$\text{trgl } DEI = \text{trgl } BEG$$

But

$$\text{trgl } BEG = \text{trgl } AEF \text{ (iii 1);}$$

therefore also

$$\text{trgl } COE = \text{trgl } DEI$$

And also

$$\text{quadr } DG = \text{quadr } CF \text{ (above)}$$

Therefore, as wholes,

$$\text{quadr } XI = \text{quadr } OT$$

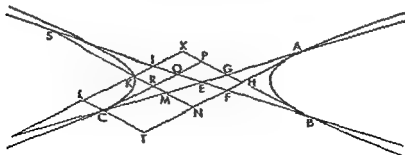
### PROPOSITION 9

With the same things supposed, if one of the points is between the diameters as  $K$ , and the other is the same with one of the points  $C$  and  $D$  for instance  $C$  and the parallels are drawn I say that

$$\text{trgl } CEO = \text{quadr } KE,$$

and

$$\text{quadr } LO = \text{quadr } LM$$



And thus is evident For since it was shown

$$\text{trgl } CEO = \text{trgl } AEF,$$

and

$$\text{trgl } AEF = \text{quadr. } KE \text{ (III 5, note),}$$

therefore also

$$\text{trgl } CEO = \text{quadr. } KE.$$

And so also

$$\text{trgl } CRM = \text{quadr. } KO,$$

and

$$\text{quadr. } KC = \text{quadr. } LO.$$

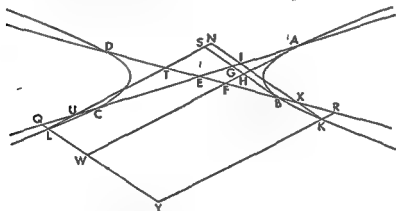
### PROPOSITION 10

With the same things supposed, let  $K$  and  $L$  be taken not as points at which the diameters hit the sections

Then it is to be shown that

$$\text{quadr. } LTRY = \text{quadr. } QYKI.$$

For since the straight lines  $AF$  and  $BG$  touch, and  $AE$  and  $BE$  are diameters



through the points of contact, and  $LT$  and  $KI$  are parallel to the tangents,

$$\text{trgl } TUE = \text{trgl } UQL + \text{trgl } EFA \text{ (I 44).}$$

And likewise also

$$\text{trgl } XEI = \text{trgl } XRK + \text{trgl } BEG.$$

But

$$\text{trgl } EFA = \text{trgl } BEG \text{ (III 1);}$$

therefore

$$\text{trgl } TUE - \text{trgl } UQL = \text{trgl } XEI - \text{trgl } XRK.$$

Therefore

$$\text{trgl } TUE + \text{trgl } XRK = \text{trgl } XEI + \text{trgl } UQL.$$

Let the common area  $KXBULY$  be added, therefore

$$\text{quadr. } LTRY = \text{quadr. } QYKI.$$

### PROPOSITION 11



for each is double the other, therefore *ex aequali*

$$CA \cdot AG = DB \cdot BF$$

And the triangles are similar because of the parallels, therefore

$$\text{trgl } CTA \sim \text{trgl } AHG \sim \text{trgl } XBD \sim \text{trgl } HBF \text{ (Eucl vi 10)}$$

And alternately, but

$$\text{trgl } AHG = \text{trgl } HBF \text{ (iii 1),}$$

therefore

$$\text{trgl } CTA = \text{trgl } XBD$$

As parts of these it was shown

$$\text{trgl } AHG = \text{trgl } HBF,$$

therefore also as remainders

$$\text{quadr } DH = \text{quadr } CH$$

And so also

$$\text{quadr } DG = \text{quadr } CF$$

And since  $CO \parallel AF$ ,

$$\text{trgl } COE = \text{trgl } AEF$$

And likewise also

$$\text{trgl } DEI = \text{trgl } BEG$$

But

$$\text{trgl } BEG = \text{trgl } AEF \text{ (iii 1),}$$

therefore also

$$\text{trgl } COE = \text{trgl } DEI$$

And also

$$\text{quadr } DG = \text{quadr } CF \text{ (above)}$$

Therefore, in wholes,

$$\text{quadr } LI = \text{quadr } OT$$

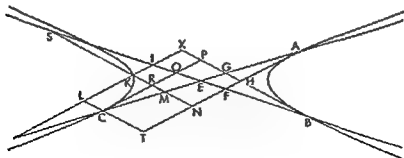
### PROPOSITION 9

With the same things supposed if one of the points is between the diameters, as  $K$ , and the other is the same with one of the points  $C$  and  $D$  for instance  $C$ , and the parallels are drawn I say that

$$\text{trgl } CEO = \text{quadr } KE,$$

and

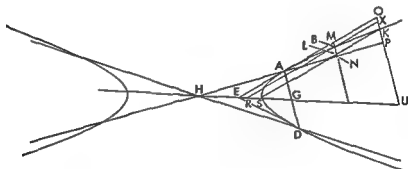
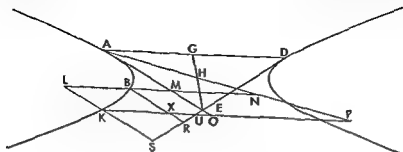
$$\text{quadr } LO = \text{quadr } LM$$



And this is evident For since it was shown

$$\text{trgl } CEO = \text{trgl } AEF,$$

For let there be the same things as before, and let the points  $B$  and  $K$  be taken at random on section  $AB$ , and through them let  $LBMN$  and  $KXOUP$



be drawn parallel to  $AD$ , and  $BXR$  and  $LKS$  parallel to  $AE$   
I say that

$$\text{quadr } BP = \text{quadr } KR$$

For since it has been shown

$$\text{trgl } AOP = \text{quadr } KOES \text{ (III 11, end)}$$

and  
 $\text{trgl } AMN = \text{quadr } BMER \text{ (III 11, end),}$

therefore, as remainders, either  
 $\text{quadr } KR - \text{quadr } BO = \text{quadr } MP$

or  
 $\text{quadr } KR + \text{quadr } BO = \text{quadr } MP$

And, with the common quadrilateral  $BO$  added or subtracted,  
 $\text{quadr } BP = \text{quadr } XS$

### PROPOSITION 13

*If in conjugate opposite sections straight lines tangent to the adjacent sections meet, and diameters are drawn through the points of contact, then the triangles*



$B$  and through it let  $\lambda RS$  be drawn parallel to  $AG$  and  $\lambda TO$  parallel to  $BE$

I say that triangle  $OHT$  differs from triangle  $XST$  by triangle  $HBF$

For let  $AU$  be drawn from  $A$  parallel to  $BF$ . Since then because of the same things as before,  $LHM$  is a diameter of the section  $AL$  and  $DHB$  is a second diameter and conjugate to it (II 20), and  $AG$  is a tangent at  $A$ , and  $AU$  has been dropped parallel to  $LM$ , therefore

*AU* *UG* comp *HU* *UA*,  
transverse side of figure on *LM* upright (1 40)

But

*AU UG XT TS.*

and

HU UA HT TO HB BF.

and

transverse side of figure on *LM* upright  
upright side of figure on *BD* transverse (r 60)

Therefore

*XT TS comp HB BF* upright side of figure on *BD* transverse

or

XT TS comp HT TO upright side of figure on BD transverse

### PROPOSITION 15

*If straight lines touching one of the conjugate opposite sections meet and diameters are drawn through the points of contact and some point is taken on any one of the conjugate sections and from it parallels to the tangents are drawn as far as the diameters then the triangle produced by them at the section is greater than the triangle produced at the center by the triangle having the tangent as base and the center of the opposite sections as vertex <sup>1</sup>*

Let there be conjugate opposite sections  $AB$ ,  $GS$ ,  $T$ , and  $X$ , whose center is  $H$  and let  $ADE$  and  $BDC$  touch the section  $AB$  and let the diameters  $AHFW$  and  $BHT$  be drawn through the points of contact  $A$  and  $B$  and let some point  $S$  be taken on the section  $GS$  and through it let  $SFL$  be drawn parallel to  $BC$  and  $SU$  parallel to  $AE$ .

I say that

$$\text{trgl } SLU \approx \text{trgl } HLF + \text{trgl } HCB$$

For let  $YHG$  be drawn through  $H$  parallel to  $BC$  and  $XIG$  through  $G$  parallel to  $AE$  and  $SO$  parallel to  $BT'$  then it is evident that  $YG$  is a diameter conju

another tangent. let it be contrived that

 $DB \cong BE \cong MN \cong 2BC.$ 

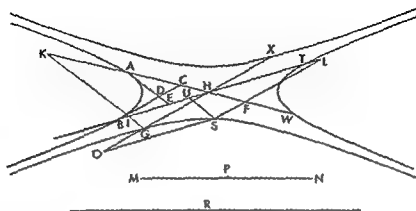
therefore  $MN$  is the so-called upright side of the figure on  $BT$  (r 50). Let  $MN$  be bisected at  $P$ , therefore

<sup>1</sup>This proposition comes as a climax to a long series and shows that the conjugate opposite sections taken as a unit have the same property as the other conic sections. The opposite sections seem to be a sort of fifth section.

$$DB \ BE \ MP : BC$$

Then let it be contrived that

$$XG \ TB \ TB \ R, \quad (\alpha)$$



then  $R$  also will be the so-called upright side of the figure applied to  $XG$  (I 16, 60)

Since then

$$DB \ BE \ MP \ BC, \quad (\beta)$$

but

$$DB \ BE \ sq \ DB \ rect \ DB, BE,$$

and

$$MP \ BC \ rect \ MP, BH \ rect \ CB, BH,$$

therefore

$$sq \ DB \ rect \ DB, BE \ rect \ MP, BH \ rect \ CB, BH$$

And

$$rect \ MP, BH = sq \ HG,$$

because

$$sq \ XG = rect \ TB, MN \text{ (I 16)} \quad (\gamma)$$

and

$$rect \ MP \ BH = fourth \ rect \ TB \ MN$$

and

$$sq \ HG = fourth \ sq \ XG,$$

therefore

$$sq \ DB \ rect \ DB, BE \ sq \ HG \ rect \ CB, BH$$

Alternately

$$sq \ DB \ sq \ HG \ rect \ DB, BE \ rect \ CB, BH$$

But

$$sq \ DB \ sq \ HG \ trgl \ DBE \ trgl \ GHI,$$

for they are similar, and

$$rect \ DB, BE \ rect \ CB, BH \ trgl \ DBE \ trgl \ CBH,$$

therefore

$$trgl \ DBE \ trgl \ GHI \ trgl \ DBE \ trgl \ CBH$$

Therefore

$$trgl \ GHI = trgl \ CBH$$

Again since

$$HB \cdot BC \text{ comp } HB \cdot MP, MP \cdot BC,$$

but

$$HB \cdot MP \cdot TB \cdot MN \cdot R \cdot XG \text{ (above } \alpha \text{ and } \gamma),$$

and

$$MP \cdot BC \cdot DB \cdot BE \text{ (above, } \beta)$$

therefore

$$HB \cdot BC \text{ comp } DB \cdot BE, R \cdot XG$$

And since  $BC$  is parallel to  $SL$ , and triangle  $HCB$  is similar to triangle  $HLF$ , and

$$HB \cdot BC \cdot HL \cdot LF,$$

therefore

$$HL \cdot LF \text{ comp } R \cdot XG, DB \cdot BE$$

or

$$HL \cdot LF \text{ comp } R \cdot \lambda G \cdot HG \cdot HI$$

ratios as already given, therefore

$$\text{trgl } SLU = \text{trgl } HLF + \text{trgl } HCB \text{ (I 41)}$$

### PROPOSITION 16

If two straight lines touching a section of a cone or circumference of a circle meet, and from some point of those on the section a straight line is drawn parallel to one tangent and cutting the section and the other tangent, then as the squares on the tangents are to each other so the area contained by the straight lines between the section and the tangent will be to the square cut off at the point of contact

Let there be the section of a cone or circumference of a circle  $AB$ , and let the straight lines  $AC$  and  $CB$  meeting at  $C$  touch it and let some point  $D$  be taken on the section  $AB$ , and through it let  $EDF$  be drawn parallel to  $CB$

I say that

$$\text{sq } BC \cdot \text{sq } AC \cdot \text{rect } FE, ED \cdot \text{sq } EA$$

For let the diameters  $AGH$  and  $KBL$  be drawn through  $A$  and  $B$ , and  $DMN$  through  $D$  parallel to  $AL$ , it is at once evident, that

$$DK = KF \text{ (I 46, 47)}$$

and

$$\text{trgl } AEG = \text{quadr } LD \text{ (II 2),}$$

and

$$\text{trgl } BLC = \text{trgl } ACH \text{ (III 1)}$$

Since then

$$DK = KF$$

and  $DE$  is added,

$$\text{rect } FE \cdot ED + \text{sq } DK = \text{sq } KE$$

And since triangle  $ELK$  is similar to triangle  $DNK$ ,

$$\text{sq } EK \cdot \text{sq } KD \cdot \text{trgl } EKL \cdot \text{trgl } DNK$$



## PROPOSITION 17

If two straight lines touching a section of a cone or circumference of a circle meet, and two points are taken at random on the section, and from them in the section are drawn parallel to the tangents straight lines cutting each other and the line of the section, then as the squares on the tangents are to each other, so will the rectangles contained by the straight lines taken similarly

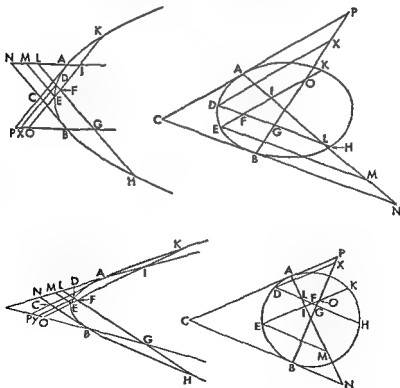
Let there be the section of a cone or circumference of a circle  $AB$ , and tangents to  $AB$ ,  $AC$  and  $CB$ , meeting at  $C$ , and let points  $D$  and  $E$  be taken at random on the section, and through them at  $EFIA$  and  $DFGH$  be drawn parallel to  $AC$  and  $CB$

I say that

$$\text{sq } CA : \text{sq } CB :: \text{rect } KF, FE : \text{rect } HF, FD$$

the  
be

Since then  $KE$  has been cut equally at  $I$  and unequally at  $F$ ,  
 $\text{rect } KF, FE + \text{sq } FI = \text{sq } EI$  (Eucl II 5)



And since the triangles are similar because of the parallels,  
 $\text{whole sq } EI : \text{whole trgl } IME$



part subtracted sq  $IF$  part subtracted trgl  $FIL$

Therefore also

remainder rect  $KF, FE$  : remainder quadr.  $FM$  :: whole  
sq  $EI$  : whole trgl  $IME$

But

sq.  $EI$  . trgl.  $IME$  :: sq  $CA$  : trgl.  $CAN$  :

Therefore

rect  $KF, FE$  quadr.  $FM$  : sq  $CA$  . trgl  $CAN$ .

But

trgl  $CAN = \text{trgl } CPB$  (III. 1),

and

quadr  $FM = \text{quadr } FX$  (III. 3);

therefore

rect  $KF, FE$  quadr  $FX$  sq  $CA$  trgl.  $CPB$ .

Then likewise it could be shown that

rect  $HF, FD$  : quadr.  $FX$  .. sq  $CB$  . trgl  $CPB$ .

Since then

rect  $KF, FE$  . quadr  $FX$  .. sq  $CA$  . trgl  $CPB$ ,

and inversely

quadr  $FX$  rect  $HF, FD$  trgl  $CPB$  sq  $CB$ ,

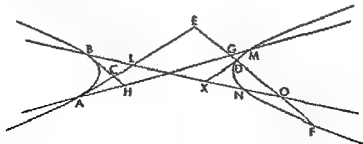
therefore *ex aequali*

sq  $CA$  sq  $CB$  . rect  $KF, FE$  rect  $HF, FD$

#### PROPOSITION 18

If two straight lines touching opposite sections meet, and some point is taken on either one of the sections, and from it some straight line is drawn parallel to one of the tangents cutting the section and the other tangent, then as the squares on the tangents are to each other, so will the rectangle contained by the straight lines between the section and the tangent be to the square on the straight line cut off at the point of contact

Let there be the opposite sections  $AB$  and  $MN$ , and tangents  $ACL$  and  $BCH$ , and through the points of contact the diameters  $AM$  and  $BN$ , and let some point  $D$  be taken at random on the section  $MN$ , and through it let  $EDF$  be drawn parallel to  $BH$



I say that

sq  $BC$  . sq  $CA$  .. rect  $FE, ED$  sq  $AE$ .

For let  $DX$  be drawn through  $D$  parallel to  $AE$  Since then  $AB$  is an hyper-

bola and  $BN$  its diameter and  $BH$  a tangent and  $DF$  parallel to  $BH$ , therefore  
 $FO=OD$  (I 48)

And  $ED$  is added therefore

$$\text{rect } FE, ED + \text{sq } DO = \text{sq } EO \text{ (Eucl II 6)}$$

And since  $EL$  is parallel to  $DX$ , triangle  $EOL$  is similar to triangle  $D\lambda O$   
 Therefore

$$\frac{\text{whole sq } EO}{\text{part subtracted sq } DO} = \frac{\text{whole trgl } EOL}{\text{part subtracted trgl } D\lambda O},$$

therefore also

$$\frac{\text{remainder rect } DE, EF}{\text{remainder quadr } DL} = \frac{\text{sq } EO}{\text{trgl } EOL}$$

But

$$\text{sq } OE \text{ trgl } EOL = \text{sq } BC \text{ trgl } BCL,$$

therefore also

$$\text{rect } FE, ED \text{ quadr } DL = \text{sq } BC \text{ trgl } BCL$$

And

$$\text{quadr } DL = \text{trgl } AEG \text{ (III 6 note),}$$

and

$$\text{trgl } BCL = \text{trgl } ACH \text{ (III 1),}$$

therefore

$$\text{rect } FE, ED \text{ trgl } AEG = \text{sq } BC \text{ trgl } ACH$$

But also

$$\text{trgl } AEG \text{ sq } EA = \text{trgl } ACH \text{ sq } AC,$$

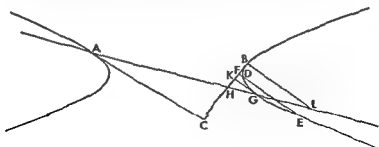
therefore ex aequali

$$\text{sq } BC \text{ sq } AC = \text{rect } FE, ED \text{ sq } EA^1$$

$$\text{sq } AC \text{ sq } CB = \text{rect } EF, FD \text{ sq } FB$$

For let  $AHG$  be drawn as a diameter through  $A$  and through  $B$  and  $G$   $GK$  and  $BL$  parallel to  $EF$  ■ now then  $BH$  touches the hyperbola at  $B$  and  $BL$  has been drawn ordinate-wise

$$AL \text{ } LG \text{ } AH \text{ } HG \text{ (I 36)}$$



But  
 and  
 therefore also  
 And alternately  
 and  
 But it was shown  
 therefore also

$$\begin{array}{l} AL \text{ } LG \text{ } CB \text{ } BK \\ AH \text{ } HG \text{ } AC \text{ } KG \\ CB \text{ } BK \text{ } AC \text{ } KG \\ AC \text{ } CB \text{ } KG \text{ } KB \\ \text{sq } AC \text{ sq } CB = \text{sq } GK \text{ sq } KB \\ \text{sq } GK \text{ sq } KB = \text{rect } EF, FD \text{ sq } FB \\ \text{sq } AC \text{ sq } CB = \text{rect } EF, FD \text{ sq } FB \end{array}$$

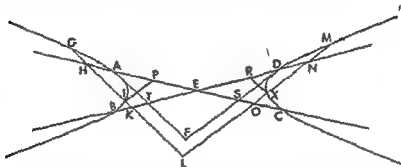
## PROPOSITION 19

If two straight lines touching opposite sections meet, and parallels to the tangents are drawn cutting each other and the section, then, as the squares on the tangents are to each other, so will the rectangle contained by the straight lines between the section and the point of meeting of the straight lines be to the rectangle contained by the straight lines taken similarly

Let there be opposite sections whose diameters are  $AC$  and  $BD$  and center in  $E$ , and let the tangents  $AF$  and  $FD$  meet at  $F$ , and let  $GHIKL$  and  $MNXOL$  be drawn from any points parallel to  $AF$  and  $FD$

I say that

$$\text{sq } AF : \text{sq } FD :: \text{rect } GL, LI : \text{rect } ML, LX.$$



Let  $IP$  and  $XR$  be drawn through  $X$  and  $I$  parallel to  $AF$  and  $FD$   
 $BH$  touches the hyperbola at  $B$ , and  $BL$  has been drawn ordinatewise,  
 $AL LG AH HG$  (I, 36)

And since

$$\text{sq } AF \text{ trgl } AFS \quad \text{sq } HL \text{ trgl } HLO \quad \text{sq } HI \text{ trgl } HIP,$$

therefore

$$\text{remainder rect } GL, LI \quad \text{remainder quadr } IPOL :: \text{sq } AF \text{ trgl } AFS$$

But

$$\text{trgl } AFS = \text{trgl } DTF \text{ (III 4),}$$

and

$$\text{quadr } IPOL = \text{quadr } KRXL \text{ (III 7),}$$

therefore also

$$\text{sq } AF \text{ trgl } DTF \quad \text{rect } GL LI \quad \text{quadr } KRXL$$

But

$$\text{trgl } DTF \text{ sq } FD :: \text{quadr } KRXL \text{ rect } ML, LX \text{ (likewise),}$$

and therefore ex aequali

$$\text{sq } AF \text{ sq } FD :: \text{rect } GL LI \text{ rect } ML, LX$$

## PROPOSITION 20

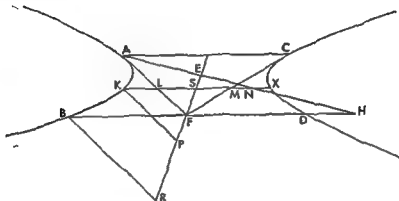
If two straight lines touching opposite sections meet, and through the point of meeting some straight line is drawn parallel to the straight line joining the points of contact and meeting each of the sections and some other straight line is drawn parallel to the same straight line and cutting the sections and the tangents, then, as the rectangle contained by the straight lines drawn from the point of meeting to cut the sections is to the square on the tangent, so is the rectangle contained by the

straight lines between the sections and the tangent to the square on the straight line cut off at the point of contact

at random, and through it let  $KLSMNX$  be drawn parallel to  $AC$ .

I say that

$$\text{rect } BF, FD \text{ sq } FA : \text{rect } KL, LX \text{ sq } AL$$



For let  $KP$  and  $BR$  be drawn from  $K$  and  $B$  parallel to  $AF$ . Since then  
 $\text{sq } BF \text{ trgl } BFR \text{ sq } KS \text{ trgl } KSP \text{ sq } LS \text{ trgl } LSF$ ,

and

$$\begin{aligned} &\text{sq } KS \text{ trgl } KSP \\ &\text{remainder rect } KL, LX \text{ (Eucl II 5)} \\ &\text{remainder quadr } KLFP \text{ (Eucl V 19)} \end{aligned}$$

and

$$\text{sq } BF = \text{rect } BF, FD \text{ (II 39, 38),}$$

and

$$\text{trgl } BRF = \text{trgl } AFH \text{ (III 11 and special case),}$$

and

$$\text{quadr } KLFP = \text{trgl } ALN \text{ (III 5),}$$

therefore

$$\text{rect } BF, FD \text{ trgl } AFH \text{ rect } KL, LX \text{ trgl } ALN$$

And

$$\text{trgl } AFH \text{ sq } AF \text{ trgl } ALN \text{ sq } AL,$$

then

$$\text{rect } BF, FD \text{ sq } FA \text{ rect } KL, LX \text{ sq } AL$$

#### PROPOSITION 21

With the same things supposed, if two points are taken on the section, and through them straight lines are drawn, the one parallel to the tangent, the other parallel to the straight line joining the points of contact, and cutting each other and the sections, then, as the rectangle contained by the straight lines drawn from the point

of meet  
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and through the  
and  $\Lambda O W I I Z$

I say that

rect  $BF, FD$  sq  $FA$  rect  $KO, OQ$  rect  $NO OG$



For since

sq  $AF$  trgl  $AFH$  sq  $AL$  trgl  $ALM$  sq  $XO$  trgl  $XOZ$

and

sq  $\Lambda O$  trgl  $\Lambda OZ$  sq  $XG$  trgl  $XGM$ ,

therefore

whole sq  $XO$  whole trgl  $\Lambda OZ$

part subtracted sq  $XG$  part subtracted trgl  $\Lambda GM$ ,

therefore also

remainder rect  $NO OG$  remainder quadr  $GOZM$  sq  $AF$  trgl  $AFH$

But

trgl  $AFH = \text{trgl } BUF$  (in 11 and special case),

and

quadr  $GOZM = \text{quadr } KORT$  (in 12),

therefore

sq  $AF$  trgl  $BFU$  rect  $NO OG$  quadr  $KORT$

But it was shown (in the course of in 20) trgl  $BUF$  sq  $BF$  or rect  $BF FD$

(in 39 38) quadr  $KORT$  rect  $\Lambda O OQ$

therefore ex aequali

sq  $AF$  rect  $BF FD$  rect  $\Lambda O OG$  rect  $\Lambda O OQ$

And inversely

rect  $BF, FD$  sq  $FA$  rect  $\Lambda O OQ$  rect  $NO OG$

# PROPOSITION 22

If two parallel straight lines in

drawn

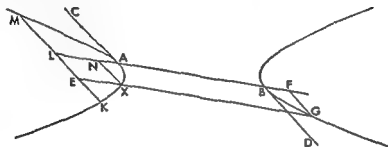
of the figure on the straight line joining the points of contact as to the upright so the rectangle contained by the straight lines between the sections and the point of meeting will be to the rectangle contained by the straight lines between the section and the point of meeting

Let there be the opposite sections  $A$  and  $B$  and let  $AC$  and  $BD$  be parallel and tangent to them and let  $AB$  be joined. Then let  $EXG$  be drawn across parallel to  $AB$  and  $KELM$  parallel to  $AC$

I say that

$AB$  upright side of the figure rect  $GE EY$  rect  $KE EM$

Let  $XN$  and  $GF$  be drawn through  $G$  and  $\lambda$  parallel to  $AC$



F  
(n)

Therefore

whole rect  $BL LA$  whole sq  $LK$   
part subtracted rect  $BN NA$  part subtracted sq  $LE$

or

rect  $BL LA$  sq  $LK$  rect  $FA AN$  sq  $LE$

for

$NA - BF$  (1 21)

therefore also

remainder rect  $FL LN$  remainder rect  $KE EM$   $AB$  upright

But

rect  $FL LN$  - rect  $GE EX$

therefore

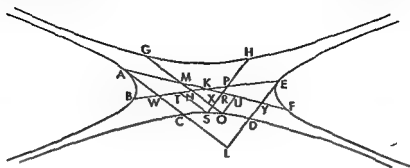
$AB$  the transverse side of figure upright

rect  $GE EY$  rect  $KE EM$

### PROPOSITION 23

If in conjugate opposite sections two straight lines touching contrary sections meet in any one section at random and any straight lines are drawn parallel to the tangents and cutting each other and the other opposite sections then as the squares on the tangents are to each other so the rectangle contained by the straight lines between the sections and the point of meeting will be to the rectangle contained by the straight lines similarly taken

Let there be the conjugate opposite sections  $AB$ ,  $CD$ ,  $EF$ , and  $GH$ , and their center  $K$ , and let  $AWCL$  and  $EYDL$ , tangents to the sections  $AB$  and  $EF$



meet at  $L$ , and let  $AK$  and  $EK$  be joined and produced to  $B$  and  $F$ , and let  $GMNXO$  be drawn from  $G$  parallel to  $AL$ , and  $HPRXS$  from  $H$  parallel to  $EL$   
I say that

$$\text{sq } EL \cdot \text{sq } LA :: \text{rect } HX, XS \text{ rect } GX, XD$$

For let  $ST$  be drawn through  $S$  parallel to  $AL$ , and  $OU$  from  $O$  parallel to  $EL$ . Since then  $BE$  is a diameter of the conjugate opposite sections  $AB$ ,  $CD$ ,  $EF$  and  $GH$ , and  $EL$  touches the section and  $HS$  has been drawn parallel to it,  
 $HP = PS$  (II 20, First Def 15),

and for the same reasons

$$GM = MO$$

And since

$$\text{sq } EL \text{ trgl } EWL : \text{sq } PS \text{ trgl } PTS :: \text{sq } PX \text{ trgl } PNX,$$

also

$$\text{remainder rect } HX, XS \text{ remainder quadr } TN, XS :: \text{sq } EL \text{ trgl } WLE.$$

But

$$\text{trgl } EWL = \text{trgl } ALY \text{ (III 4),}$$

and

$$\text{quadr } TNXS = \text{quadr } XRZO \text{ (III 15),}^1$$

therefore

$$\text{sq } EL \text{ trgl } ALY \text{ rect } HX, XS \text{ quadr } XRZO$$

But

$$\text{trgl } AYL \text{ sq } AL \text{ quadr } XRZO \text{ rect } GX, XO \text{ (same way),}$$

therefore ex aequali

$$\text{sq } EL \text{ sq } AL \text{ rect } HX, XS \text{ rect } GX, XO$$

#### PROPOSITION 24

If in conjugate opposite sections two straight lines are drawn from the center through to the sections, and one of them is taken as the transverse diameter and the other as the upright diameter and any straight lines are drawn parallel to the two diameters and meeting each other and the sections, and the point of meeting of the straight lines is the place between the four sections, then the rectangle contained by

the opposite sections

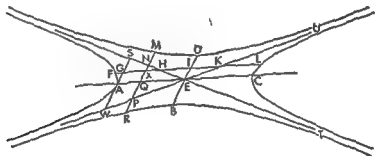
the segments of the parallel to the transverse diameter together with the rectangle to which the rectangle contained by the segments of the parallel to the upright diameter has the ratio which the square on the upright diameter has to the square on the transverse, will be equal to twice the square on the half of the transverse

or the angle  $UET$

For let the asymptotes of the sections  $SET$  and  $UEW$  be drawn, and through  $A$ ,  $SGAW$  tangent to the section

Since then,

rect  $SA, AV \approx sq DE$  (I 60, II 1).



therefore

rect SA,AW sq EA sq DE sq EA

And

$$\text{rect } SA, AW \text{ sq } AE \text{ comp } SA \quad AE, WA \quad AE$$

But

SA AE NX XH

and

WA AE PX XK.

therefore

$$\text{sq } DE \quad \text{sq } AE \text{ comp } NX \quad XH, PX \quad XK$$

**Rest**

$$\text{rect } PX, XN \quad \text{rect } KX, XH \quad \text{comp } NX \quad XH, PX \quad XK.$$

therefore

$$\text{sq } DE \quad \text{sq } AE \quad \text{rect } PX, XN \quad \text{rect } KX, XH$$

Therefore also

$$\text{sq } DE \text{ sq } AL' \text{ sq } DE \vdash \text{rect } P\lambda, XN \text{ sq } AE \vdash \text{rect } KX, XH$$

And

sq  $DE = \text{rect } PM, MN$  (π 11) =  $\text{rect } RN, NM$  (π 16).

and

$$\text{sq } AE = \text{rect } KF, FH \text{ (II 11)} = \text{rect } LH, HF \text{ (II 16)}.$$



therefore

$$\text{sq } DE \text{ sq } AE \text{ rect } PX, XN + \text{rect } RN, NM \\ \text{rect } KX, XH + \text{rect } LH, HF$$

And

$$\text{rect } PX, XN + \text{rect } RN, NM = \text{rect } RX, XM,<sup>1</sup>$$

therefore

$$\text{sq } DE \text{ sq } AE \text{ rect } RX, XM \text{ rect } KX, XH + \text{rect } KF, FH$$

Then it must be shown that

$$\text{rect } FX, XL + \text{rect } KX, XH + \text{rect } KF, FH = 2 \text{ sq } AE$$

Let the common square  $AE$  that is rectangle  $KF, FH$ , be subtracted, therefore it remains to be shown that

$$\text{rect } FX, XL + \text{rect } KX, XH = \text{sq } AE$$

And this is so, for

$$\text{rect } FX, XL + \text{rect } KX, XH = \text{rect } LH, HF, \dagger$$

$$\text{rect } FX, XL + \text{rect } KX, XH = \text{rect } KF, FH \text{ (II 16),} \\ = \text{sq } AE \text{ (II 11)}$$

Then let the straight lines  $FL$  and  $MR$  meet on one of the asymptotes at  $H$ . Then

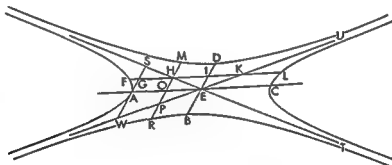
$$\text{rect } FH, HL = \text{sq } AE,$$

and

$$\text{rect } MH, HR = \text{sq } DE \text{ (II 11, 16),}$$

therefore

$$\text{sq } DE \text{ sq } AE \text{ rect } MH, ER \text{ rect } FH, HL$$



And so we want twice rectangle  $FH HL$  to equal twice the square on  $AE$ . And it does

<sup>1</sup>For  
and  
therefore

$$RP = NM \text{ (II 8),}$$

$$RO = OM \text{ (II 3),}$$

$$PO = ON$$

But

$$\text{rect } PX, \lambda\lambda + \text{sq } O\lambda = \text{sq } ON \text{ (Eucl II 5),}$$

and for the same reasons

$$\text{rect } RN, \lambda M + \text{sq } ON = \text{sq } OM,$$

and

$$\text{rect } R\lambda, \lambda M + \text{sq } O\lambda = \text{sq } OM$$

Hence

$$\text{rect } R\lambda, \lambda M + \text{sq } O\lambda = \text{rect } R\lambda, \lambda M + \text{sq } OX,$$

and adding equals to equals,

$$\text{rect } R\lambda, \lambda M + \text{sq } O\lambda + \text{rect } P\lambda, \lambda V + \text{sq } O\lambda = \text{rect } R\lambda, \lambda M + \text{sq } O\lambda + \text{rect } P\lambda, \lambda V + \text{sq } O\lambda$$

Subtracting the common squares

$$\text{rect } R\lambda, \lambda M + \text{rect } P\lambda, \lambda V = \text{rect } R\lambda, \lambda M$$

<sup>†</sup>By the same manner of proof as in the note above, but using also Euclid II 6 because of the different position of the point  $\lambda$

And let the point  $X$  be within the angle  $SEK$  or angle  $WET$ . Then likewise by the composition of ratios

$$\text{sq } DE \text{ sq } AE \text{ rect } PX, XN \text{ rect } KX, XH$$

And

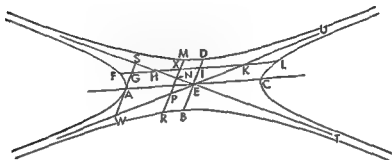
$$\text{sq } DE = \text{rect } PM, RN = \text{rect } RN, NM,$$

and

$$\text{sq } AL = \text{rect } FH, HL,$$

therefore

$$\text{rect } RN, NM \text{ rect } FH, HL$$



part subtracted rect  $PX, XN$  · part subtracted rect  $KX, XH$

Therefore also

$$\text{rect } RN, NM \text{ rect } FH, HL$$

remainder rect  $RX, XM$  remainder (sq  $AE$  - rect  $KX, XH$ )

Therefore it must be shown that

$$\text{rect } FX, XL + (\text{sq } AE - \text{rect } KX, XH) = 2 \text{ sq } AE$$

Let the common square on  $AE$ , that is rectangle  $FH, HL$ , be subtracted, therefore it remains to be shown that

$$\text{rect } AX, XH + (\text{sq } AE - \text{rect } KX, XH) = \text{sq } AE$$

And this is so, for

$$\text{rect } KX, XH + \text{sq } AE - \text{rect } KX, XH = \text{sq } AE$$

### PROPOSITION 25

With the same things supposed let the point of meeting of the parallels to  $AC$  and  $BD$  be within one of the sections  $D$  and  $B$ , as set out below, at  $X$

I say that the rectangle contained by the segments of the parallel to the transverse, that is rectangle  $OX, XN$ , will be greater than the rectangle to

the half of the transverse

For, for the same reasons

$$\text{sq } DE \text{ sq } AE \text{ rect } PX, XH \text{ rect } SX, XL$$

and

$$\text{sq } DE = \text{rect } PM, MH,$$

and

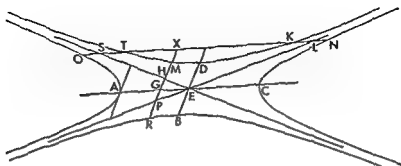
$$\text{sq } AE = \text{rect } LO, OS \text{ (II 11);}$$

therefore also

$$\text{sq } DE \text{ sq } AE \text{ rect } PM, MH \text{ rect } LO, OS$$

And since

$$\text{whole rect } PX, XH : \text{whole rect } LX, XS ::$$



part subtracted rect  $PM, MH$  part subtracted rect  $LO, OS$ ,  
or rect  $ST, TL$  (II 22),

therefore also

$$\text{remainder rect } RX, SM \text{ remainder rect } TX, XK \\ (\text{first note to III 24, II 8}) : \text{sq } DE : \text{sq } AE$$

Therefore it must be shown that

$$\text{rect } OX, XN = \text{rect } TX, XK + 2 \text{ sq } AE$$

Let the common rectangle  $TX, XK$  be subtracted, therefore it must be shown that

$$\text{rect } OT, TN \text{ (first note to III 24)} = 2 \text{ sq } AE$$

And it is (II 23)

### PROPOSITION 26

And if the point of meeting of the parallels at  $X$  is within one of the sections  $A$  and  $C$ , as set out below, then the rectangle contained by the segments of the

other has to the square on the transverse by twice the square on half of the transverse

For, since for the same reasons as before

$$\text{sq } DE \text{ sq } AE \text{ rect } WX, XS \text{ rect } KX, XH,$$

therefore also

$$\text{whole rect } RX, XG^1 \text{ whole rect } KX, XH + \text{sq } AE :: \\ \text{sq upright diameter sq transverse}$$

Therefore it must be shown that

$$\text{rect } LX, XF + 2 \text{ sq } AE = \text{rect } KX, XH + \text{sq } AE$$

<sup>1</sup>For by II 11

$$\text{rect } HG, GS = \text{sq } DE,$$

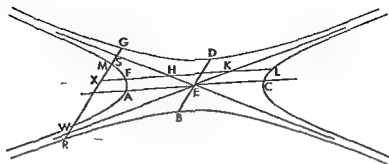
and

$$RW = GS \text{ (II 16)}$$

Therefore by the first note to III 24, and II 16

$$\text{rect } HX, AS + \text{sq } DE = \text{rect } HX, AS + \text{rect } WG, GS = \text{rect } RX, XG$$

Let the common square on  $AE$  be subtracted; therefore it remains to be shown that



$$\text{rect } LX, XF + \text{sq } AE = \text{rect } KX, XH$$

or

$$\text{rect } LX, XF + \text{rect } LH, HF = \text{rect } KX, XH \quad (\text{II } 16, 11)$$

And it is, for

$$\text{rect } LH, HF + \text{rect } LX, XF = \text{rect } KX, XH^1$$

#### PROPOSITION 27

If the conjugate diameters of an ellipse or circumference of a circle are drawn, and one of them is called the upright diameter and the other the transverse, and two straight lines, meeting each other and the line of the section, are drawn parallel to them, then the squares on the straight lines cut off on the straight line drawn parallel to the transverse between the point of meeting of the straight lines and the line of the section plus the figures described on the straight lines cut off on the straight line drawn parallel to the upright diameter between the point of meeting of the straight lines and the line of the section, figures similar and similarly situated to the figure on the upright diameter, will be equal to the square on the transverse diameter

For let there be the ellipse or circumference of a circle  $ABCD$ , whose center

$FM$ , similar and similarly situated to the figure on  $AC$  will be equal to the square on  $BD$

therefore also

$$BP \cdot BD = \text{sq } AC + \text{sq } BD$$

And

$$\text{sq } BD = \text{figure on } AC;$$

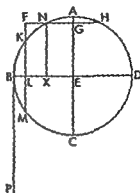
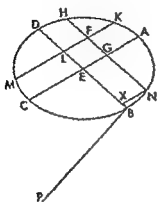
therefore

$$BP \cdot BD :: \text{sq } AC : \text{figure on } AC$$

<sup>1</sup>This is another case of the first note to III ■■

And

sq  $AC$  figure on  $AC$   
 sq  $NX$  figure on  $NX$  similar to the figure on  $AC$  (Eucl vi 22),



therefore also

$BP \cdot BD$  sq  $NX$  figure on  $NX$  similar to the figure on  $AC$

And also

$BP \cdot BD$  sq  $NX$  rect  $BX \cdot XD$  (i 21)

therefore

figure on  $NX$  or  $FL$  similar to the figure on  $AC = \text{rect } BX \cdot XD$

Then likewise we could show that

figure on  $AL$  similar to the figure on  $AC = \text{rect } BL \cdot LD$

And since the straight line  $AH$  has been cut equally at  $G$  and unequally at  $F$ ,

sq  $HF + \text{sq } FN = 2[\text{sq } HG + \text{sq } GF] = 2[\text{sq } NG + \text{sq } GF]$  (Eucl vi 9)

Then for the same reasons also

sq  $MF + \text{sq } FA = 2[\text{sq } KL + \text{sq } LF]$

and the figure on  $MF$  and  $FA$  similar to the figure on  $AC$  are double the similar figures on  $KL$  and  $LF$

And

figure on  $KL + \text{figure on } FL = \text{rect } BX \cdot XD + \text{rect } BL \cdot LD$  (above)

and

sq  $NG + \text{sq } GF = \text{sq } LE + \text{sq } EL$

therefore

sq  $VF + \text{sq } FH + \text{figures on } AF \text{ and } FM \text{ similar to the figure on } AC =$   
 $2[\text{rect } BX \cdot XD + \text{rect } BL \cdot LD + \text{sq } LE + \text{sq } EL]$

And since the straight line  $BD$  has been cut equally at  $E$  and unequally at  $F$ ,  
 rect  $BF \cdot FD + \text{sq } FE = \text{sq } BE$  (Eucl ii 5)

And likewise also

rect  $BL \cdot LD + \text{sq } LE = \text{sq } BE$

and so

rect  $BX \cdot XD + \text{rect } BL \cdot LD + \text{sq } LE + \text{sq } LE = 2 \text{ sq } BE$

Therefore the squares on  $NF$  and  $FH$  together with figures on  $AF$  and  $FM$  similar to the figure on  $CA$  are double the square on  $BE$ . But also

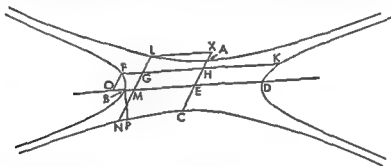
sq  $BD = 2 \text{ sq } BE$

therefore the squares on  $NF$  and  $FH$  plus the figures on  $AF$  and  $FM$  similar to the figure on  $AC$  are equal to the square on  $BD$

## PROPOSITION 28

If in conjugate opposite sections conjugate diameters are drawn, and one of them is called the upright, and the other the transverse, and two straight lines are drawn parallel to them and meeting each other and the sections, then the squares on the straight lines cut off on the straight line drawn parallel to the upright between the point of meeting of the straight lines and the sections have to the squares on the straight lines cut off on the straight line drawn parallel to the transverse between the point of meeting of the straight lines and the sections the ratio which the square on the upright diameter has to the square on the transverse diameter

Let there be the conjugate opposite sections  $A, B, C$ , and  $D$  and let  $AEC$  be the upright diameter and  $BED$  the transverse, and let  $FGHK$  and  $LGMN$



be drawn parallel to them and cutting each other and the sections  
I say that

$$\text{sq } LG + \text{sq } GN : \text{sq } FG + \text{sq } GK :: \text{sq } AC : \text{sq } BD$$

For let  $LX$  and  $FO$  be drawn ordinatewise from  $F$  and  $L$ , therefore they are parallel to  $AC$  and  $BD$ . And from  $B$  let the upright side for  $BD$ ,  $BP$ , be drawn, then it is evident that

$$\begin{aligned} & PB \cdot BD : \text{sq } AC :: \text{sq } BD : \text{sq } AC \quad (\text{I 15}) \\ & \text{sq } FO : \text{rect } BO, OD : \text{rect } CX, XA :: \text{sq } LX : \text{sq } LX \quad (\text{I 60, 21}) \end{aligned}$$

Therefore

$$+ \text{sq } LX$$

or

$$\begin{aligned} & \text{sq } AC : \text{sq } BD :: \text{rect } CX, XA + \text{sq } AE + \text{sq } EH \\ & \quad : \text{rect } DO, OB + \text{sq } BE + \text{sq } ME \end{aligned}$$

But

$$\text{rect } CX, XA + \text{sq } AE = \text{sq } XE,$$

and

$$\text{rect } DO, OB + \text{sq } BE = \text{sq } OE \quad (\text{Eucl II 6}),$$

therefore

$$\begin{aligned} & \text{sq } AC : \text{sq } BD :: \text{sq } XE + \text{sq } EH : \text{sq } OE + \text{sq } EM :: \\ & \quad \text{sq } LM + \text{sq } MG : \text{sq } FH + \text{sq } HG \end{aligned}$$

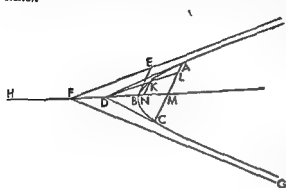
And, as has been shown,

$$\text{sq } NG + \text{sq } GL = 2[\text{sq } LM + \text{sq } MG],$$



## PROPOSITION 30

If two straight lines touching an hyperbola meet, and through the points of contact a straight line is produced, and through the point of meeting a straight line is drawn parallel to some one of the asymptotes and cutting both the section and the straight line joining the points of contact, then the straight line between the point of meeting and the straight line joining the points of contact will be bisected by the section.



Let there be the hyperbola  $ABC$ , and let  $AD$  and  $DC$  be tangents and  $EF$  and  $FG$  asymptotes, and let  $AC$  be joined, and through  $D$  parallel to  $FE$  let  $DKL$  be drawn

I say that

$$DK = KL$$

For let  $FDBM$  be joined and produced both ways, and let  $FH$  be made equal to  $BF$ , and through the points  $B$  and  $K$  let  $BE$  and  $KN$  be drawn

(a)

And

$$\text{sq } BF = \text{sq } BE - HB \text{ upright (r 1),}$$

therefore also

$$\text{sq } DN = \text{sq } NK - HB \text{ upright}$$

But

$$HB \text{ upright} = \text{rect } HN, NB = \text{sq } NK \text{ (r 21);}$$

therefore also

$$\text{sq } DN = \text{sq } NK - \text{rect } MN, NB = \text{sq } NK$$

Therefore

$$\text{rect } HN, NB = \text{sq } DN$$

these ways

In proposition 37 we have the line  $CF$  divided by the section at  $D$  and  $F$ , and at  $E$  by the straight line joining the points of contact in such a way that

$$CF : CD = FE : ED$$

This is the same form of the harmonic proportion as we found in r 34, and  $DF$  is the harmonic mean between  $CF$  and  $FE$

ED



And also

$$\text{rect } MF, FD = \text{sq } FB \text{ (I 37),}$$

because  $AD$  touches and  $AM$  has been dropped ordinatewise, and so also

$$\text{rect } HN, NB + \text{sq } FB = \text{rect } MF, FD + \text{sq } DN$$

But

$$\text{rect } HN, NB + \text{sq } FB = \text{sq } FN \text{ (Eucl II 6),}$$

and therefore

$$\text{rect } MF, FD + \text{sq } DN = \text{sq } FN$$

Therefore  $DM$  has been bisected at  $N$  with  $DF$  added (Eucl II 6) And  $KN$  and  $LM$  are parallel, therefore

$$DK = KL$$

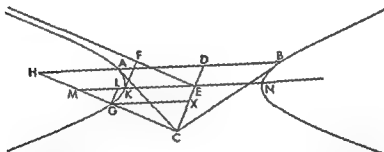
### PROPOSITION 31

If two straight lines touching opposite sections meet, and a straight line is produced through the points of contact, and through the point of meeting a straight line is drawn parallel to the asymptote and cutting both the section and the straight line joining the points of contact, then the straight line between the point of meeting and the straight line joining the points of contact will be bisected by the section

and  $CB$ , and through  $C$  let

I say that

$$CO = GH$$



Let  $CE$  be joined and produced to  $D$ , and through  $E$  and  $G$  let  $NEKM$  and  $GK$  be drawn parallel to  $AB$ , and through  $G$  and  $K$  let  $KF$  and  $GL$  be drawn parallel to  $CD$

Since triangle  $KFE$  is similar to triangle  $MLG$ ,

$$\text{sq } KE : \text{sq } KF :: \text{sq } ML : \text{sq } LG$$

And it has been shown

$$\text{sq } KE : \text{sq } KF :: \text{rect } NL, LK : \text{sq } LG \text{ (}\alpha \text{ and } \beta \text{ of III 30),}$$

therefore

$$\text{rect } NL, LK = \text{sq } ML$$

Let the square on  $KE$  be added to each, therefore

$$\text{rect } NL, LA + \text{sq } KE = \text{sq } LE = \text{sq } GX = \text{sq } ML + \text{sq } KE$$

And

$$\text{sq } GX : \text{sq } ML + \text{sq } KE :: \text{sq } XC : \text{sq } LG + \text{sq } KF \text{ (Eucl VI 4, V 12),}$$

therefore

$$\text{sq } AC = \text{sq } LG + \text{sq } KF$$

And

$$\text{sq } LG = \text{sq } XE$$

and

$$\begin{aligned} \text{sq } KF &= \text{sq on half of second diameter (II 1),} \\ &= \text{rect } CE, ED \text{ (I 38),} \end{aligned}$$

therefore

$$\text{sq } XC = \text{sq } XE + \text{rect } CE, ED$$

Therefore the straight line  $CD$  has been cut equally at  $X$  and unequally at  $E$  (Eucl II 5)

And  $DH$  is parallel to  $GX$ , therefore

$$CG = GH$$

### PROPOSITION 32

If two straight lines touching an hyperbola meet, and a straight line is produced through the points of contact, and a straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact, and a straight line is drawn through the midpoint of the straight line joining the points of contact parallel to one of the asymptotes, then the straight line cut off between this midpoint and the parallel will be bisected by the section

Let there be the hyperbola  $ABC$ , whose center is  $D$ , and asymptote  $DE$ , and let  $AF$  and  $FC$  touch, and let  $CA$  and  $FD$  be joined and produced to  $G$  and  $H$ , then it is evident that

$$AH = HC$$

Then let  $FK$  be drawn through  $F$  parallel to  $AC$ , and  $HLK$  through  $H$  parallel to  $DE$

I say that

$$KL = HL$$

Let  $LM$  and  $BE$  be drawn through  $B$  and  $L$  parallel to  $AC$ , then as has been already shown (III 30,  $\alpha$ ,  $\beta$ , and conclusion)

$$\text{sq } DB = \text{sq } BE = \text{sq } HM = \text{sq } ML = \text{rect } BM, MG = \text{sq } ML,$$

therefore

$$\text{rect } GM, MB = \text{sq } MH$$

And also

$$\text{rect } HD, DF = \text{sq } DB$$

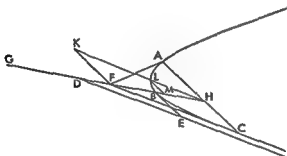
because  $AF$  touches and  $AH$  has been dropped ordinately (I 37), therefore

$$\text{rect } GM, MB + \text{sq } DB = \text{rect } HD, DF + \text{sq } MH = \text{sq } DM \text{ (Eucl II 6)}$$

Therefore  $FH$  has been bisected at  $M$  with  $DF$  added. And  $KF$  and  $LM$  are parallel, therefore

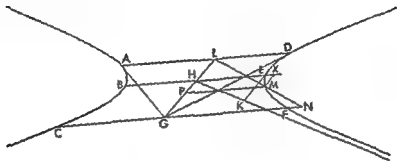
$$KL = LH$$

### PROPOSITION 33



contact, and still another straight line is drawn through the midpoint of the straight line joining the points of contact parallel to one of the asymptotes and meeting the

also let  $ALD$  be joined, then it is evident that it is bisected at  $L$  (II 30). Then let  $BHE$  and  $CGF$  be drawn through  $G$  and  $H$  parallel to  $AD$ , and  $LMN$  through  $L$  parallel to  $HK$ .



I say that

$$LM = MN$$

For let  $EK$  and  $MX$  be dropped from  $E$  and  $M$  parallel to  $GH$ , and  $MP$  through  $M$  parallel to  $AD$ .

Since then through things already shown (III 30,  $\alpha$  and  $\beta$ )

$$sq\ HE = sq\ EK = rect\ BL, NE = sq\ XM,$$

therefore

$$sq\ HE = sq\ EK = rect\ BX, LE + sq\ HE = sq\ KE + sq\ XM \text{ (Eucl v 12)}$$

$$\text{or } sq\ HE = sq\ EK = sq\ HX = sq\ KE + sq\ XM \text{ (Eucl II 6)}$$

But it has been shown (I 38, II 1)

$$sq\ EK = rect\ GH, HL$$

and

$$sq\ XM = sq\ HP,$$

therefore

$$sq\ HE = sq\ EK = sq\ HX \text{ or } sq\ MP = rect\ GH, HL + sq\ HP$$

And

$$sq\ HE = sq\ EK = sq\ MP = sq\ PL \text{ (Eucl VI 4),}$$

therefore

$$sq\ MP = sq\ PL = sq\ MP = rect\ GH, HL + sq\ HP$$

Therefore

$$sq\ PL = rect\ GH, HL + sq\ HP$$

Therefore the straight line  $LG$  has been cut equally at  $P$  and unequally at  $H$  (Eucl II 5)

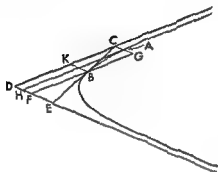
And  $MP$  and  $GN$  are parallel, therefore

$$LM = MN$$

### PROPOSITION 34

*If some point is taken on one of the asymptotes of an hyperbola, and a straight line from it touches the section, and through the point of contact a parallel to the asymptote is drawn then the straight line drawn from the point taken parallel to the other*

asymptotes  $CD$  and  $DE$ , and let  $\square$



let  $CAG$  be drawn parallel to  $DE$

I say that

$CA = AG$

For let  $AH$  be drawn through  $A$  parallel to  $CD$  and  $BK$  through  $B$  parallel to  $DE$ . Since then

$CB = BE$  (II 3).

therefore also

$$CK=KD$$

and

$DF = FE$

And since

$\text{rect } KB, BF = \text{rect } CA AH \text{ (II 12).}$

and

$$BF = DK = CK.$$

and

$AH = DC.$

therefore

$$\text{rect } DC, CA = \text{rect } GC, CK$$

Therefore

DC CK GC CA

And

$$CD=2CK.$$

therefore also

$$GC=2\ CA$$

Therefore

$CA = AG$

### PROPOSITION 35

With the same things being so if from the point taken some straight line is drawn cutting the section at two points then as the whole straight line is to the straight line cut off outside so will the segments of the straight line cut off inside be to each other

For let there be the hyperbola  $AB$  and the asymptotes  $CD$  and  $DE$  and  $CBE$  touching and  $HB$  parallel and through  $C$  let some straight line  $CALFG$  be drawn across cutting the section at  $A$  and  $F$

I say that

FC CA FL AL

For let  $CN$ ,  $KA$ ,  $OP$  and  $FU$  be drawn through  $C$ ,  $A$ ,  $B$  and  $F$

parallel to  $DE$ , and  $APS$  and  $TFRMX$  through  $A$  and  $F$  parallel to  $CD$

Since then

$$AC \approx FG \text{ (II 8),}$$

therefore also

$$KA \approx TG \text{ (Eucl VI 4).}$$

But

$$KA \approx DS;$$

therefore also

$$TG \approx DS$$

And so also

$$CK \approx DU.$$

And since

$$CK \approx DU,$$

also

$$DK \approx CU;$$

therefore

$$DK : CK :: CU : CK.$$

And

$$CU : CK :: FC : AC,$$

and

$$FC : AC :: MK : KA,$$

and

$$MK : KA :: \text{pllg } MD : \text{pllg } DA \text{ (Eucl VI 1),}$$

and

$$DK \cdot CK \cdot \text{pllg } HK \cdot \text{pllg } KN,$$

therefore also

$$\text{pllg } MD : \text{pllg } DA :: \text{pllg } HK : \text{pllg } KN.$$

But

$$\text{pllg } DA \approx \text{pllg } DB \text{ (II 12)} \approx \text{pllg } ON,$$

for

$$CB \approx BE \text{ (II 3),}$$

and

$$DO \approx OC,$$

therefore

$$\text{pllg } MD : \text{pllg } ON :: \text{pllg } HK : \text{pllg } KN.$$

and

remainder  $\text{pllg } MH$  remainder  $\text{pllg } BK :: \text{whole pllg } MD : \text{whole pllg } ON$

And since

$$\text{pllg } DA \approx \text{pllg } DB,$$

let the common parallelogram  $DP$  be subtracted,

therefore

$$\text{pllg } KP \approx \text{pllg } PH$$

Let the common parallelogram  $AB$  be added, therefore

$$\text{whole pllg } BK \approx \text{whole pllg } AH$$

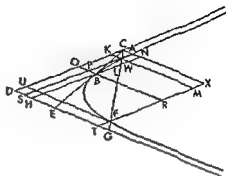
Therefore

$$\text{pllg } MD : \text{pllg } DA :: \text{pllg } MH : \text{pllg } AH$$

But

$$\text{pllg } MD : \text{pllg } DA :: MK : KA :: FC : AC,$$

and



pllg  $MH$  pllg  $AH$   $MW$   $WA$   $FL$   $LA$ ,

therefore also

$FC$   $AC$   $FL$   $LA$

### PROPOSITION 36

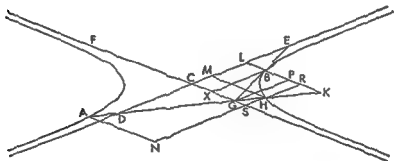
With the same things being so if the straight line drawn across from the point neither cuts the section at two points nor is parallel to the asymptote it will meet the opposite section, and as the whole straight line is to the straight line between the section and the parallel through the point of contact, so will the straight line between the opposite section and the asymptote be to the straight line between the asymptote and the other section.

Let there be the opposite sections  $A$  and  $B$  whose center is  $C$  and asymptotes  $DE$  and  $FG$ , and let some point  $G$  be taken on  $CG$ , and from it let  $GBE$  be drawn tangent, and  $GH$  neither parallel to  $CE$  nor cutting the section in two points (r 26)

It has been shown that  $GH$  produced meets  $CD$  and therefore also section  $A$  let it meet it at  $A$ , and let  $KBL$  be drawn through  $B$  parallel to  $CG$

I say that

$AK$   $KH$   $AG$   $GH$



For let  $HM$  and  $AN$  be drawn from the points  $A$  and  $H$  parallel to  $CG$ , and  $BX$   $GP$  and  $RHSN$  from  $B$ ,  $G$  and  $H$  parallel to  $DE$  Since then

$$AD = GH \text{ (r 16),}$$

$$AG \cdot GH = DH \cdot HG$$

But

$$AG \cdot GH = NS \cdot SH,$$

and

$$DH \cdot GH = CS \cdot SG,$$

And therefore

$$NS \cdot SH = CS \cdot SG$$

But

$$NS \cdot SH \text{ pllg } NC \text{ pllg } CH,$$

and

$$CS \cdot SG \text{ pllg } RC \text{ pllg } RG,$$

therefore also

$$\text{pllg } NC \text{ pllg } CH \text{ pllg } RC \text{ pllg } RG$$

And as one is to one so are all to all, therefore

pllg NC : pllg CH . whole pllg NL whole pllg CH + pllg RG.

And since

$$EB = BG,$$

also

$$LB = BP$$

and

$$\text{pllg } LX = \text{pllg } BG$$

And

$$\text{pllg } LX = \text{pllg } CH \text{ (II 12);}$$

therefore also

$$\text{pllg } HG = \text{pllg } CH$$

Therefore

$$\text{pllg } NC \text{ pllg } CH \text{ whole pllg } NL \text{ whole pllg } BG + \text{pllg } RG$$

or

$$\text{pllg } NC \text{ pllg } CH \cdot \text{pllg } NL \text{ pllg } RX.$$

But

$$\text{pllg } RX = \text{pllg } LH,$$

since also

$$\text{pllg } CH = \text{pllg } BC \text{ (II 12),}$$

and

$$\text{pllg } MB = \text{pllg } XH$$

Therefore

$$\text{pllg } NC \text{ pllg } CH \text{ pllg } NL \text{ pllg } LH$$

But

$$\text{pllg } NC \text{ pllg } CH \cdot NS \text{ SH } AG \text{ GH},$$

and

$$\text{pllg } NL \text{ pllg } LH \cdot NR \text{ RH } AK \text{ KH},$$

therefore also

$$AK \text{ KH } AG \text{ GH}$$

### PROPOSITION 37

If two straight lines touching a section of a cone or circumference of a circle or opposite sections meet and a straight line is joined to their points of contact and from the point of meeting of the tangents some straight line is drawn across cutting the line (of the section) at two points, then as the whole straight line is to the straight line cut off outside, so will the segments produced by the straight line joining the points of contact be to each other

Let there be the section of a cone AB and tangents AC and CB and let AB be joined and let CDEF be drawn across

I say that

$$CF \text{ CD } FE \text{ FD}$$

Let the diameters CH and AK be drawn through C and A, and through F and D, DP, FR LFM and NDO parallel to AH and LC Since then LFM is parallel to ADO

$$FC \text{ CD } LF \text{ AD } FM \text{ DO } LM \text{ XO},$$

and therefore

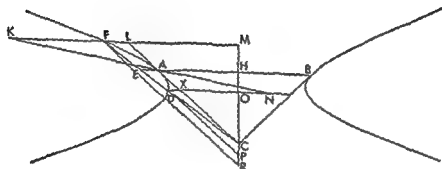
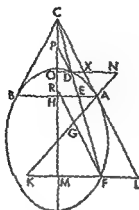
$$\text{sq } LM \text{ sq } AO \text{ sq } FM \text{ sq } DO$$

But

$$\text{sq } LM \text{ sq } AO \text{ trgl } LMC \text{ trgl } XCO \text{ (Eucl vi 19),}$$







## PROPOSITION 38

With the same things being so, if some straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact, and a straight line drawn through the midpoint of the straight line joining the points of contact cuts the section in two points and the straight line through the point of meeting of the tangents, then as the line cut off outside between the tangents is to the line cut off outside between the tangents by the straight line joined

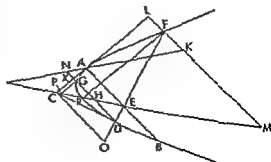
to the points of contact

Let there be the section  $AB$  and tangents  $AC$  and  $BC$  and  $AB$  the straight line joining the points of contact and  $AN$  and  $CM$  diameters, then it is evident that  $AB$  has been bisected at  $E$  (II 30, 39)

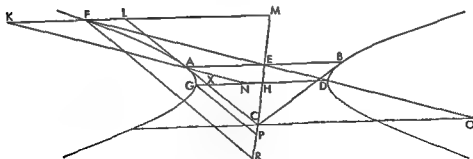
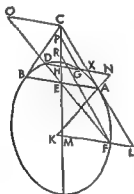
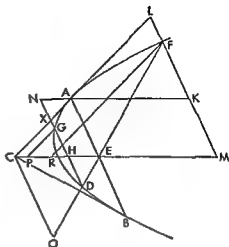
Let  $CO$  be drawn from  $C$  parallel to  $AB$ , and let  $FEDO$  be drawn across through  $E$

I say that

$$FO : OD = FE : ED$$



For let  $LFAM$  and  $DHGAN$  be drawn through  $F$  and  $D$  parallel to  $AB$  and through  $F$  and  $G$   $FR$  and  $GP$  parallel to  $LC$ . Then likewise as before (III 37) it will be shown that  
 $\text{sq } LM = \text{sq } XH = \text{sq } LA = \text{sq } AX$   
 And  
 $\text{sq } LM = \text{sq } XH = \text{sq } LC = \text{sq } CL$   
 $\text{sq } FO = \text{sq } OD$   
 and  
 $\text{sq } LA = \text{sq } AX = \text{sq } FE = \text{sq } ED$ ,  
 therefore  
 $\text{sq } FO = \text{sq } OD = \text{sq } FE = \text{sq } ED$   
 and  
 $FO = OD = FE = ED$



## PROPOSITION 39

If two straight lines touching opposite sections meet and a straight line is produced through the points of contact and a straight line drawn from the point of meeting of the tangents cuts both of the sections and the straight line joining the points of contact then as the whole straight line drawn across is to the straight line

cut off outside between the section and the straight line joining the points of contact, so will the segments of the straight line produced by the segments and the point of meeting of the tangents be to each other

Let there be the opposite sections  $A$  and  $B$  whose center is  $C$  and tangents  $AD$  and  $DB$ , and let  $AB$  and  $CD$  be joined and produced, and through  $D$  let some straight line  $EDFG$  be drawn across

I say that

$EG \quad GF \quad ED \quad DF$

For let  $AC$  be joined and produced, and through  $E$  and  $F$  let  $EHS$  and  $FLMNXO$  be drawn parallel to  $AB$ , and parallel to  $AD$ ,  $EP$  and  $FR$

Since then  $FX$  and  $ES$  are parallel, and  $EF$ ,  $XS$ , and  $HM$  have been drawn through to them.

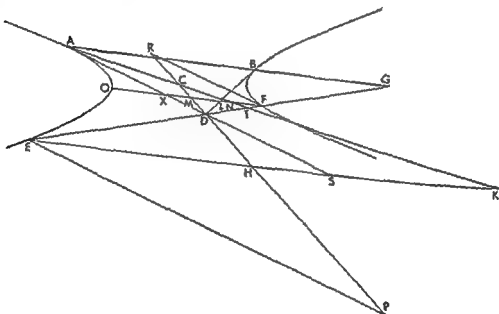
EH HS FM MX

And alternately

EH FM HS MX.

therefore also

sq *EH* sq *FM* sq *HS* sq *MX*



But

sq EH sq FM trgl EHP trgl FRM.

and

sq HS sq VX trgl DHS trgl AMD,

therefore also

trgl EHP	trgl FRM	trgl DHS	trgl AMD

And

$$\text{trgl } EHP \approx \text{trgl } ASK + \text{trgl } DHS \text{ (iii 11),}$$

and

$$\text{trgl } FRM = \text{trgl } AXN + \text{trgl } XMD \text{ (m 11)}$$

therefore

$$\text{trgl } DHS \text{ trgl } \lambda MD \text{ trgl } ASK + \text{trgl } DHS \text{ trgl } A \lambda N + \text{trgl } \lambda MD,$$

and

$$\text{remainder trgl } ASK \text{ remainder trgl } A \lambda X \text{ trgl } DHS \text{ trgl } \lambda MD$$

But

$$\text{trgl } ASK \text{ trgl } ANX \text{ sq } KA \text{ sq } AN \text{ sq } EG \text{ sq } FG *$$

and

$$\text{trgl } DHS \text{ trgl } \lambda MD \text{ sq } HD \text{ sq } DM \text{ sq } ED \text{ sq } DF$$

Therefore also

$$EG \text{ } FG \text{ } ED \text{ } DF$$

### PROPOSITION 40

With the same things being so if a straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact and if a straight line drawn from the midpoint of the straight line joining the points of contact cuts both of the sections and the straight line parallel to the straight line joining the points of contact then as the whole straight line drawn across is to the straight line cut off outside between the parallel and the section so will the straight line's segments produced by the sections and the straight line joining the points of contact be to each other

Let there be the opposite sections  $A$  and  $B$  whose center is  $C$ , and tangents  $AD$  and  $DB$  and let  $AB$  and  $CDE$  be joined therefore

$$AE = EB \text{ (n 39)}$$

And from  $D$  let  $FDG$  be drawn parallel to  $AB$  and from  $E$   $LE$  at random

I say that

$$HL \text{ } LK \text{ } HE \text{ } EK$$

From  $H$  and  $K$  let  $NMHX$  and  $KOP$  be drawn parallel to  $AB$  and  $HR$  and  $AS$  parallel to  $AD$  and let  $XACT$  be drawn through

Since then  $XAU$  and  $MAP$  have been drawn across the parallels  $AM$  and  $AP$

$$XA \text{ } AU \text{ } MA \text{ } AP$$

But

$$XA \text{ } AU \text{ } HE \text{ } EA$$

and

$$HE \text{ } EA \text{ } HV \text{ } KO$$

because of the similarity of the triangles  $HEN$  and  $KEO$  therefore

$$HN \text{ } KO \text{ } MA \text{ } AP,$$

therefore also

$$\text{sq } HN \text{ sq } KO \text{ sq } MA \text{ sq } AP$$

\*For

$$EG \text{ } TG \text{ } KA \text{ } TA$$

and

$$TG \text{ } TF \text{ } TA \text{ } TN$$

and

$$TG - TF \text{ } TG \text{ } TA - TN \text{ } TA$$

therefore ex aequalis

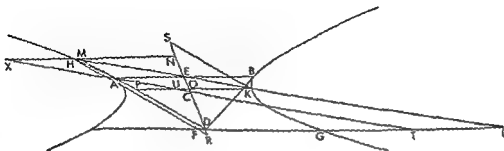
$$EG \text{ } FG \text{ } KA \text{ } AV$$

But

sq HN sq KO trgl HRN trgl KSO

and

$$\text{sq } MA \quad \text{sq } AP \quad \text{trgl } XMA \quad \text{trgl } AUP,$$



therefore also

trgl	HRN	trgl	ASO	trgl	AMA	trgl	AUP

And

$$\text{trgl } HNR = \text{trgl } \backslash MA + \text{trgl } MND \quad (\text{III } 11)$$

and

$$\text{trgl } LSO = \text{trgl } AUP + \text{trgl } DOP \quad (\text{in } 11)$$

Therefore also

trgl  $\lambda MA + \text{trgl } UND$	trgl  $AUP + \text{trgl } DOP$	trgl  $\lambda MA$	trgl  $AUP$ ,

therefore also

remainder trgl	$\lambda$	MD	remainder trgl	DOP	whole	whole
----------------	-----------	----	----------------	-----	-------	-------

But

$$\text{trgl } \lambda \text{ VA} \quad \text{trgl } AUP \quad \text{sq } \lambda A \quad \text{sq } AU$$

and

trgl	VMD	trgl	DOP	sq	MN	sq	PO,

therefore also

sq MN sq PO sq XA sq AU

But

sq MN sq PO sq ND sq OD

and

sq YA sq AU sq HF sq EK,

and

sq ND sq DO sq HL sq LK,

therefore also

$sq\ HE\ sq\ EK\ sq\ HL\ sq\ LK$

Therefore

HE EA HL LA

### PROPOSITION 41

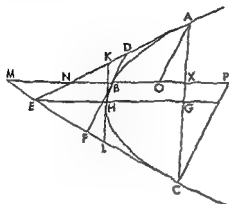
If three straight lines touching a parallel line cut each other they will be cut in the same ratio

Let there be the parabola  $ABC$  and tangents  $ADE$   $EFC$  and  $DBF$

I say that

$$CF \quad FE \quad ED \quad DA \quad FB \quad BD$$

For let  $AC$  be joined and bisected at  $G$



Then it is evident that the straight line from  $E$  to  $G$  is a diameter of the section (II 29)

If then it goes through  $B$ ,  $DF$  is parallel to  $AC$ , (II 5) and will be bisected by  $EG$ , and therefore

$$AD = DE \text{ (I 35),}$$

and

$$CF = FE \text{ (I 35),}$$

and what was sought is apparent

Let it not go through  $B$ , but through  $H$ , and let  $KHL$  be drawn through  $H$  parallel to  $AC$ , therefore it will touch the section at  $H$  (I 32), and because of things already said (I 35),

$$AK = KE$$

and

$$LC = LE$$

Let  $MNBX$  be drawn through  $B$  parallel to  $EG$ , and  $AO$  and  $CP$  through  $A$  and  $C$  parallel to  $DF$ . Since then  $MB$  is parallel to  $EH$ ,  $MB$  is a diameter (I 40, I 51 end), and  $DF$  touches at  $B$ , therefore  $AO$  and  $CP$  have been dropped ordinately (II 5, First Def 1 4). And since  $MB$  is a diameter, and  $CM$  a tangent, and  $CP$  an ordinate

$$MB = BP \text{ (I 35),}$$

and so also

$$MF = FC$$

And since

$$MF = FC$$

and

$$EL = LC,$$

$$MC \quad CF \quad EC \quad CL,$$

and alternately

$$MC \quad EC \quad CF \quad CL$$

But

$$MC \quad EC \quad XC \quad CG,$$

therefore also

$$CF \quad CL \quad XC \quad CG$$

And

$$CL \quad EC \quad CG \quad CA,$$

therefore *ex aequalis*

$$CA \quad XC \quad EC \quad CF,$$

and *convertendo*

$$EC \quad FE \quad CA \quad AX,$$

*separando*

$$CF \quad FE \quad XC \quad AX$$

Again since  $MB$  is a diameter and  $AN$  a tangent and  $AO$  an ordinate,

$$NB=BO \text{ (r 35),}$$

and

$$ND=DA$$

And also

$$EK=KA,$$

therefore

$$AE \quad KA \quad NA \quad DA,$$

alternately

$$AE \quad NA \quad KA \quad DA$$

But

$$AE \quad NA \quad GA \quad AX,$$

therefore also

$$KA \quad DA \quad GA \quad AX$$

And also

$$AE \quad KA \quad CA \quad GA,$$

therefore, *ex aequali*,

$$AE \quad DA \quad CA \quad AX,$$

*separando*

$$ED \quad DA \quad AC \quad AX$$

And it was also shown

$$XC \quad AX \quad CF \quad FE,$$

therefore

$$CF \quad FF \quad ED \quad DA$$

Again since

$$XC \quad AX \quad CP \quad AO,$$

and

$$CP=2 \text{ BF,}$$

and

$$CM=2 \text{ MF,}$$

and

$$AO=2 \text{ BD,}$$

and

$$AN=2 \text{ ND,}$$

therefore

$$XC \quad AX \quad FB \quad BD \quad CF \quad FE \quad ED \quad DA$$

#### PROPOSITION 42

If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of the diameter parallel to an ordinate and some other straight line at random is drawn tangent it will cut off from them straight lines containing a rectangle equal to the fourth part of the figure to the same diameter

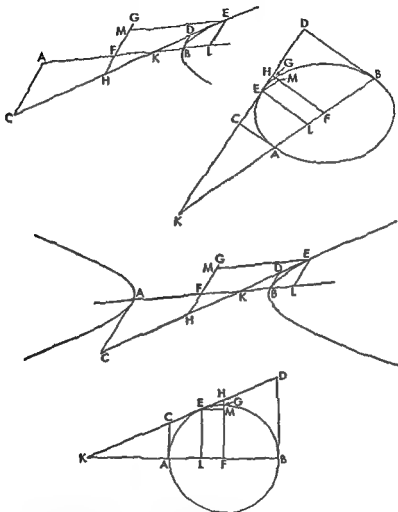
For let there be some one of the aforesaid sections whose diameter is  $AB$  and from  $A$  and  $B$  let  $AC$  and  $DB$  be drawn parallel to an ordinate and let some other straight line  $CED$  be tangent at  $E$

I say that

$$\text{rect. } AC, BD = \text{fourth part of figure to } AB$$

For let its center be  $F$ , and through it let  $FG$  be drawn parallel to  $AC$  and  $BD$ . Since then  $AC$  and  $BD$  are parallel, and  $FG$  is also parallel, therefore it is the diameter conjugate to  $AB$  (First Def, 1 6), and so

sq  $FG$  = fourth part of figure to  $AB$  (Sec Def 1 3)



If then  $FG$  goes through  $E$  in the case of the ellipse and circle,

$$AC = FG = BD \text{ (II 7),}$$

and it is immediately evident that

$$\text{rect } AC, BD = \text{sq } FG = \text{sq } AB \text{ (I 37),}$$

let

Since then

$$\begin{aligned} \text{rect } AF, FL &= \text{sq } AF \text{ (I 37),} \\ KF \cdot AF &= AF \cdot FL, \end{aligned}$$



and  $KA : AL :: KF : AF$  or  $FB$  (Eucl v. 19);

inversely

$$FB : KF :: AL : KA;$$

componendo or separando

$$BK : KF :: LK : KA.$$

Therefore also

$$DB : FH :: EL : CA.$$

Therefore

$$\begin{aligned} \text{rect } DB, CA &= \text{rect } FH, EL, \\ &= \text{rect. } HF, FM. \end{aligned}$$

But

$$\begin{aligned} \text{rect } HF, FM &= \text{sq } FG \text{ (I 38),} \\ &= \text{fourth figure to } AB \text{ (Sec Def I 11);} \end{aligned}$$

therefore also

$$\text{rect } DB, CA = \text{fourth figure to } AB.$$

### PROPOSITION 43

*If a straight line touch an hyperbola, it will cut off from the asymptotes, beginning with the center of the section, straight lines containing a rectangle equal to the rectangle contained by the straight lines cut off by the tangent at the section's vertex at its axis*

Let there be the hyperbola  $AB$ , and asymptotes  $CD$  and  $DE$ , and axis  $BD$ , and let  $FBG$  be drawn through  $B$  tangent, and some other tangent at random,  $CAH$

I say that

$$\text{rect } FD, DG = \text{rect } CD, DH.$$

For let  $AK$  and  $BL$  be drawn from  $A$  and  $B$  parallel to  $DG$ , and  $AM$  and  $BN$  parallel to  $CD$ . Since then  $CAH$  touches,

$$CA = AH \text{ (II 3);}$$

and so

$$CH = 2AH$$

and

$$CD = 2AM$$

and

$$DH = 2AK.$$

Therefore

$$\text{rect } CD, DH = 4 \text{ rect } KA, AM$$

Then likewise it could be shown

$$\text{rect } FD, DG = 4 \text{ rect } LB, BN.$$

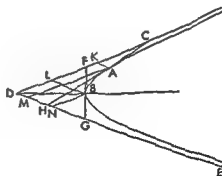
But

$$\text{rect } KA, AM = \text{rect } LB, BN \text{ (II 12)}$$

Therefore also

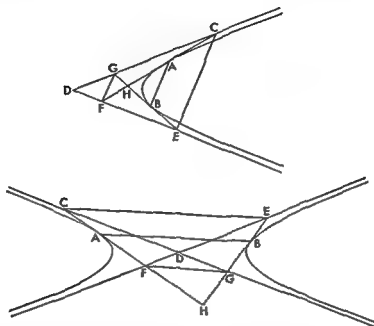
$$\text{rect } CD, DH = \text{rect } FD, DG$$

Then likewise it could be shown, even if  $DB$  were some other diameter and not the axis.



## PROPOSITION 44

If two straight lines touching an hyperbola or opposite sections meet the asymptotes then the straight lines drawn to the sections will be parallel to the straight line joining the points of contact



For let there be either the hyperbola or the opposite sections  $AB$ , and asymptotes  $CD$  and  $DE$  and tangents  $CAHF$  and  $EBHG$ , and let  $AB$ ,  $FG$ , and  $CE$  be joined

I say that they are parallel

For since

$$\text{rect } CD, DF = \text{rect } GD, DE \text{ (III 43),}$$

therefore

$$\frac{CD}{DE} = \frac{GD}{DF},$$

therefore  $CE$  is parallel to  $FG$  And therefore

$$\frac{HF}{FC} = \frac{HG}{GE}$$

And

$$\frac{FC}{AF} = \frac{GE}{GB},$$

for each is double (II 3), therefore *ex aequali*

$$\frac{HG}{GB} = \frac{HF}{FA}$$

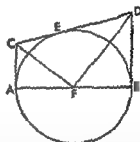
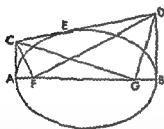
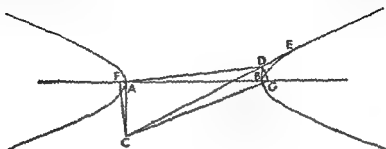
Therefore  $FG$  is parallel to  $AB$

## PROPOSITION 45

If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of the axis at right angles, and a rectangle equal to the fourth part of the figure is applied to the axis on each side and exceed-

ing by a square figure in the case of the hyperbola and  
 cent in the case of the ellipse, and some  
 and meeting the perpendicular  
 points of meeting to the points  
 aforesaid points<sup>1</sup>

Let there be one of the sections mentioned whose axis is  $AB$ , and  $AC$  and  $BD$



<sup>1</sup>"The points of application" are in modern terminology the foci of the conics. The circle is seen here as an ellipse whose two foci or focal points coincide with the center. This theory is, of course, a special application of Euclid vi 28 and 29, two theorems on which depends one whole side of Greek geometry.

Apollonius never speaks of the focus of the parabola, but it can be found by analogy with the ellipse.

Thus in the ellipse above

$$\text{rect } AF, FB = \text{fourth rect } AB, R$$

where  $R$  is the parameter. Or

$$\text{rect } AF, (AB - AF) = \text{fourth rect } AB, R$$

or

$$AF \text{ fourth } R = AB (AB - AF)$$

Then if we consider the ellipse as its axis,  $AB$  gets as large as we please, we can think of it as approaching as near as we please to a parabola with parameter  $R$ . The ratio  $AB (AB - AF)$  approaches as near as we please to equality and hence also the ratio  $AF \text{ fourth } R$ . At the limit we can think of the ellipse as the parabola, its axis  $AB$  as infinite and  $AB$  as equal to  $AB - AF$ . Then  $AF$  will be equal to a fourth  $R$ . Thus the focus of a parabola will be defined as the point on its axis at a distance from the vertex equal to one quarter of the parameter. Then many of the properties of the foci of the ellipse can be proved analogously for the parabola. Thus in the case of this proposition,  $FD$  will become parallel to  $CE$ . Hence any straight line from the focus of a parabola parallel to a tangent will make a right angle with the straight line drawn from the focus to the intersection of the tangent and the perpendicular to the axis at the vertex.



at right angles and  $CED$  tangent and let the rectangle  $AF$   $FB$  and the rectangle  $CG$   $GD$  be joined on each side

For since it has been shown  
rect  $AC$   $BD$  = fourth figure on  $AB$  (III 42)  
and since also

rect  $AF$   $FB$  = fourth figure on  $AB$

therefore

rect  $AC$   $BD$  = rect  $AF$   $FB$

Therefore

$AC$   $AF$   $FB$   $BD$

And the angles at points  $A$  and  $B$  are right therefore  
angle  $ACF$  = angle  $BFD$  (Eucl VI 6),

and

angle  $AFC$  = angle  $FDB$

And since angle  $CAF$  is right therefore

angle  $ACF$  + angle  $AFC$  = 1 rt angle

And it has also been shown that

angle  $ACF$  = angle  $DFB$

therefore

angle  $AFC$  + angle  $DFB$  = 1 right angle

Therefore

angle  $DFC$  = 1 right angle

Then likewise it could also be shown

angle  $CGD$  = 1 right angle

### PROPOSITION 46

With the same things being so the straight lines joined make equal angles with the tangents

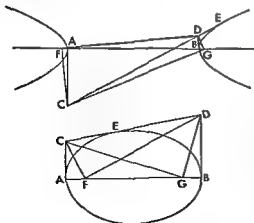
For with the same things supposed I say that

angle  $ACF$  = angle  $DCG$

and

angle  $CDF$  = angle  $BDG$

For since it has been shown  
that both angle  $CFD$  and angle  $CGD$



angle  $DCG$  = angle  $DFG$

for they are on the same segment of the circle And it was shown  
angle  $DFG$  = angle  $ACF$  (III 45),  
and so

angle  $DCG$  = angle  $ACF$

And likewise also

angle  $CDI$  = angle  $BDG$

## APOLLONIUS OF PERGA

## PROPOSITION 47

With the same things being so, the straight line drawn from the point of meeting of the joined straight lines to the point of contact will be perpendicular to the tangent

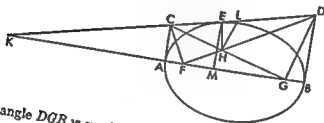
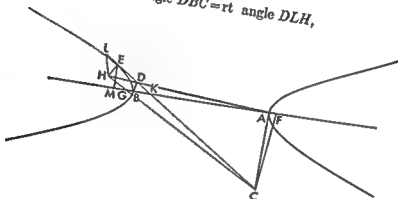
For let the same things as before be supposed and let  $CG$  and  $FD$  meet each other at  $H$ , and let  $CD$  and  $BA$  produced meet at  $K$ , and let  $EH$  be joined

I say that  $LH$  is perpendicular to  $CD$

For if not, let  $HL$  be drawn from  $H$  perpendicular to  $CD$  Since then

$$\text{angle } CDF = \text{angle } GBD \text{ (III 46),}$$

$$\text{rt angle } DBC = \text{rt angle } DLH,$$



therefore triangle  $DGB$  is similar to triangle  $LHD$  Therefore

But

$$GD \text{ } DH \text{ } BD \text{ } DL$$

because the angles at  $F$  and  $G$  are right angles (III 45) and the angles at  $H$  are equal, but

$$GD \text{ } DH \text{ } FC \text{ } CH$$

because of the similarity of the triangles  $AFC$  and  $LCH$  (III 46), therefore also

$$FC \text{ } CH : AC \text{ } CL$$

Alternately

$$BD \text{ } DL : AC \text{ } CL$$

But

$$BD \text{ } AC : DL \text{ } CL$$

therefore also

$$BD \text{ } AC : BK \text{ } KA$$

$$DL \text{ } CL : BK \text{ } KA$$

Let  $EM$  be drawn from  $E$  parallel to  $AC$ , therefore it will have been dropped ordinatewise to  $AB$  (II 7), and

$$BK \cdot KA \cdot BM \cdot MA$$

And

$$BM \cdot MA :: DE \cdot EC;$$

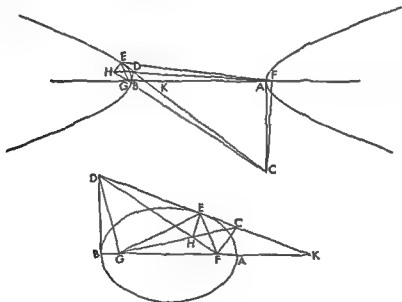
therefore also

$$DL \cdot CL :: DE \cdot EC,$$

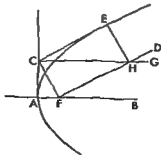
and thus is absurd. Therefore  $HL$  is not perpendicular nor is any other straight line except  $HE$  \*

### PROPOSITION 48

*With the same things being so, it must be shown that the straight lines drawn from the point of contact to the points produced by the application make equal angles with the tangent*



\*There is the analogous theorem for the parabola.  $FD$  becomes a straight line parallel to



$CE$  and  $CG$  a straight line parallel to  $AB$ . Again  $HE$  is perpendicular to  $CE$ , and this can be proved rigorously as well as understood by analogy

## APOLLONIUS OF PERGA

For let the same things be supposed, and let  $EF$  and  $EG$  be joined  
I say that

For since angles  $DGH$  and  $DEH$  are right angles (in 45 47), the circle described about  $DH$  as a diameter will pass through the points  $E$  and  $G$  (Eucl in 31), and so

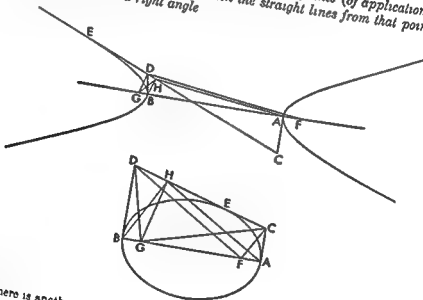
angle  $DHG$  = angle  $DEG$  (Eucl in 21),  
for they are in the same segment Likewise then also

angle  $CEF$  = angle  $CHF$

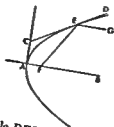
But  
angle  $CHF$  = angle  $DHG$ ,  
for they are vertical angles, therefore also  
angle  $CEF$  = angle  $DEG$  \*

## PROPOSITION 49

With the same things being so, if from one of the points (of application) a perpendicular is drawn to the tangent, then the straight lines from that point to the ends of the axis make a right angle



\*Here there is another and important analogous theorem for the parabola.  $EG$  becomes parallel to  $AB$  and



angle  $DEG$  = angle  $CEF$

For let the same things be supposed, and let the perpendicular  $GH$  be drawn from  $G$  to  $CD$ , and let  $AH$  and  $BH$  be joined

I say that angle  $AHB$  is a right angle

For since angle  $DBG$  is a right angle and also angle  $DHG$ , the circle described about  $DG$  as a diameter will pass through  $H$  and  $B$ , and  
 angle  $BHG = \text{angle } BDG$

But it was shown

$$\text{angle } AGC = \text{angle } BDG \text{ (III 46),}$$

therefore also

$$\text{angle } BHG = \text{angle } AGC = \text{angle } AHC \text{ (Eucl III 21) }$$

And so also

$$\text{angle } CHG = \text{angle } AHB$$

But angle  $CHG$  is a right angle, therefore also angle  $AHB$  is a right angle

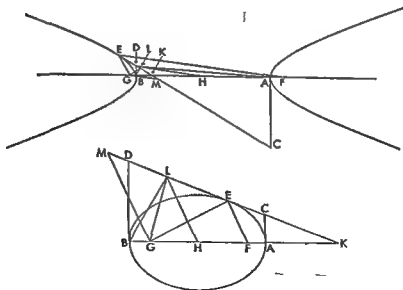
### PROPOSITION 50

With the same things being so, if from the center of the section there falls to the tangent a straight line parallel to the straight line drawn through the point of contact and one of the points (of application), then it will be equal to one half the axis

For let there be the same things as before and let  $H$  be the center, and let  $EF$  be joined, and let  $DC$  and  $BA$  meet at  $K$ , and through  $H$  let  $HL$  be drawn parallel to  $EF$

I say that

$$HL = HB$$



For let  $EG$ ,  $AL$ ,  $LG$  be joined and through  $G$  let  $GM$  be drawn parallel to  $EF$ . Since then

$$\text{rect } AF, FB = \text{rect } AG, GB \text{ (See III 45),}$$



therefore

$$AF = GB$$

But also

$$AH = HB,$$

therefore also

$$FH = HG$$

And so also

$$EL = LM$$

And since it was shown (III 48)

$$\text{angle } CEF = \text{angle } DEG,$$

and

$$\text{angle } CEF = \text{angle } EMG,$$

therefore also

$$\text{angle } EMG = \text{angle } DEG$$

And therefore

$$EG = GM$$

But it was also shown

$$EL = LM$$

therefore  $GL$  is perpendicular to  $EM$  And so through what was shown before (III 49) angle  $ALB$  is a right angle and the circle described about  $AB$  as a diameter will pass through  $L$  And

$$HA = HB,$$

therefore also, since  $HL$  is a radius of the semicircle,

$$HL = HB$$

### PROPOSITION 51

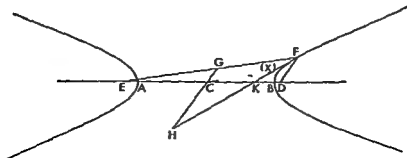
*If a rectangle equal to the fourth part of the figure is applied from both sides to the*

*much as the axis*

For let there be an hyperbola or opposite sections whose axis is  $AB$  and center  $C$ , and let each of the rectangles  $AD$   $DB$  and  $AE$ ,  $EB$  be equal to the fourth part of the figure and from points  $E$  and  $D$  let the straight lines  $EF$  and  $FD$  be deflected to the line of the section

I say that

$$EF = FD + AB$$



Let  $FKH$  be drawn tangent through  $F$  and  $GCH$  through  $C$  parallel to  $FD$ ,  
therefore

$$\text{angle } KHG = \text{angle } KFD,$$

for they are alternate And

$$\text{angle } KFD = \text{angle } GFH \text{ (III 48),}$$

therefore

$$GF = GH$$

But

$$GF = GE,$$

since also

$$AE = BD$$

and

$$AC = CB$$

and

$$EC = CD,$$

and therefore

$$GH = EG$$

And so

$$FE = 2GH$$

And since it has been shown (III 50)

$$CH = CB *$$

therefore

$$FE = 2(GC + CB)$$

But

$$FD = 2GC,$$

and

$$AB = 2CB,$$

therefore

$$FE = FD + AB$$

And so  $EF$  is greater than  $FD$  by  $AB$

### PROPOSITION 52

*If in an ellipse a rectangle equal to the fourth part of the figure is applied from both sides to the major axis and deficient by a square figure and from the points resulting from the application straight lines are deflected to the line of the section then they will be equal to the axis*

Let there be an ellipse whose major axis is  $AB$  and let each of the rectangles  $AC \cdot CB$  and  $AD \cdot DB$  be equal to the fourth of the figure and from  $C$  and  $D$  let the straight lines  $CE$  and  $ED$  have been deflected to the line of the section

I say that

$$CE + ED = AB$$

\*For

$$GF = GH$$

and by III 50 a line  $C(X)$  drawn parallel to  $GF$  is equal to  $CB$  But also

$$C(X) = CH$$

Hence

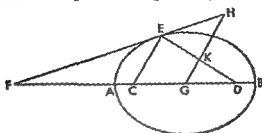
$$CH = CB$$

Let  $FEH$  be drawn tangent, and  $G$  be center and through it let  $GKH$  be drawn parallel to  $CE$ . Since then

$$\text{Angle } CEF = \text{angle } HEK \text{ (in 48),}$$

and

$$\text{angle } CEF = \text{angle } EHK,$$



therefore also

$$\text{angle } EHK = \text{angle } HEK.$$

Therefore also

$$HK = KE$$

And since

$$AG = GB,$$

and

$$AC = DB,$$

therefore also

$$CG = GD,$$

and so also

$$EK = KD$$

And for this reason

$$ED = 2HK,$$

and

$$EC = 2KG,$$

and

$$ED + EC = 2GH$$

But also

$$AB = 2GH \text{ (in 50);}$$

therefore

$$AB = ED + EC$$

#### PROPOSITION 53

If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of a diameter parallel to an ordinate, and straight lines drawn from the same ends to the same point on the line of the section cut the parallels then the rectangle contained by the straight lines cut off is equal to the figure on that same diameter

Let there be one of the aforesaid sections  $ABC$  whose diameter is  $AC$ , and let  $AD$  and  $CE$  be drawn parallel to an ordinate, and let  $ABE$  and  $CBD$  be drawn across

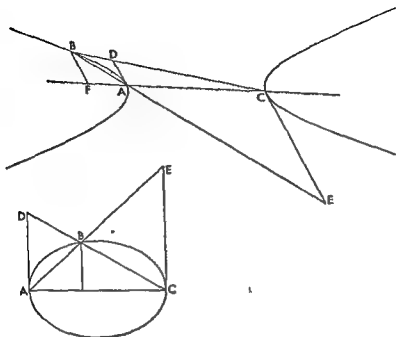
I say that

$$\text{rect } AD \cdot FC = \text{figure on } AC$$

For let  $BF$  be drawn from  $B$  parallel to an ordinate

Therefore

rect  $AI', FC$  sq  $FB$  transverse side upright side  
sq  $AC$  . the figure (1 21)



But

rect  $AF, FC$  sq  $FB$  comp  $AF$   $FB$   $FC$   $FB$ ,

therefore

figure sq  $AC$  comp  $FB$   $AF, FB$   $FC$

But

$AF$   $FB$   $AC$   $CE$ ,

and

$FC$   $FB$   $AC$   $AD$ ,

therefore

figure sq  $AC$  comp  $CE$   $AC, AD$   $AC$

And also

rect  $AD, CE$  sq  $AC$  comp  $CE$   $AC, AD$   $AC$ ,

therefore

figure sq  $AC$  rect  $AD, CE$  sq  $AC$

Therefore

rect  $AD, CE$  = figure on  $AC$

#### PROPOSITION 51

If two tangents to a section of a cone or to a circumference of a circle meet and through the points of contact parallels to the tangents are drawn and from the points of contact to the same point of the line of the section straight lines are drawn across cutting the parallels, then the rectangle contained by the straight lines cut

off to the square on the straight line joining the points of contact has a ratio compounded of the ratio which the inside segment line joining the point of meeting of the tangents and the midpoint of the straight line joining the points of contact has in square to the remainder, and of the ratio which the rectangle contained by the tangents has to the fourth part of the square on the straight line joining the points of contact

Let there be a section  $AD$  and  $CD$ , joined, and let  $AF$

$AD$ , and let some point  $H$  on the section be taken, and let the straight lines  $AH$  and  $CH$  be joined and produced to  $G$  and  $F$

I say that

rect  $AF, CG$  sq  $AC$  comp sq  $EB$  sq  $BD$ , rect  $AD, DC$   
fourth sq  $AC$  or rect  $AE, EC$

For let  $KHOXL$  be drawn from  $H$  parallel to  $AC$ , and from  $B$ ,  $MBN$  parallel to  $AC$ , then it is evident that  $MN$  is tangent (II 29, 5, 6) Since then

$$AE = EC,$$

also

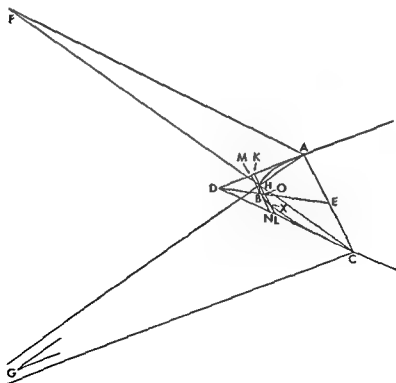
$$MB = BN$$

and

$$KO = OL$$

and

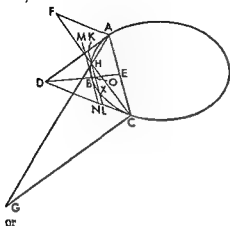
$$HO = OX \text{ (II 7)}$$



and

$$KH = XL$$

Since then  $MB$  and  $MA$  are tangents and  $KHL$  has been drawn parallel to  $MB$ ,



$$\text{sq } AM \text{ sq } MB \text{ sq } AK \text{ rect } XK, KH \text{ (iii 16)}$$

or

$$\text{sq } AM \text{ rect } MB, BN \text{ sq } AK \text{ rect } LH, HK$$

And

$$\text{rect } NC, AM \text{ sq } AM \text{ rect } LC, AK \text{ sq } AK \text{ (Eucl vi 2, v, 18),}$$

therefore *ex aequali*

$$\text{rect } NC, AM \text{ rect } MB, BN$$

$$\text{rect } LC, AK \text{ rect } LH, HK$$

But

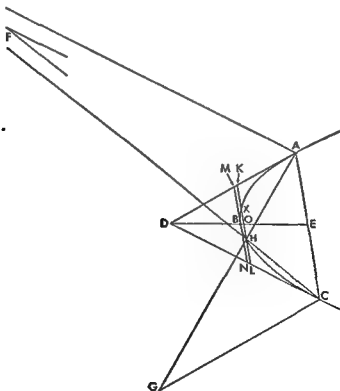
$$\text{rect } LC, AK \text{ rect } LH, HK \text{ comp } LC \text{ } LH, AK \text{ } HK$$

or

$$\text{rect } LC, AK \text{ rect } LH, HK \text{ comp } FA \text{ } AC, GC \text{ } CA$$

which is the same as

$$\text{rect } GC, FA \text{ sq } CA$$



Therefore

rect  $NC, AM$  rect  $MB, BN$  rect  $GC, FA$  sq  $CA$

But, with the rectangle  $ND, DM$  taken as a mean

rect  $NC, AM$  rect  $MB, BN$  comp

rect  $NC, AM$  rect  $ND, DM$ , rect  $ND, DM$  rect  $MB, BN$ ,

therefore

rect  $GC, FA$  sq  $CA$  comp

rect  $NC, AM$  rect  $ND, DM$ , rect  $ND, DM$  rect  $MB, BN$

But

rect  $NC, AM$  rect  $ND, DM$  sq  $EB$  sq  $BD$ ,

and

rect  $ND, DM$  rect  $NB, BM$  rect  $CD, DA$  rect  $CE, EA$ ,

therefore

rect  $GC, FA$  sq  $CA$  comp sq  $BE$  sq  $BD$ , rect  $CD, DA$  rect  $CE, EA$

### PROPOSITION 55

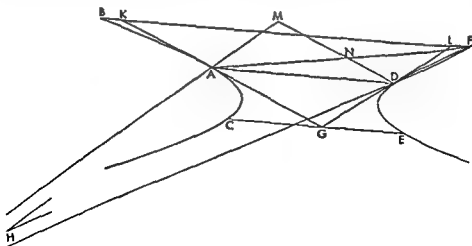
If two straight lines touching opposite sections meet, and through the point of meeting a straight line is drawn parallel to the straight line joining the points of contact, and from the points of contact parallels to the tangents are drawn across, and straight lines are produced from the points of contact to the same point of one of the sections cutting the parallels, then the rectangle contained by the straight lines cut off will have to the square on the straight line joining the points of contact the ratio which the rectangle contained by the tangents has to the square on the straight line drawn through the point of meeting parallel to the straight line joining the points of contact as far as the section.

Let there be the opposite sections  $ABC$  and  $DEF$ , and tangents to them  $AG$  and  $GD$  and let  $AD$  be joined and from  $G$  let  $CGE$  be drawn parallel to  $AD$ , and from  $A$   $AM$  parallel to  $DG$  and from  $D$ ,  $DM$  parallel to  $AG$ , and let some point  $F$  be taken on the section  $DF$ , and let  $ANF$  and  $FDH$  be joined

I say that

sq  $CG$  rect  $AG, GD$  sq  $AD$  rect  $HA, DN$ .

For let  $ILAB$  be drawn through  $F$  parallel to  $AD$



Since then it has been shown that

$$\text{sq } EG \text{ sq } GD \text{ rect } BL \text{ } LF \text{ sq } DL \text{ (III 20),}$$

and

$$CG = EG \text{ (II 38),}$$

and

$$BK = LF \text{ (II 38),}$$

therefore

$$\text{sq } CG \text{ sq } GD \text{ rect } KF, FL \text{ sq } DL$$

And also

$$\text{sq } GD \text{ rect } AG, GD \text{ sq } DL \text{ rect } DL, AK \text{ (Eucl VI 2, 1);}$$

therefore *ex aequali*

$$\text{sq } GC \text{ rect } AG, GD \text{ rect } KF, FL \text{ rect } DL, AK$$

But

$$\text{rect } KF, FL \text{ rect } DL, AK \text{ comp } KF \text{ } AK, FL \text{ } DL$$

But

$$KF \text{ } AK \text{ } AD \text{ } DN,$$

and

$$FL \text{ } DL \text{ } AD \text{ } HA,$$

therefore

$$\text{sq } CG \text{ rect } AG, GD \text{ comp } AD \text{ } DN, AD \text{ } HA$$

And also

$$\text{sq } AD \text{ rect } HA, DN \text{ comp } AD \text{ } DN, AD \text{ } HA,$$

therefore

$$\text{sq } CG \text{ rect } AG, GD \text{ sq } AD \text{ rect } HA, DN.$$

#### PROPOSITION 56

*If two straight lines touching one of the opposite sections meet, and parallels to the tangents are drawn through the points of contact, and straight lines cutting the parallels are drawn from the points of contact to the same point of the other section, then the rectangle contained by the straight lines cut off will have to the square on the straight line joining the points of contact the ratio compounded of the ratio which, of the straight line joining the point of meeting and the midpoint that part between the midpoint and the other section has in square to that part between the same section and the point of meeting, and of the ratio which the rectangle contained by the tangents has to the fourth part of the square on the straight line joining the points of contact*

*Let there be the - - - - -*

get

be

an

*CL*, and let *CBM* and *CAN* be joined

I say that

$$\text{rect } MA, BN \text{ sq } AB \text{ comp sq } LD \text{ sq } DE, \text{ rect } AE, EB$$

$$\text{fourth sq } AB \text{ or rect } AL, LB$$

For let *GCK* and *HDF* be drawn from *C* and *D* parallel to *AB*, then it is evident that

$$HD = DF,$$

and

$$AX = XG,$$



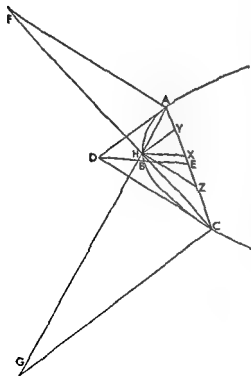


TRANSLATOR'S APPENDIX ON THREE- and  
FOUR LINE LOCI

The three-line locus property of conics is easily deduced for the ellipse, hy-

in a constant ratio to the rectangle contained by the other two distances

It is shown in III 54 that in the case of conic sections and circles  
rect  $AF, CG$  sq  $AC$  comp sq  $EB$  sq  $BD$ , rect  $AD, DC$  fourth sq  $AC$ .  
Now if we  
therefore  
that the  
them and the rectangles contained by them, are also fixed and given. Then



represents the distance from  $H$  to  $AD$  at another given angle, and  $ZC$  represents the distance from  $H$  to  $DC$  at another given angle. Then by similar triangles

$$\begin{array}{l} CZ : ZH :: AC : AF, \\ AY : YH :: AC : CG, \end{array}$$

therefore compounding  
rect  $CZ, AY$  rect  $ZH, YH$  sq  
 $AC$  rect  $AF, CG$

Now we have seen that the rectangle  $AF, CG$  is a constant magnitude as the point  $H$  changes and the square on  $AC$  is constant, therefore their ratio is constant. Therefore

rect  $CZ, AY$  rect  $ZH, YH$  is a constant ratio (1)

Again by similar triangles

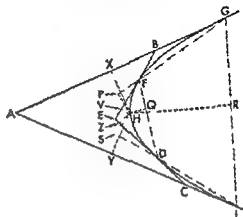
$$\begin{array}{l} ZH : HX :: CD : DE, \\ YH : HX :: AD : DE, \end{array}$$

therefore compounding  
rect  $ZH, YH$  sq  $HX$  rect  $CD, AD$  sq  $DE$

But rectangle  $CD, AD$  and the square on  $DE$  are constant magni-

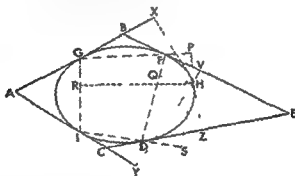
tudes as the point  $H$  changes, therefore their ratio is constant. Therefore  
 $\text{rect } ZH, YH = \text{sq } HX$  is a constant ratio (2)  
 Compounding (1) and (2), we get a constant ratio, that is  
 $\text{rect } CZ, AY = \text{sq } HX$  is a constant ratio

In other words, as the point  $H$  changes, the rectangle contained by the distances from  $H$  to two of the given straight lines (at given angles to those straight lines) has a constant ratio to the square on the distance to the third straight line (at a given angle to that straight line). And it can easily be proved by means of similar triangles that if any other three angles are chosen for the distances than those chosen here for the demonstration, then the corresponding ratio will be constant although not equal.



The four-line locus property can be easily deduced from the three-line. If to any conic section we construct four tangents  $AG, BE, AI,$  and  $EC$ , and the straight lines  $FG, GI, ID$  and  $DF$ , joining the points of contact, and draw the distances from any point  $H$  on the conic to these straight lines at any given angles (perpendiculars are convenient), then by the three-line locus property with respect to triangle  $FBG$  for any point  $H$  on the conic

$$\text{rect } HX, HV = \text{sq } HP \text{ is constant,} \quad (\alpha)$$



with respect to triangle  $AIG$

$$\text{rect } HX, HY = \text{sq } HR \text{ is constant,} \quad (\beta)$$

with respect to triangle  $DCI$

$$\text{rect } HY, HZ = \text{sq } HS \text{ is constant,} \quad (\gamma)$$

with respect to triangle  $IFD$

$$\text{rect } HZ, HV = \text{sq } HQ \text{ is constant} \quad (\delta)$$

It will be noticed that we have taken in succession a pair of adjacent tangents and the straight line joining their points of contact. It will also be noticed that the rectangles in the four ratios present a cyclic arrangement, so that if the

inverse of  $(\alpha)$  is compounded with  $(\beta)$ , and the inverse of  $(\gamma)$  with  $(\delta)$ , we would have two constant ratios

$$\text{pllpd } HY, HP, HP \quad \text{pllpd } HV, HR, HR, \quad (\epsilon)$$

$$\text{pllpd } HV, HS, HS \quad \text{pllpd } HY, HQ, HQ \quad (\zeta)$$

Again compounding the first of these with the second, we would have finally  
rect  $HP, HS$  rect  $HQ, HR$ , a constant ratio

And this is the property of the four line locus, namely the locus of points  $H$  such that the rectangle contained by the distances from points  $H$  to any two given fixed straight lines  $FG$  and  $ID$  has to the rectangle contained by the distances from  $H$  to two other fixed straight lines  $IG, FD$  = constant ratio

The rigorous method of effecting these compoundings is as follows For inverting  $(\alpha)$ , by Eucl xi, 32 we have the constant ratios

$$\text{sq } HP \text{ rect } HX, HV \quad \text{pllpd } HP, HP, HY \quad \text{pllpd } HX, HV, HY,$$

$$\text{rect } HX, HY \quad \text{sq } HR \quad \text{pllpd } HX, HY, HV \quad \text{pllpd } HR, HR, HV$$

Hence by definition the ratio (a constant one) compounded of these two is

$$\text{pllpd } HY, HP, HP \quad \text{pllpd } HV, HR, HR$$

And in the same way we find the constant ratio compounded of the inverse of  $(\gamma)$  and  $(\delta)$  Now

$$\text{pllpd } HY, HP, HP \quad \text{pllpd } HV, HR, HR \text{ comp } HY \quad HV, \text{sq } HP \quad \text{sq } HR,$$

$$\text{pllpd } HV, HS, HS \quad \text{pllpd } HY, HQ, HQ \text{ comp } HV \quad HY, \text{sq } HS \quad \text{sq } HQ$$

IF then we take two lines  $M$  and  $N$  such that

$$\frac{HP}{HS} = \frac{HR}{HQ} = \frac{M}{N}, \quad (\eta)$$

$$\frac{HP}{HS} = \frac{HR}{HQ} = \frac{M}{N}, \quad (\theta)$$

then

$$\text{sq } HP \quad \text{sq } HR \quad HP \quad M,$$

$$\text{sq } HS \quad \text{sq } HQ \quad HS \quad N$$

Hence

$$\text{ratio comp } HY \quad HV, \text{sq } HP \quad \text{sq } HR \text{ ratio comp } HY \quad HV, HP \quad M$$

$$\text{ratio comp } HV \quad HY, \text{sq } HS \quad \text{sq } HQ \text{ ratio comp } HV \quad HY, HS \quad N$$

But

$$\text{rect } HY \quad HP \quad \text{rect } HV \quad M \text{ comp } HY \quad HV, HP \quad M,$$

$$\text{rect } HV, HS \quad \text{rect } HY, N \text{ comp } HV \quad HY, HS \quad N,$$

and

$$\text{pllpd } HV \quad HP \quad HS \quad \text{pllpd } HV \quad HS \quad HS \quad N$$

ratio

$$\text{rect } HP, HS \quad \text{rect } N, M$$

Now taking  $L$  and  $O$  as some constants

$$\text{rect } HP \quad HS \quad \text{rect } N, M \quad L \quad O$$

and

$$\text{rect } HP \quad HS \quad \text{rect } HR \quad HQ \quad \text{rect } HR, HQ \quad \text{rect } M, N$$

by compounding  $(\eta)$  and  $(\theta)$  But equal ratios have equal duplicate ratios (Heath's note to Euclid vi 22) and hence

$$\text{rect } HP \quad HS \quad \text{rect } HR \quad HQ \quad \text{rect } HR, HQ \quad \text{rect } M, N$$

magnitudes  $MA$  and  $BN$  may change, but the rectangle  $MA, BN$  is a constant magnitude

For as before let  $CX$  be drawn parallel to  $DE$ ,  $CY$  to  $EA$ ,  $CZ$  to  $EB$  By similar triangles

$$\frac{AY}{BZ} = \frac{YC}{ZC} = \frac{AB}{AB} = \frac{BN}{MA},$$

therefore compounding

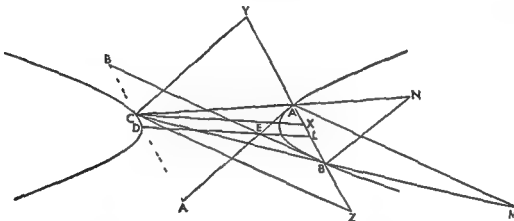
$$\text{rect } AY, BZ : \text{rect } YC, ZC :: \text{sq } AB : \text{rect } MA, BN$$

Since rectangle  $MA, BN$  is constant as  $C$  changes and also the square on  $AB$  is constant therefore

$$\text{rect } AY, BZ : \text{rect } YC, ZC \text{ is a constant ratio} \quad (1)$$

Again by similar triangles

$$\frac{ZC}{CX} = \frac{EB}{EL},$$



$$\frac{YC}{CX} = \frac{EA}{EL}$$

therefore compounding

$$\text{rect } YC, ZC : \text{sq } CX :: \text{rect } EB, EA : \text{sq } EL$$

Hence

$$\text{rect } YC, ZC : \text{sq } CX \text{ is a constant ratio} \quad (2)$$

Compounding (1) and (2) we have a constant ratio

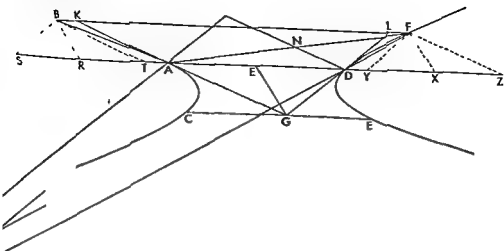
$$\text{rect } AY, BZ : \text{sq } CY$$

But  $AY$  and  $BZ$  are equal to  $CA$  and  $CB$  the distances from  $C$ . This is the property of the three-line locus of section  $C$  with respect to the straight lines  $EA$  and  $EB$  tangents to the other section and  $LB$  the straight line joining

Again from III 55 we can conclude that both of the opposite sections to

Now since the three last terms of this proportion are evidently constants as the point  $F$  changes, therefore also although  $HA$  and  $DN$  change with  $F$ , yet rectangle  $HA, DN$  remains constant in magnitude. Then reproducing the figure of

in 55 we drop  $YF$  parallel to  $DL$ , and  $FZ$  to  $KA$ , and  $FX$  to  $GE$ , where  $E$  is the midpoint of  $AD$ . Then by similar triangles



$$\begin{aligned} YD \cdot FY &: AD \cdot HA, \\ AZ \cdot FZ &: AD \cdot DN; \end{aligned}$$

therefore compounding

$$\text{rect } YD, AZ \cdot \text{rect } FY, FZ \cdot \text{sq } AD \cdot \text{rect } HA, DN.$$

But the last two terms are constant, therefore

$$\text{rect } YD, AZ \cdot \text{rect } FY, FZ \text{ is a constant ratio}$$

(1)

Again by similar triangles

$$\begin{aligned} FY \cdot FX &\cdot DG \cdot EG, \\ FZ \cdot FX &\cdot AG \cdot EG, \end{aligned}$$

therefore compounding

$$\text{rect } FY, FZ \cdot \text{sq } FX \cdot \text{rect } DG, AG \cdot \text{rect } ED, EG.$$

But the last two terms are constant, therefore

$$\text{rect } FY, FZ \cdot \text{sq } FX \text{ is a constant ratio}$$

(2)

Compounding (1) and (2), we see that

$$\text{rect } YD, AZ \cdot \text{sq } FX \text{ is a constant ratio}$$

and  $FY$  is the distance from  $F$  to  $AD$ . And so the

$$\begin{aligned} DN &= LA, \\ AZ &= KF, \end{aligned}$$

and  $FY$  is the distance from  $F$  to  $AD$ . And so the

1  
1

$$\begin{aligned} DN &= LA, \\ KF &= AZ, \\ TA &= BK \end{aligned}$$

magnitudes  $MA$  and  $BN$  may change, but the rectangle  $MA \cdot BN$  is a constant magnitude

For as before let  $CX$  be drawn parallel to  $DE$ ,  $CY$  to  $EA$ ,  $CZ$  to  $EB$ . By similar triangles

$$\frac{AY}{BZ} = \frac{YC}{ZC} = \frac{AB}{AB} = \frac{BN}{MA},$$

therefore compounding

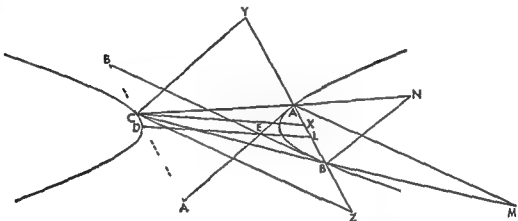
$$\text{rect } AY, BZ = \text{rect } YC, ZC = \text{sq } AB = \text{rect } MA, BN$$

Since rectangle  $MA \cdot BN$  is constant as  $C$  changes and also the square on  $AB$  is constant, therefore

$$\text{rect } AY, BZ = \text{rect } YC, ZC \text{ is a constant ratio} \quad (1)$$

Again by similar triangles

$$\frac{ZC}{CX} = \frac{EB}{EL},$$



$$\frac{YC}{CX} = \frac{EA}{EL}$$

therefore compounding

$$\text{rect } YC, ZC = \text{sq } CX = \text{rect } EB, EA = \text{sq } EL$$

Hence

$$\text{rect } YC, ZC = \text{sq } CX \text{ is a constant ratio} \quad (2)$$

Compounding (1) and (2) we have a constant ratio

$$\text{rect } AY, BZ = \text{sq } CA$$

But  $AY$  and  $BZ$  are equal to  $CA'$  and  $CB$  the distances from  $C$ . This is the property of the three line locus of section  $C$  with respect to the straight lines  $EA$  and  $EB$ .

their points of

tangents to

could be shown in the same way as before

Again from III 55 we can conclude that both of the opposite sections together are a three line locus to the triangle formed by a tangent to each of the sections and the straight line joining their points of contact. For by III 55

$$\text{rect } HA, DV = \text{sq } AD = \text{rect } AG, GD = \text{sq } CG$$

Now since the three last terms of this proportion are evidently constants as the point  $F$  changes, therefore also although  $HA$  and  $DV$  change with  $F$ , yet rectangle  $HA, DV$  remains constant in magnitude. Then reproducing the figure of

## INTRODUCTION TO ARITHMETIC



But it was shown in the course of III 55 that

$$BK = LF,$$

$$BL = KF.$$

Hence

$$TA = BK = YD = LF,$$

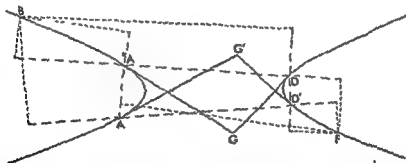
$$AZ = KF = BL$$

Therefore

$$\text{rect } LF, KF \text{ sq } FX :: \text{rect } BK, BL : \text{sq } BR.$$

Hence any point  $B$  on one opposite section fulfills the same constant ratio with respect to its distances from the three fixed lines  $AD$ ,  $GD$ , and  $AG$  as any point  $F$

on each section (the points being four points of contact of four tangents, and the straight lines, the straight lines joining them)



To sum up, a parabola, ellipse, circle, and hyperbola are three-line loci with respect to any two tangents to them and a straight line joining the points of contact tangents

points of with respect to two tangents, each to one of the sections, together with the

sections together are a four-line locus to any four straight lines joining four points, two lying on each opposite section.

## INTRODUCTION TO ARITHMETIC



## BIOGRAPHICAL NOTE

NICOMACHUS, *fl.* c. A.D. 100

canism Jamblichus says of Nicomachus "The man is great in mathematics, and had as instructors those that were most skilled in the subject "

Nothing is known of the personal life of Nicomachus except what is said or implied in the dedication of the *Introduction to Harmonics* to an unknown lady: "But I must spur on all my zeal, most noble and august lady, since it is you that bid me And, if the gods are willing just as soon as I shall have leisure and rest from my journeyings, I will compile for you a better and more detailed *Introduction* dealing with this very subject and, so that you may the will take my beginning, say, from the same our instruction when I was expounding the

which is one of the best sources on Neo-Pythagoreanism, extracts and paraphrases of this work survive in a later anonymous work of the same name and in the *Bibliotheca* a collection of extracts from ancient works made in the ninth century by Photius, patriarch of Constantinople Nicomachus also wrote an

also an *Introduction to Astronomy*, thereby completing the quadrivial series  
The success of the *Introduction to Arithmetic* must have been immediate It

was used as a text book throughout later antiquity and, in the Latin paraphrase of Boethius throughout the Middle Ages. It had a host of commentators. In the *Philopatriis*, attributed to Lucian, a character says "You reckon like Nicomachus." This remark lends itself to more than one interpretation, but in any case it is evidence of his fame. Nicomachus also appears to have been considered one of the 'golden chain,' or succession, of true philosophers, for Proclus, the fifth century Neo-Platonist, who belonged to that "chain," claimed, on the basis of a dream, that he had within him the soul of Nicomachus.

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# BOOK ONE

## CHAPTER I

[1] The ancients, who under the leadership of Pythagoras first made science systematic, defined philosophy as the love of wisdom. Indeed the name itself means this, and before Pythagoras all who had knowledge were called "wise" indiscriminately—a carpenter, for example, a cobbler, a helmsman, and in a word anyone who was versed in any art or handi-

craft. But when the desire and pursuit of this knowl-

edge was clearly distinguished from the desire for the clear sense of the term and the thing defined. This "wisdom" he defined as the knowledge, or science, of the truth in real things, conceiving "science" to be a steadfast and firm apprehension of the underlying substance, and "real things" to be those which continue uniformly and the same in the universe and never

[4] Now it is with such things that "wisdom" is particularly concerned, but accidentally also with things that share in them, that is, bodies

## CHAPTER II

[1] Those things, however, are immaterial, eternal, without end, and it is their nature to persist ever the same and unchanging, abiding by their own essential being, and each one of them is called real in the proper sense. But what are involved in birth and destruction,

them, they are not actually real by their own nature, for they do not abide for even the shortest moment in the same condition, but are always passing over in all sorts of changes. [2] To quote the words of Timaeus, in Plato,<sup>1</sup> "What is that which always is, and has no

[3] Therefore, if we crave for the goal that

ceive in connection with or together with matter, such as qualities, quantities, configurations, largeness, smallness, equality, relations, actualities, dispositions, places, times, all those things, in a word, whereby the qualities found in each body are comprehended—all these are of themselves immovable and unchangeable, but accidentally they share in and partake of the affections of the body to which they belong

<sup>1</sup>Timaeus, 27

<sup>2</sup>The word used by Nicomachus, *ἐξωτάτα*, once employed by Aristotle in the *Physics* 1098<sup>b</sup> 20 ff



the like, which are properly and peculiarly called 'magnitudes', others are discontinuous, in a side-by-side arrangement, and, as it were, in heaps, which are called "multitudes," a flock, for instance, a people, a heap, a chorus, and the like

[5] Wisdom, then, must be considered to be

starts from a definite root and never ceases in ceasing and magnitude, when division beginning with a limited whole is carried on, cannot bring the dividing process to an end, but proceeds therefore to infinity—and since sciences

of the parts About geometry, indeed, and arithmetic and astronomy, they have handed down to us a clear understanding and not least also about music For these seem to be sister sciences, for they deal with sister subjects, the

from multitude and size set off from magnitude

### CHAPTER III

half times," 'one and one-third times' and so forth it is clear that

[2] And once more, inasmuch as part of "size" is in a state of rest and stability and another part in motion and revolution, two other sciences in the same way will accurately treat of 'size,' geometry the part that abides and is at rest, astronomy that which moves and revolves

[3] Without the aid of these, then, it is not possible to deal accurately with the forms of

ness of theory, so in truth lines numbers, harmonic intervals, and the revolutions of circles

pear one to him who studies rightly, and what we say will properly appear if one studies all

the way I describe, him I for my part will not call that a true man

to those comprehended by the mind and un

souls, and above all to the reason which is in our souls

[7] And likewise in Plato's *Republic*, when the interlocutor of Socrates appears to bring certain plausible reasons to bear upon the mathematical sciences, to show that they are useful to human life arithmetic for reckoning distributions, contributions, exchanges, and partnerships, geometry for sieges the founding of cities and sanctuaries, and the partition

of land, music for festivals, entertainment, and the worship of the gods, and the doctrine of the

because you seem to fear that these are useless studies that I recommend, but that is very difficult, nay, impossible. For the eye of the soul, blinded and buried by other pursuits, is rekindled and aroused again by these and these alone, and it is better that this be saved than thousands of bodily eyes, for by it alone is the truth of the universe beheld!"

## CHAPTER IV

[1] Which then of these four methods must we first learn? Evidently, the one which naturally exists before them all, is superior and takes the place of origin and root and, as it were, of mother to the others. [2] And thus is arithmetic not solely because we said that it existed before all the others in the mind of the creating God like some universal and exemplary plan,

other sciences with itself," but is not abolished together with them. For example, "animal" is naturally antecedent to "man," for abolish "animal" and "man" is abolished, but if "man" be abolished, it no longer follows that "animal" is abolished at the same time. And again, "man" is antecedent to "schoolteacher," for if "man" does not exist, neither does "schoolteacher," but if "schoolteacher" is nonexistent, it is still possible for "man" to be. Thus since it has the property of abolishing the other ideas with itself, it is likewise the older.

[3] For this always implies man. Again, take "horse," "animal" is always implied along with "horse," but not the reverse, for if "animal" exists, it is not necessary that "horse" should exist, nor if "man" exists, must "musician" also be implied.

[4] So it is with the foregoing sciences, if geometry exists, arithmetic must also needs be implied, for it is with the help of this latter that we can speak of triangle, quadrilateral, octahedron, icosahedron, double, eightfold, or one and one-half times, or anything else of the sort which is used as a term by geometry, and such things cannot be conceived of without the numbers that are implied with each one. For how can "triple" exist, or be spoken of, unless the number 3 exists beforehand, or "eightfold" without 8? But on the contrary 3, 4, and the rest might be without the figures existing to which they give names. [5] Hence arithmetic abolishes geometry along with itself, but is not abolished by it, and while it is implied by geometry, it does not itself imply geometry.

## CHAPTER V

[1] And once more is this true in the case of music, not only because the absolute is prior to the relative, as "great" to "greater" and "rich" to "richer" and "man" to "father," but also because the musical harmonies, diatessaron, diapente, and diapason, are named for numbers, similarly all of their harmonic ratios are arithmetical ones, for the diatessaron is the ratio of 4 3, the diapente that of 3 2, and the diapason the double ratio, and the most perfect, the didiapason, is the quadruple ratio.

[2] More evidently still astronomy attains through arithmetic the investigations that pertain to it, not alone because it is later than geometry in origin—for motion naturally comes after rest—nor because the motions of the stars have a perfectly melodious harmony, but also because risings, settings, progressions, retrogressions, increases, and all sorts of phases are governed by numerical cycles and quantities.

sake of clearness

## CHAPTER VI

[1] All that has by nature with systematic method been arranged in the universe seems both in part and as a whole to have been determined and ordered in accordance with number, by the forethought and the mind of him that

<sup>1</sup>Republic 527 ff

<sup>2</sup>Plato Rep 522

<sup>3</sup>Cf below II 22 3 Cf Aristotle, Met., 1019<sup>a</sup> 1 ff

<sup>4</sup>Cf Aristotle e.g., Top., VI 144<sup>b</sup> 17 also Top., II 4 111<sup>a</sup> 25 ff

<sup>5</sup>Cf Plato, Rep., 528

created all things, for the pattern was fixed, like a preliminary sketch, by the domination of the mind of the world-

true and the eternal

[2] It must needs be, number, being set over such things as these, is only constituted, in accordance with itself

ed is knit together in course, of real things, for neither can now existent things be set in harmony, nor can things that exist, but are like one another, nor yet things that are different, but have no relation one to another. It remains, accordingly, that those things out of which a harmony is made are both real, different, and things with some relation to one another

[4] Of such things therefore, scientific number consists for the most fundamental species in it are two, embracing the essence of quantity, different from one another and not of a wholly different genus odd and even, and they are reciprocally woven into harmony with each other, inseparably and uniformly, by a wonderful and divine Nature as straightway we shall see

## CHAPTER VII

[1] Number is limited multitude or a combination of units or a flow of quantity made up of units and the first division of number is even and odd

that which can be divided into two equal parts by the intervention of a unit

[3] Now this is the definition after the ordinary conception by the Pythagorean doctrine, however, the even number is that which admits of division into the greatest and the smallest parts at the same operation, greatest in size and smallest in quantity in accordance with the natural contrariety of these two genera and the odd is that which does not allow this to be done to it, but is divided into two unequal parts

[4] In still another way, by the ancient defini-

tion, the even is that which can be divided alike into two equal and two unequal parts except that the dyad, which is its elementary form, admits but one division, that into equal parts, and in any division whatsoever it brings to light only one species of number, however it may be divided, independent of the other. The odd is a number which in any division whatsoever, which necessarily is a division into unequal parts, shows both the two species of number together, never without intermixture one with another, but always in one another's company

[5] By the definition in terms of each other, the odd is that which differs by a unit from the even in either direction, that is, toward the greater or the less, and the even is that which differs by a unit in either direction from the odd, that is, is greater by a unit or less by a unit

## CHAPTER VIII

[1] Every number is at once half the sum of the two on either side of itself, and similarly half the sum of those next but one in either direction, and of those next beyond them and so on as far as it is possible to go [2] Unity alone, because it does not have two numbers on either side of it, is half merely of the adjoining number hence unity is the natural starting point of all number

[3] By subdivision of the even, there are the even-times even the odd times even, and the even-times odd. The even times even and the even times odd are opposite to one another, like extremes, and the odd times even is common to them both like a mean term

the even times even is a number divided into two properties

of its genus and is similarly capable of division and again in the same way each of their parts divisible into two equals until the division of the successive subdivisions reaches the naturally indivisible unit [5] Take for example 64 one half of this is 32 and of this 16, and of this the half is 8, and of this 4, and of this 2, and then finally unity is half of the latter, and this is naturally indivisible and will not admit of a half

[6] It is a property of the even times even that, whatever part of it be taken, it is always

Euclid's definition is 'The even times even number is that which is measured by an even number an even number of times' *Elements* VII, Def 8

even times even in designation, and at the same time, by the quantity of the units in it, even-times even in value, and that neither of these two things will ever share in the other class [7] Doubtless it is because of this that it is called even times even, because it is itself even and always has its parts, and the parts of its parts down to unity, even both in name and in value in other words every part that it has is even times even in name and even-times even in value

[8] There is a method of producing the even-times even, so that none will escape, but all

instance, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512

[10] Now each of the numbers set forth was produced by the double ratio, beginning with unity, and is in every respect even times even, and every part that it may be found to have is always named from some one of the numbers before it in the series, and the sum of units in this part is the same as one of the numbers before it, by a system of mutual correspondence, indeed and interchange If there is an even number of terms

on either side of the means then to the next on either side, until it comes to the extreme terms, so that the whole will correspond in value to unity and unity to the whole For example, if

extremes unity = one one hundred twenty-eighth, and conversely 128 = the whole, to correspond with unity

[11] If, however, the series consists of an odd number of terms seven for example, and we deal with 64, there will be of necessity one mean term in accordance with the nature of the odd, the mean term will correspond to it-

self because it has no partner, and those on either side of it in turn will correspond to one another until this correspondence ends in the

eighth part, with nothing else to correspond to it

[12] It is the property of all these terms when they are added together successively to be equal to the next in the series, lacking one unit, so that of necessity their summation in any way whatsoever will be an odd number, for that which fails by a unit of being equal to an even number = odd [13] This observation will be of use to us very shortly in the construction of perfect numbers<sup>1</sup> But to take an exam-

the sums of those below them are similarly related Thus unity itself<sup>2</sup> is within one unit of equaling the next term, which is 2, and these two together fail by 1 of equaling the next, and the three together are within 1 of the next in order, and you will find that this goes on with out interruption to infinity

[14] This too it is very needful to recall If the number of terms of the even times even series dealt with is even, the product of the extremes will always be equal to the product of the means if there is an odd number of terms, the product of the extremes will be equal to the square of the mean For, in the case of an even number of terms, 1 times 128 is equal to 8 times 16 and further to 2 times 64 and again to 4 times 32, and this = so in every case, and with an odd number of terms, 1 times 64 equals 8 times 8, and this equals 4 times 16, and this again equals 2 times 32, the mean term alone multiplied by itself

## CHAPTER IX

[1] The even times odd number is one which

the genus common to it and the even, the halves are not

<sup>1</sup>See Chapter 16

<sup>2</sup>Cf on I 8 7

[2] It is the property of the even times odd that whatever factor it may be discovered to

part never by any means is of the same genus as its name. To take a single example, the number 18, its half, with an even name, is 9, odd in

as you care to proceed. The greater terms always differ by 4 from the next smaller ones, the reason for which is that their original basic

by a difference of 4, passing over three terms, and produced by the multiplication of the odd numbers by 2.

[6] They are said to be opposite in properties to the even times even, because of these

parts from extremes to mean term or terms makes the product of the former equal to the

terms, or if there should be two means their sum equals that of the two extremes.

## CHAPTER X

tioned species like a single mean between two extremes, for in one respect it resembles the even times even, and in another the even times odd and that property wherein it varies from the one it shares with the other, and by that property which it shares with the one it differs from the other.

[2] The odd times even number is an even number which can be divided into two equal parts, whose parts also can so be divided, and sometimes even the parts of its parts, but it cannot carry the division of its parts as far as

as far as the naturally indivisible unit

even

[4] It alone has at once the proper qualities of each of the former two<sup>2</sup> and then again properties which belong to neither of them, for of them one had only the highest term divisible, and the other only the smallest indivisible, but this neither, for it is observed to have more divisions than one in the greater term, and more than one indivisible in the lesser.

[6] Furthermore, there are in it certain parts whose names are not opposed to their values nor of the opposite genus,<sup>3</sup> after the fashion of the even times even and there are also always other parts of a name opposite and contrary in kind to their values after the fashion of the even times odd. For example, in 24, there are parts not opposed in name to their values, the fourth part, 6, the half, 12, the sixth, 4, and the twelfth, 2 but the third part, 8, the eighth, 3, and the twenty-fourth, 1, are opposed, and so it is with the rest.

[6] This number is produced by a somewhat complicated method, and shows, after a fashion, even in its manner of production, that it is a mixture of both other kinds. For whereas the even times even is made from even numbers, the doubles from unity to infinity, and the even times odd from the odd numbers from 3, progressing to infinity, this must be woven

<sup>2</sup>Cf I 9 6

<sup>3</sup>Cf I 8 7, 9 2

series

3, 5, 7, 9, 11, 13, 15, 17, 19, . . .

and the even times even, beginning with 4, again one after another in a second series after their own order

4, 8, 16, 32, 64, 128, 256, . . .

the resulting numbers, then again multiply by the second number of the same series the same numbers once more, as far as you can, and write down the results, then with the third number again multiply the same terms anew, and however far you go you will get nothing but the odd times odd, . . .

we, which we must note down in one line. Then taking a new group of six.

Odd numbers	3	5	7	9	11	13	15
Even times even	4	8	16	32	64	128	256
Odd times even numbers,	12	24	48	96	192	384	768
	20	40	80	160	320	640	1280
	28	56	112	224	448	896	1792
	36	72	144	288	576	1152	2304
	44	88	176	352	704	1408	2816

Length

112, 224, and in the same way as far as you care to go, you will get similar results

[10] Now when you arrange the products of multiplication by each term in its proper line, making the lines parallel, in marvelous fashion there will appear along the breadth of the table the peculiar property of the even times odd,

term, or their product, should there be two. Thus this one species has the peculiar properties of them both, because it is a natural mixture of them both

## CHAPTER XI

[1] Again, while the odd is distinguished over against the even in classification and has nothing in common with it, since the latter is divisible into equal halves and the former is not thus divisible, nevertheless there are found three species of the odd,<sup>1</sup> differing from one another, of which the first is called the prime and incomposite,<sup>2</sup> that which is opposed to it the secondary and composite, and that which is midway between both of these and is viewed as a mean among extremes, namely, the variety which, in itself, is secondary and composite, but relatively is prime and incomposite

[2] Now the first species, the prime and incomposite, is found whenever an odd number admits of no other factor save the one with the number itself as denominator, which is always unity, for example, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31. None of these numbers will by any chance be found to have a fractional part with a denominator different from the number itself, but only the one with this as denominator, and this part will be unity in each case, for 3 has only a third part, which has the same denominator as the number and is of course uni-

ty, 5 a fifth, 7 a seventh, and 11 only an eleventh part, and in all of them these parts are unity

[3] It has received this name because it can be measured only by the number which is first and common to all, unity, and by no other,

accordance with their own quantity. To be sure, when they are combined with themselves, other

<sup>1</sup> Cf. Euclid, *Elem.* VII, Def. 11-14

<sup>2</sup> Cf. Euclid, *Elem.*, VII, Def. 11, Aristotle, *VIII* 3 17<sup>a</sup> 39

square of the mean, should there be one mean

numbers might be produced originating from them as from a fountain and a root, wherefore they are called "prime," because they exist beforehand as the beginnings of the others. For

anything

## CHAPTER XII

elementary quality, for it gets its origin by the combination of something else. For this reason it is characteristic of the secondary number to have, in addition to the fractional part with the number itself as denominator, yet another part or parts with different denominators, the former always, as in all cases, unity, the latter never unity, but always either that number or those numbers by the combination of which it was produced. For example, 9, 15, 21, 25, 27, 33, 35, 39 each one of these is measured by unity, as other numbers are, and like them has a fractional part with the same denominator as the number itself, by the nature of the class common to them all but by exception and more peculiarly they also employ a part, or parts with a different denominator. 9, in addition to the ninth part has a third part besides. 15 a third and a fifth besides a fifteenth. 21 a seventh and a third besides a twenty first, and 25, in addition to the twenty fifth, which has as a denominator 25 itself, also a fifth with a different denominator.

[2] It is called secondary, then, because it can employ yet another measure along with unity, and because it is not elementary, but is produced by some other number combined with itself or with something else in the case of 9, 3 in the case of 15, 5 or, by Zeus, 3, and those following in the same fashion. And it is called composite for this or some such reason that it may be resolved into those numbers out of which it was made, since it can also be measured by them. For nothing that can be broken down is incomposite but by all means composite.

## CHAPTER XIII

[1] Now while these two species of the odd are opposed to each other a third one<sup>2</sup> is con-

ceived of between them, deriving as it were, its specific form from them both, namely the number which is in itself secondary and composite, but relatively to another number is prime and incomposite. This exists when a number, in addition to the common measure, unity, is measured by some other number and is therefore able to admit of a fractional part, or parts with denominator other than the number itself, as well as the one with itself as denominator. When this is compared with another number of similar properties, it is found that it cannot be measured by a measure common to the other, nor does it have a fractional part with the same denominator as those in the other. As an illustration, let 9 be compared with 25. Each in itself is secondary and composite, but relatively to each other they have only unity as a common measure, and no factors in them have the same denominator, for the third part in the former does not exist in the latter nor is the fifth part in the latter found in the former.

[2] The production of these numbers is called by Eratosthenes the "sieve," because we take the odd numbers mingled together and indiscriminate and out of them by this method of production separate, as by a kind of instrument or sieve, the prime and incomposite by themselves, and the secondary and composite by themselves and find the mixed class by themselves.

[3] The method of the "sieve" is as follows. I set forth all the odd numbers in order, beginning with 3, in as long a series as possible, and then starting with the first I observe what ones it can measure, and I find that it can measure the terms two places apart, as far as we care to proceed. And I find that it measures not as it chances and at random but that it will measure the first one that is the one two places removed, by the quantity of the one that stands first in the series that is by its own quantity, for it measures it 3 times and the one two places from this by the quantity of the second in order, for this it will measure 5 times and again the one two places further on by the quantity of the third in order or 7 times, and the one two places still farther on by the quantity of the fourth in order or 9 times and so ad infinitum in the same way.

four places apart the first by the quantity of the first in order, or 3 times the second by that of the second, or 5 times, the third by that of

<sup>1</sup>Cf Euclid, *Elements* VII Def 14

<sup>2</sup>Cf Euclid, *Elements*, VII Def 13

the third, or 7 times, and in this order *ad infinitum*

[5] Again, as before, the third term 7, taking over the measuring function, will measure terms six places apart, and the first by the quantity of 3, the first of the series, the second by that of 5, for this is the second number, and the third by that of 7, for this has the third position in the series

[6] And analogously throughout, this process will go on without interruption, so that the numbers will succeed to the measuring function in accordance with their fixed position in the series, the interval separating terms measured is determined by the orderly progress of the even numbers from 2 to infinity, or by the doubling of the position in the series occupied by the measuring term, and the number of times a term is measured is fixed by the orderly advance of the odd numbers in series from 3

[7] Now

Therefore

ure in accordance with its own quantity will have but one fractional part with denominator

are measured by two measures at the same time, will have several fractional parts with other denominators besides the one with the same as the number itself, these will be secondary and composite

[8] The third division, the one common to both the former, which is in itself secondary and composite but primary and incomposite in relation to another, will consist of the numbers

its own quantity, for it is 5 times 5, these numbers have no common measure except unity

[10] We shall now investigate how we may have a method of discerning whether numbers are prime and incomposite, or secondary and composite, relatively to each other, since of the former unity is the common measure, but of the latter some other number also besides uni-

us to determine whether they are prime and incomposite relatively to each other or secondary and composite, and if they are secondary and composite, what number is their common measure We must compare the given numbers and subtract the smaller from the larger as many times as possible, then after this subtraction, subtract in turn from the other as many times as possible for this changing about and subtraction from one and the other in turn will necessarily end either in unity or in some one and the same number, which will necessarily be odd [12] Now when the subtractions terminate in unity they show that the numbers are prime and incomposite relatively to each other, and when they end in some other number, odd in quantity and twice produced, then say that they are secondary and composite relatively to each other, and that their common measure is that very number which twice appears

For example, if the given numbers were 23 and 45, subtract 23 from 45, and 22 will be the remainder, subtracting this from 23, the remainder is 1, subtracting this from 22 as many times as possible you will end with unity Hence they are prime and incomposite to one another, and unity, which is the remainder, is their common measure

secondary and composite relatively to each



other, and 7 their common measure in addition to the universal unit

## CHAPTER XIV

[1] To make again a fresh start, of the simple even numbers some are superabundant, some deficient, like extremes set over against each other, and some are intermediary between them and are called perfect [2] Those which are said

reactions of the greater and the less for apart from these no other form of inequality could be conceived, nor could evil,<sup>2</sup> disease, disproportion, unseemliness nor any such thing, save in terms of excess or deficiency I or in the realm of the greater<sup>3</sup> there arise excesses, overreaching, and superabundance, and in the less need, deficiency, privation, and lack, but in that which lies between the greater and the less, namely, the equal, are virtues, wealth, moderation, propriety, beauty, and the like, to which the aforesaid form of number, the perfect, is most akin

[3] Now the superabundant number is one which has, over and above the factors which

the poet says,<sup>4</sup> and ten mouths, or with nine lips, or three rows of teeth, or a hundred hands, or too many fingers on one hand Similarly if, when all the factors in a number are examined and added together in one sum, it proves upon investigation that the number's own factors exceed the number itself, this is called a superabundant number, for it oversteps the symmetry which exists between the perfect and its

they make 36, which, compared to the original number, 24, is found to be greater than it, although made up solely of its factors Hence in this case also the parts are in excess of the whole

<sup>1</sup>Cf I 17 2, 4, 6, also I 23 4

<sup>2</sup>Cf Arist., *Ethics* II 6 1106<sup>b</sup> 33

<sup>3</sup>Cf Arist., *Ethics* II 6 1106<sup>b</sup> 24, 33.

<sup>4</sup>Homer, *Odyssey*, XII 85 ff

## CHAPTER XV

[1] The deficient number is one which has qualities the opposite of those pointed out, and whose factors added together are less in comparison than the number itself It is ■ if some animal should fall short of the natural number ■ of limbs or parts or as if a man should have but one eye, as in the poem, "And one round orb was fixed in his brow", or ■ though one should be one-handed, or have fewer than five fingers on one hand, or lack a tongue, or some such member Such a one would be called deficient and so to speak maimed, after the peculiar fashion of the number whose factors are less than itself, such as 8 or 14 For 8 has the factors half, fourth, and eighth, which are 4, 2, and 1, and added together they make 7, and

make 10, less than the original number so this number also is deficient in its parts, with respect to making up the whole out of them

## CHAPTER XVI

[1] While these two varieties are opposed after the manner of extremes, the so-called perfect number<sup>5</sup> appears as a mean, which is discovered to be in the realm of equality, and

were, moderation between excess and deficiency, and that which is in tune, between pitches too high and too low

[2] Now when a number, comparing with itself the sum and combination of all the factors whose presence it will admit, neither exceeds them in multitude nor is exceeded by them, then such a number ■ properly said to be perfect, as one which is equal to its own parts Such numbers are 6 and 28 for 6 has the factors half, third, and sixth, 3, 2, and 1, respectively, and these added together make 6

<sup>5</sup>Fuelid's definition *Flem*, I II 22 is "A perfect number is one that is equal to its own parts."

parts greater than the whole nor the whole greater than the parts, but their comparison is in equality, which is the peculiar quality of the perfect number

[3] It comes about that even as fair and excellent things are few and easily enumerated, while ugly and evil ones are widespread, so also the perfect numbers are few and easily enumerated.

erated and arranged with suitable order, for only one is found among the units, 6, only one other among the tens, 28, and a third in the rank of the hundreds, 496 alone and a fourth within the limits of the thousands, that is, below ten thousand, 8,128. And it is their accompanying characteristic to end alternately in 6 or 8 and always to be even.

[4] There is a method of producing them,<sup>1</sup> neat and unailing, which neither passes by any of the perfect numbers nor fails to differentiate any of those that are not such, which is carried out in the following way

You must set forth the even times even numbers from unity, advancing in order in one line, as far as you please 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024, 2,048, 4,096. Then you must add them together, one at a time, and each time you make a summation observe the result to see what it is. If you find that it is a

prime and incomposite, multiply it by the last term added, and the result will be a perfect

demonstrations, for it has no factor with denominator different from the number itself but only that with denominator agreeing. Therefore I multiply it by the last number to be taken into the sum that is, 2. I get 6, and thus I declare to be the first perfect number in actuality, and to have those parts which are beheld

in the numbers of which it is composed. For it will have unity as the factor with denominator the same as itself, that is, its sixth part, and 3 as the half, which is seen in 2, and conversely 2 as its third part.

[5] Twenty-eight likewise is produced by the same method when another number, 4, is added to the previous ones. For the sum of the three, 1, 2, and 4, is 7, and is found to be prime and incomposite, for it admits only the factor with denominator like itself, the seventh part. Therefore I multiply it by the quantity of the term last taken into the summation and my result is 28, equal to its own parts, and having its factors derived from the numbers already adduced, a half corresponding to 2, a fourth to 7, a seventh to 4, a fourteenth to offset the half, and a twenty-eighth, in accordance with its own nomenclature, which is 1 in all numbers.

[6] When these have been discovered, 6 among the units and 28 in the tens, you must do the same to fashion the next. [7] Again add the next number, 8, and the sum is 15. Observing this, I find that we no longer have a prime and incomposite number, but in addition to the factor with denominator like the number itself, it has also a fifth and a third, with unlike denominators. Hence I do not multiply it by 8, but add the next number, 16, and 31 results. As this is a prime, incomposite number, of necessity it will be multiplied in accordance with the general rule of the process by the last number added, 16, and the result is 496, in the hundreds and then comes 8,128 in the thousands, and so on, as far as it is convenient for one to follow.

[8] Now unity is potentially a perfect number but not actually, for taking it from the series as the very first I observe what sort it is, according to the rule, and find it prime and incomposite for it is so in very truth, not by participation like the rest, but it is the primary number of all, and alone incomposite. [9] I multiply it, therefore by the last term taken

## CHAPTER XVII

[1] Now that we have given a preliminary systematic account of absolute quantity we come in turn to relative quantity.

[2] Of relative quantity, then the highest generic divisions are two, equality and inequality.

<sup>1</sup> Cf. Euclid (IX 36)

ity, for everything viewed in comparison with another thing is either equal or unequal, and there is no third thing besides these

[3] Now the equal is seen, when of the things compared one neither exceeds nor falls short in comparison with the other, for example, 100 compared with 100, 10 with 10, 2 with 2, a mina with a mina, a talent with a talent, a cubit with a cubit, and the like, either in bulk, length, weight, or any kind of quantity [4] And as a peculiar characteristic, also this relation is of itself not to be divided or separated, as being most elementary, for it admits of no difference For there is no such thing as this kind of equality and that kind, but the equal exists in one and the same manner [5] And that which corresponds to an equal thing to be sure, does not have a different name from it, but the same, like "friend," "neighbor," "comrade," so also "equal," for it is equal to an equal

[6] The unequal, on the other hand, is split up by subdivisions, and one part of it is the greater, the other the less, which have opposite names and are antithetical to one another in their quantity and relation For the greater is greater than some other thing, and the less again is less than another thing in comparison, and their names are not the same, but they each have different ones, for example, "father" and "son," "striker" and "struck," "teacher" and "pupil," and the like

[7] Moreover, of the greater, separated by a second subdivision into five species, one kind is the multiple another the superparticular, another the superpartient, another the multiple superparticular, and another the multiple superpartient [8] And of its opposite, the less, there arise similarly by subdivision five species, opposed to the foregoing five varieties of the greater, the submultiple, subsuperparticular, subsuperpartient submultiple superparticular, and submultiple superpartient for as a whole answers to a whole, smaller to greater, so also the varieties correspond, each to each, in the aforesaid order, with the prefix sub-

## CHAPTER XVIII

[1] Once more then the multiple is the species of the greater first and most original by nature, as straightway we shall see, and it is a number which, when it is observed in comparison with another, contains the whole of that number more than once For example, compared with unity, all the successive numbers beginning with 2 generate in their proper order the regular forms of the multiple for 2, in the

first place, is and is called the double, 3 triple, 4 quadruple, and so on, for "more than once" means twice, or three times, and so on in succession as far as you like

[2] Answering to this is the submultiple, which is itself primary in the smaller division of inequality It is the number which, when it is compared with a larger, is able to measure it completely more than once, and "more than once" starts with twice and goes on to infinity [3] If then it measures the larger number that is being compared twice only, it is properly called the subdouble, as 1 is of 2, if thrice, subtriple, as 1 of 3, if four times, subquadruple, as 1 of 4, and so on in succession

[4] While each of these, the multiple and the submultiple, is generically infinite, the varieties by subdivision and the species also are observed naturally to make an infinite series For the double, beginning with 2, goes on through all the even numbers, as we select alternate numbers out of the natural series and these will be called doubles in comparison with the even and odd numbers successively placed beginning with unity [5] All the numbers from the beginning two places apart, and third in order, are triples, for example, 3, 6, 9, 12, 15, 18, 21, 24 It is their property to be alternately odd and even, and they themselves in the regular series from unity are triples of all the numbers in succession as far as one wishes to go on with the process

[6] The quadruples are those in the fourth places, three apart, for instance, 4, 8, 12, 16, 20, 24, 28, 32, and so on These are the quadruples of the regular series of numbers from unity going on as far as one finds it convenient to follow. It belongs to them all to be even, for one needs only to take the alternate terms out of the even numbers already selected Thus necessarily it is true that the even numbers, with no further designation, are all doubles, the alternate ones quadruples, those two places apart sextuples and those three places apart octuples, and this series will go on, on this same analogy, indefinitely

[7] The quintuples will be seen to be those four places apart, placed fifth from one another, and themselves the quintuples of the successive numbers beginning with unity Alternately they are odd and even, like the triples

## CHAPTER XIX

[1] The superparticular, the second species of the greater both naturally and in order, is a

number that contains within itself the whole  
of the —

ter, and the smaller subequalter, if it is a third, sesquitertian and subsesquitertian, and as you go on throughout it will always thus agree so that these species also will progress to infinity, even though they are species of an unlimited genus.

For it comes about that the first species, the sesquialter ratio, has as its consequents the even numbers in succession from 2, and no other at all, and as antecedents the triples in succession from 3, and no other [9] These must be joined together regularly, first to first, second to second, third to third—3 2, 6 4 9 6, 12 8—and the analogous numbers to the ones corresponding to them in position.

[4] If we care to investigate the second species of the superparticular, the sesquitertian (for the fraction naturally following after the half is the third), we shall have this definition of it—a number which contains the whole of the number compared, and a third of it in addition to the whole. We may have examples of it, in the proper order, in the successive quadruples beginning with 4 joined to the triples from 3, each term with the one in the corresponding position in the series, for example, 4 3 8 6, 12 9, and so on to infinity [5] It is plain that that which corresponds to the sesquitertian but is called with the prefix sub-, subsesquitertian, is the number, the whole of which is contained and a third part in addition, for example, 3 4, 6 8, 12 16, and the similar pairs of numbers.

number the third two the fourth three the fifth four and so on as far as you like [7]

greater

[8] That by nature and by no disposition of ours the multiple is a more elementary and an older form than the superparticular we shall shortly learn through a somewhat intricate process. And here, for a simple demonstration, we must prepare in regular and parallel lines the multiples specified above according to their

varieties, first the double in one line, then in a second the triple, then the quadruple in a third, and so on as far as the tenfold multiples, so that we may detect their order and variety, their regulated progress, and which of them is naturally prior, and indeed other corollaries delightful in their exactness. Let the diagram be as follows [9]

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

[10] Let there be set forth in the first row the natural series from unity, and then in order those species of the multiple which we were bidden to insert.

[11] Now then in comparison with the first rows beginning with unity, if we read both across and up and down in the form of the letter gamma, the next rows both ways, themselves in the form of a gamma beginning with 4, are multiples according to the first form of the multiple, for they are doubles. The first differs by unity from the first, the second from the second by 2, the third from the third by 3, the next by 4, those following by 5, and you

structure of the diagram in places just above the quadruples, and in the subsequent forms of the multiple the analogy will hold throughout

[14] In comparison with the second line

order beneath display the first species of the superparticular, that is, the sesquialter, between terms occupying corresponding places. Thus by divine nature, not by our convention or agreement, the superparticulars are of later origin than the multiples. For illustration, 3 is the sesquialter of 2, 6 of 4, 9 of 3, 12 of 6, 15 of 10, and throughout thus. They have as a difference the successive numbers from unity, like those before them.

[15] The sesquiterterians, the second species of superparticular, proceed with a regular, even advance from 4, 3, 8, 6, 12, 9, 16, 12, and so on, having also a regular increase of their differences. [16] And in the other multiple and superparticular relations you will see that the results are in harmony and not by any means

the corners are units, the one at the beginning a simple unit, that at the end the unit of the third course, and the other two units of the

either way there is an even progress from unity to the tens and again on the opposite sides two other progressions from 10 to 100.

[17] The terms on the diagonal from 1 to 100 are all square numbers, the products of equals by equals and those flanking them on either side are all heteromecic unequal and the products of sides of which one is greater than the other by unity and so the sum of two successive squares and twice the heteromecic numbers between them is always a square and conversely a square is always produced from the two heteromecic numbers on the sides and twice the square between them.

[20] An ambitious person might find many other pleasing things displayed in this diagram upon which it is not now the time to dwell for we have not yet gained recognition of them from our Introduction and so we must turn to the next subject. For after these two generic relations of the multiple and the superparticular and the other two, of points to them, with the prefix sub-, the submultiple and the sub-

superparticular, there are in the greater division of inequality the superpartient, and in the less its opposite, the subsuperpartient.

## CHAPTER XX

[1] It is the superpartient relation when a number contains within itself the whole of the number compared and in addition more than one part of it, and "more than one" starts with 2 and goes on to all the numbers in succession. Thus the root-form of the superpartient is naturally the one which has in addition to the whole two parts of the number compared, and as a species will be called superbipartient, after this the one with three parts besides the whole will be called supertripartient as a species then comes the superquadrupartient, the superquintupartient, and so forth.

[2] The parts have their root and origin with the third, for it is impossible in this case to begin with the half. For if we assume that any number contains two halves of the compared number, besides the whole of it, we shall inadvertently be setting up a multiple instead of a superpartient, because each whole, plus two

not the problem laid before us nor in accord with the systematic construction of our science.

[3] After the superpartient the subsuperpartient immediately is produced, whenever a number is completely contained in the one compared with it, and in addition several parts of it, 2, 3, 4, or 5, and so on.

## CHAPTER XXI

[1] The regular arrangement and orderly production of both species are discovered when we set forth the successive even and odd numbers in the following manner:

—that is, 7 to 4,—third to third—that is, 9 to 5,—fourth to fourth—that is, 11 to 6,—and so on in the same order as far as you like. In this way the forms of the super-

quanti-partient, and further in succession in similar manner, for after the root-forms of each species the ones which follow them will be produced by doubling, or tripling, both the terms, and in general by multiplying after the regular forms of the multiple

in the first part, that is, the multiple, it will have double, triple, quadruple, quintuple, and

TABLE OF THE SUPERPARTIENTS

Root-forms	5	3	7	4	9	5	11	11	13	7
	10	6	14	8	18	10	22	12	26	14
	15	9	21	12	27	15	33	18	39	21
	20	12	28	16	36	20	44	24	52	28
	25	15	35	20	45	25	55	30	65	35
	30	18	42	24	54	30	66	36	78	42
	35	21	49	28	63	35	77	42	91	49
	40	24	56	32	72	40	88	48	104	56
	45	27	63	36	81	45	99	54	117	63

[2] It must be observed that from the two parts in addition to the whole which are contained in the greater term, we are to understand "third," in the case of three parts, "fourth," with four parts, "fifth," with five, "sixth" and so on, so that the ratio of the

so forth, and in the second part, generically from the superparticular, its specific forms in due order, the sesquialter, sesquitercian, sesquiquartan, sesquiquintan, and so on, so that the combination will proceed in somewhat this order

Double sesquialter, double sesquitercian, double sesquiquartan, double sesquiquintan, double sesquiseptan, and analogously

Beginning once more triple sesquialter, triple sesquitercian, triple sesquiquartan, triple sesquiquintan

Again quadruple sesquialter, quadruple sesquitercian, quadruple sesquiquartan, quadruple sesquiquintan

Again quintuple sesquialter, quintuple sesquitercian, quintuple sesquiquartan, quintuple sesquiquintan, and the forms analogous to these *ad infinitum*. Whatever number of times the greater contains the whole of the smaller, by this quantity the first part of the ratio of the terms joined together in the multiple superparticular is named, and whatever may be the factor, in addition to the whole several times contained, that is, in the greater term, from the same

tions of relative quantity are these which have been enumerated. Those which are compounded of them and as it were woven out of two into one are the following, of which the antecedents are the multiple superparticular and multiple superpartient, and the consequents the ones that immediately arise in connection with each of the former, named with the prefix

prefix sub-

## CHAPTER XXII

[2] As a compound, such a number is doubly diversified after the peculiarities of nomenclature of its components on either side, for inasmuch as the multiple superparticular is composed of the multiple and superparticular ge-

The successive terms beginning with 5 and differing by 5 will be without exception double sesquialters of all the successive even numbers from 2 on, when terms in the same position in the series are compared, and beginning with 3, if all those with a difference of 3 be set forth, as 3, 6, 9, 12, 15, 18, 21, and in another series there be set forth those that differ by 7, to infinity, as 7, 14, 21, 28, 35, 42, 49, and the greater be compared with the smaller, first to first, second to second, third to third, fourth to fourth, and so on, the second species will appear, the double sesquiquartian, disposed in its proper order

[4] Then again, to take a fresh start, if the simple series of quadruples be set forth, 4, 8, 12, 16, 20, 24, 28, 32, and then there be placed beside it in another series the successive numbers beginning with 9, and increasing by 9, as 9, 18, 27, 36, 45, 54, we shall have revealed once more the multiple superparticular in a specific form, that is, the double sesquiquartian in its proper order and any one who desires can contrive this to an unlimited extent

[5] The second kind begins with the triple sesquialter, such as 7 2, 14 4, and in general the numbers that advance by steps of 7 compared with the even numbers in order from 2

[6] Then once more, 10 3 is the first triple sesquiquartian, 20 6 the second and, in a word, the multiples of 10 in succession, compared with the successive triples. Thus indeed we can observe with greater exactitude and clearness in the table studied above, for in comparison with the first row the succeeding rows in order, compared as whole rows, display the forms of the multiple in regular order up to infinity when they are all compared in each case to the same first row and when each row is compared to all those above it, in succession, the second row being taken as our starting point, all the forms of the superparticular are produced in their proper order and if we start with the third row, all of those beginning with the fifth that are odd in the series when they are compared with this same third row, and those following it, will show all the forms of the superpartient in proper order. In the case of the multiple superparticular, the comparisons will have a natural order of their own if we start with the second row and compare the terms from the fifth, first to first, second to second, third to third, and so on, and then the terms of the seventh row to the third, those of the ninth to the

fourth, and follow the corresponding order as far as we are able to go [7] It is plain that here too the smaller terms have names corresponding to the larger ones, with the prefix sub-, according to the nomenclature given them all

## CHAPTER XXIII

[1] The multiple superpartient in the remaining relation of number. This, and the relation called by a corresponding name with the prefix sub-, exist when a number contains the whole of the number compared more than once (that is, twice, three, or any number of times) and certain parts of it, more than one, either two, three, or four, and so on, besides [2] These parts are not halves, for the reasons mentioned above, but either thirds, fourths, or fifths, and so on

[3] From what has already been said it is not hard to conceive of the varieties of this relation, for they are differentiated in the same way as, and consistently with, those that precede, double superpartient, double superpartient, double superquadrupartient, and so on. For example, 8 is the double superbipartient of 3, 16 of 6, and in general the numbers beginning with 3 and differing by 8 are double superbipartients of those beginning with 3 and differing by 3, when those in corresponding places in the series are compared, and in the case of the other varieties one could ascertain their proper sequence by following out what has already been said. In this case, too, we must conceive that the nomenclature of the number compared goes along and suffers corresponding changes, with the addition of the prefix sub-

[4] Thus we come to the end of our speculation upon the ten arithmetical relations for a first Introduction. There is, however, a method very exact and necessary for all discussion of the nature of the universe which very clearly and indisputably presents to us the fact that that which is fair and limited, and which subjects itself to knowledge, is naturally prior to the unlimited incomprehensible, and ugly, and furthermore that the parts and varieties of the infinite and unlimited are given shape and boundaries by the former, and through it attain to their fitting order and sequence, and like objects brought beneath some sort of measure, all gain a share of likeness to it and similarity of name when they fall under its influence. For thus it is reasonable that the rational

<sup>1</sup>Referring to the table in Chapter 19

<sup>2</sup>See 20 2 above

<sup>3</sup>Cf 1 2 5

part of the soul will be the agent which puts in order the irrational part, and passion and ap-

the principle that pertains to these universal matters. It is capable of proving that all the complex species of inequality and the varieties of these species are produced out of equality, first and alone, as from a mother and root.

[7] Let there be given us equal numbers in three terms, first, units, then two's in another group of three, then three's, next four's, five's, and so on as far as you like. For them, as the setting forth of these terms has come about by

and then the quintuple, and, following the order we have previously recognized, *ad infinitum*, second, the superparticular, and here again the first form, the sesquialter, will lead, and the next after it, the sesquitercian, will follow, and after them the next in order, the sesquiquartan, the sesquiquintan, the sesquisextan, and so on *ad infinitum*, third, the superpartient, which once more the superbipartient will lead, the supertripartient will follow immediately upon it, and then will come the superquadrupartient, the superquintupartient, and according to the foregoing as far as one may proceed.

[8] Now you must have certain rules, like invariable and inviolable natural laws, following which the whole aforesaid advance and progress from equality may go on without failure. These are the directions. Make the first equal to the first, the second equal to the sum of the first and second, and the third to the sum of the first, twice the second, and the third. For if you fashion according to these rules you

your paying any heed or offering any aid. From equality you will first get the double, from the double the triple, from the triple successively

start, if the superparticulars are set forth in the order of their production, but with terms

tient from the sesquitercian, the superquadrupartient from the sesquiquartan, and so on *ad infinitum*. [11] If, however, the superparticu-

from the third, the sesquiquartan, and so on. [12] From those produced by the reversal of the superparticular, that is, the superpartients, and from those produced without such revers-

remaining numerical relations.

[13] The following must suffice as illustrations of all that has been said hitherto, the production of these numbers and their sequence, and the use of direct and of reverse order. [14] From the relation and proportion in terms of the sesquialter, reversed so as to begin with the largest term, there arises a relation in superpartient ratios, the superbipartient, and from

relation in terms of sesquitercians, beginning with the greatest term, is derived a superpartient, the supertripartient, beginning with the smallest term, a double sesquitercian. For example, from 16, 12, 9 comes either 16, 28, 49 or 9, 21, 49. And from the relation in terms of sesquiquartans, when it is arranged with the largest term, is derived a

<sup>1</sup>Cf. Aristotle, *Ethics*, 1107<sup>b</sup> 4 ff. See *ibid.*, 1108<sup>a</sup> 4 ff., 1145<sup>b</sup> 8 ff.



tient, the superquadrupartient, when it starts with the smallest term, a multiple superparticular, the double sesquiquantal, for instance, from 25, 20, 16 comes either 25, 45, 81 or 16, 36, 81

[16] In the case of all these relations that are thus differentiated, and of the one from which both of the differentiated ones are derived the last term is always the same and a square, the first term becomes the smallest, and invariably the extremes are squares

[16] Moreover the multiple superpartients and superpartients of other kinds are made to appear in yet another way out of the superpartients, for example, from the superbipartient relation arranged so as to begin with the small-

est term comes the double superbipartient, but, arranged so as to start with the greatest, the superpartient ratio of 8 5 Thus from 9, 15, 25 comes either 9, 24, 64 or 25 40, 64 From the supertripartient, beginning with the smallest term, we have the double supertripartient, and, beginning with the largest, the ratio of 11 7 Thus, from 16, 28, 49 comes either 16, 44, 121 or 49, 77, 121 [17] Again, from the superquintupartient, as, for example, 25, 45, 81, beginning with the lesser term we derive the double superquintupartient in the terms 25, 70, 196 but beginning with the greater = superpartient again, the ratio of 14 9, in the terms 81, 126, 196 And you will find the results analogous and in agreement with the foregoing in all successive cases to infinity

## BOOK TWO

### CHAPTER I

[1] An element is said to be and is the smallest thing which enters into the composition of an object and the least thing into which it can be analyzed. Letters for example are called the —

verse in general are simple bodies fire water air and earth for out of them in the first instance we account for the constitution of the universe and into them finally we conceive of it as being resolved

We wish also to prove that equality is the elementary principle of relative number for

as the  $\frac{1}{2}$  we have however demonstrated that in the realm of inequality advance and increase have their origin in equality and go on to absolutely all the relations with a certain regularity through the operation of the three rules. It remains then in order to make it an element in very truth to prove that analyses also finally come to an end in equality. Let this then be considered our procedure

### CHAPTER II

[1] Suppose then you are given three terms in any relation whatsoever and in any ratio whether multiple superparticular superpartient or a compound of these multiple superparticular or multiple superpartient provided only that the mean term is seen to be in the same ratio to the lesser as the greater to the mean and vice versa. Subtract always from

the mean the lesser term whether it be first or last in order and set down the lesser term itself as the first term of your new series then put as your second term what remains from the second after the subtraction then after having subtracted the sum of the new first term and twice the new second term from the remaining number—that is the greater of the numbers originally given you—make the re-

way you subtract the remainder from these same terms it will be found that your three terms have passed back into three others more primitive and you will find that this always takes place as a consequence until they are reduced to equality whence by every necessity it appears evident that equality is the elementary principle of relative quantity

[3] There follows upon this speculation a most elegant principle extremely useful in its application to the Platonic psychogony<sup>1</sup> and the problem of all harmonic intervals for in

### CHAPTER III

der any circumstances

<sup>1</sup>See on I 23 4

<sup>2</sup>See on I 23 8.

<sup>3</sup>See Plato *Timæus* 35 ff

<sup>4</sup>Refers to the table in

[2] The doubles, then, will produce<sup>1</sup> sesquialters, the first one, the second two, the third three, the fourth four, the fifth five, the sixth six, and neither more nor less, but by every necessity when the superparticulars that are generated attain the proper number, that is, when their number agrees with the multiples that have generated them, at that point by a divine device, as it were, there is found the

Sixteen, the fourth double, will stand at the head of four sesquialters, 24, 36, 54, and finally 81, so that they may of necessity be equal in  
 $1 : 2 :: 2 : 4 :: 4 : 8 :: 8 : 16 :: 16 : 24 :: 24 : 36 :: 36 : 54 :: 54 : 81$   
 infinity.

[4] For the sake of illustration let there be set down the table of the doubles, thus:

The double ratio in the breadth of the table

	1	2	4	8	16	32	64	
		3	6	12	24	48	96	
The triple ratio along the hypotenuse			9	18	36	72	144	The sesquialter ratio in the depth
				27	54	108	216	
					81	162	324	
						243	486	
							729	

number which terminates them all because 11 naturally is not divisible by that factor whereby the progression of the superparticular ratios went on.

#### CHAPTER IV

[1] We must make a similar table in illustration of the triple:

The triple ratio in the breadth

	1	3	9	27	81	243	729	
		4	12	36	108	324	972	
The quadruple ratio on the hypotenuse			16	48	144	432	1296	The sesquitercian ratio in the depth
				64	192	576	1728	
					256	768	2304	
						1024	3072	
							4096	

From the triples all the sesquitertercians will proceed, likewise equal in number to the number of the progression.

In the foregoing table we shall observe that in the same way the first triple, 3, stands at the head of but one sesquitertercian ratio, 4, its own

sesquiquartans come from the quadruples, reaching a culmination after their independent progression in a number that is not divisible by 4.

[2] In the same way, the first quadruple, 16, stands at the head of one sesquiquartan ratio only, 5, which number is not divisible by 4.

in order is the triple 27, three times removed from 1, for the triples progress thus 1, 3, 9, 27. Therefore this number will stand at the head of three sesquitertercian ratios and no more. The first is its own, 36, the second the sesquitertercian

course, five.

as we found to be true in our previous discussion.

<sup>1</sup>See I. 19. 2.

mon, Nature shows us that the doubles are more nearly original than the triples, the triples than the quadruples, these latter than the quintuples and so on throughout. For the highest rows of figures, across the breadth of the tables, if they are doubles, will have doubles lying parallel to them, and the numbers lying diagonally on the hypotenuse, will be of the next succeeding variety, greater by 1, that is, triples, seen also in a series of parallel lines. If, however, there are triples across the breadth, the diagonals will by all means be quadruples, if the former are quadruples, then the latter are quintuples, and so forth.

### CHAPTER V

[1] It remains, after we have explained what other ratios are produced by combination of ratios, to pass on to the succeeding topics of the *Introduction*.

[2] Now the first two ratios of the superparticular, combined, produce the first ratio of the multiple, namely 1 2 3 4 5 6

a double  
For example 2 4 6 8 10 12

indeed 4, lying between 6 and 3, gives the sesquitercian ratio to 3 and the sesquialter to 6

[3] It was rightly said, then, that the double, when resolved, is resolved into the sesquialter and the sesquitercian, and that when sesquialter and sesquitercian are combined there arises the double, and that the first two forms of the superparticular combined make the first form of the multiple

[4] But again, to take another start this first form of the multiple which has thus been produced, together with the first form of the superparticular, will produce the next form of the same class, that is, the second multiple, the triple, for from every multiple and sesquialter combined a triple of necessity arises. For example, as the double of 6 is 12, and the sesquialter of this is 18, then immediately 18 is the triple of 6 and to take another method if I do not care to make 12 the mean term, but rather 9, the sesquialter of 6, the same result will come about, without deviation and harmoniously,

for while 18 is the double of 9 it will preserve the triple ratio to 6. Hence from the sesquialter and the double, the first forms of the superparticular and the multiple, there arises by combination the second form of the multiple, the triple, and into them it is always resolved [5] For look you, 6, which is the triple of 2, will have a mean term 3, which will exhibit two ratios, the sesquialter with regard to 2, and the double ratio of 3 to itself

But if this triple ratio, likewise, the second form of the multiple, is combined with the sesquitercian, which is the second form of the superparticular, there would be produced from them the next form of the multiple, namely, the quadruple, and thus also will of necessity be resolved into them after the same fashion as the cases previously set forth and the quadruple, taking into combination the sesquiquartan, will make the quintuple, and, once more, the latter with the sesquiquintan will make the sextuple, and so on to the end. Thus the multiples in regular order from the beginning with the superparticulars in regular order from the beginning will be found to produce the next larger multiples. For the double with the sesquialter makes the triple, the triple with the sesquitercian the quadruple, the quadruple with the sesquiquartan the quintuple, and as far as you wish to proceed no contrary result will appear

### CHAPTER VI

[1] Up to this point then we have sufficiently discussed relative number, by a process of selection measuring out what is easily compre-

subjects which we must first survey and observe are concerned with 1 2 3 4 5 6

since they are more closely related to magnitude, is properly given in the *Geometrical Introduction*. Yet the germs of these ideas are taken

with itself, but is not abolished by them, and conversely  $\equiv$  of necessity implied by them but does not itself imply them

[2] First, however, we must recognize that each letter by which we indicate a number, such as *iota*, the sign for 10, *kappa* for 20, and *omega* for 800, designates that number by man's convention and agreement, not by nature. On the other hand, the natural, unartificial, and therefore simplest indication of numbers would be the setting forth one beside the other of the units contained in each. For example, the writing of one unit by means of one alpha will be the sign for 1, two units side by side, that is,  $\equiv$  series of two alphas, will be the sign for 2, when three are put in a line it will be the character for 3, four in a line for 4, five for 5, and so on. For by means of such a notation and indication alone could the schematic arrangement of the plane and solid numbers mentioned be made clear and evident, thus

- The number 1,  $\alpha$
- The number 2,  $\alpha \alpha$
- The number 3,  $\alpha \alpha \alpha$
- The number 4,  $\alpha \alpha \alpha \alpha$
- The number 5,  $\alpha \alpha \alpha \alpha \alpha$

and further in similar fashion

[3] Unity, then, occupying the place and character of a point, will be the beginning of intervals and of numbers, but not itself an interval or a number, just as the point is the beginning of a line, or an interval, but is not itself line or interval. Indeed, when a point is

case of equality among the relatives for a proportion is preserved—as the first is to the second, so the second is to the third—but no interval  $\equiv$  generated in the relation of the extremes to each other as there is in all the other relations with the exception of equality. In exactly the same way unity alone out of all number, when it multiplies itself, produces nothing greater than itself

Unity, therefore, is non-dimensional and elementary, and dimension first is found and seen  $\equiv$  2, then in 3, then in 4 and  $\equiv$  succession in the following numbers, for "dimension" is that

which is conceived of as between two limits

[4] The first dimension  $\equiv$  called "line," for "line" is that which is extended in one direction. Two dimensions are called "surface," for a "surface" is that which  $\equiv$  extended in two directions. Three dimensions are called "solid," for a "solid" is that which is extended in three directions, and it is by no means possible to conceive of a solid which has more than three dimensions, depth, breadth, and length. By these are defined the six directions which are said to exist in connection with every body and

upon one, forward and backward upon the second, and right and left upon the third

[5] The statement, also, as it happens, can be made conversely thus. If a thing  $\equiv$  solid, it has by all means three dimensions, length, depth and breadth, and conversely, if it has the three dimensions, it is always a solid, and nothing else

[6] That which has but two dimensions, therefore, will not be a solid, but a surface, for the latter admits of but two dimensions. Here too it is possible similarly to reverse the statement. directly stated, a surface  $\equiv$  that which has two dimensions, and conversely, that which has two dimensions is always a surface

[7] The surface, then, is exceeded by the solid by one dimension, and the line is exceeded by the surface by one, for the line is that which is extended in but one direction and has only one dimension, and it falls short of the solid by two dimensions. The point falls short of the latter by one dimension, and hence it has al-

one

CHAPTER VII

[1] The point, then, is the beginning of dimension, but not itself a dimension, and likewise the beginning of a line, but not itself a line. the line is the beginning of surface but

[3] Exactly the same in numbers, unity is the beginning of all number that advances unit by unit in one direction, linear number is the beginning of plane number, which spreads out like a plane in one more dimension, and plane number is the beginning of solid number, which possesses a depth<sup>1</sup> in the third dimension, besides the original ones. To illustrate and classify, linear numbers are all those which begin with 2 and advance by the addition of 1 in one and the same dimension, and plane numbers are those<sup>2</sup> that begin with 3 as their most elementary root and proceed through the next succeeding numbers. They receive their names also in the same order, for there are first the

five numbers, for the side of the one potentially first is unity, that of the one actually first, that is, 3, is 2, that of 6, which is actually second, 3, that of the third, 4, the fourth, 5, the fifth, 6, and so on.

[3] The triangular number  $\equiv$  produced from the natural series of number set forth in a line, and by the continued addition of successive terms, one by one, from the beginning, for by the successive combinations and additions of

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, I take the first term and have the triangular number which is potentially first, 1,



; then adding the

with 3

[4] The triangle, therefore, is found to be the most original and elementary form of the plane number. Thus we can see from the fact that, among plane figures,<sup>3</sup> graphically represented, if lines are drawn from the angles to the centers each rectilinear figure will by all

set beneath one unit and the number three  $\equiv$  made a triangle



one — — — — —

other. It is therefore the element of the others, and has itself no element. [5] Likewise, as the argument proceeds in the realm of numerical forms, it will confirm this statement.

## CHAPTER VIII

[1] Now a triangular number  $\equiv$  one which,



Again, the number that naturally follows, 4, added in and set down below the former, reduced to units, gives the one in order next after the aforesaid, 10, and takes a triangular form.



find that such a numerical series as far as you like takes the triangular form, if you put as the most elementary form the one that arises from unity, so that unity may appear to be potentially a triangle, and 3 the first actually.

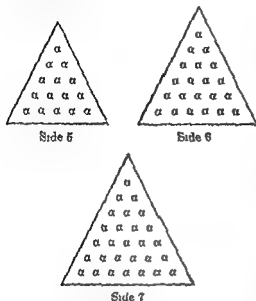
[2] Their sides will increase by the succes-

<sup>1</sup>Cf. Plato, *Tymaeus*  $\equiv$

<sup>2</sup>But cf. Euclid in *Elements* VII, Def. 17

<sup>3</sup>Cf. Plato, *Tymaeus*, 53  $\equiv$

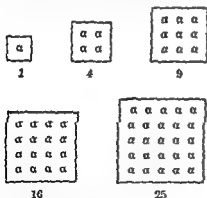
numbers as have been added from the natural series to produce it



### CHAPTER IX

[1] The square is the next number after this, no longer 3, like the former,

1, 4, 9, 16, 25, 36, 49, representations of these numbers are equilateral, square figures, as here shown, and it will be similar as far as you wish to go



[2] It is true of these numbers as it was also of the preceding, that the advance in their sides progresses with the natural series. The side of the square potentially first, 1 is 1 that of 4, the first in actuality, is that of 9, actually the second, 3, that of 16, the next, actually the

third, 4, that of the fourth 5, of the fifth, 6, and so on in general with all that follow

[3] This number also is produced if the natural series is extended in a line, increasing by 1, and no longer the successive numbers are added to the numbers in order, as was shown before, but rather all those in alternate places, that is, the odd numbers. For the first, 1 is potentially the first square, the second, 1 plus 3, actually, the third, 1 plus 3

numbers, and so on

[4] In these cases also, it is a fact that the side of each consists of as many units as there are numbers taken into the sum to produce it

### CHAPTER X

[1] The pentagonal number is one which likewise upon its resolution into units and depiction as a plane figure assumes the form of an equilateral pentagon. 1, 5, 12, 22, 35, 51, 70, and analogous numbers are examples. [2] Each side of the first actual pentagon, 5 is 2, for 1 is potentially first, 1, 3

are the numbers 1, 4, 9, 16, 25, 36, 49, chosen out of the natural arithmetical series set forth in a row. For in a like and similar manner, there are added together to produce the pentagonal numbers the terms beginning with 1 to any extent whatever that are two places apart, that is, those that have a difference of 3

Unity is the first pentagon, potentially, and is thus depicted



5, made up of 1 plus 4, is the second, similarly represented



12 the third, is made up out of the two former numbers with 7 added to them so that it may have 3 as a side as three numbers have been

added to make it Similarly the preceding pentagon, 5, was the combination of two numbers and had 2 as its side The graphic representation of 12 is this



The other pentagonal numbers will be produced by adding together one after another in due order the terms after 7 that have the difference 3, as, for example, 10, 13, 16, 19, 22, 25, and so on The pentagons will be 22, 35, 51, 70, 92, 117, and so forth

# CHAPTER XI

[1] The hexagonal, heptagonal, and succeeding numbers will be set forth in their series by following the same process, if from the natural series of number there be set forth series with their differences increasing by 1 For as the triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, and so on, the squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, and so on, the pentagons are 1, 5, 10, 15, 21, 28, 36, 45, 55, and so on, the hexagons are 1, 6, 12, 19, 27, 36, 46, 57, 69, 82, 96, and so on, the heptagons are 1, 7, 13, 20, 28, 37, 47, 58, 70, 83, 97, and so on, the octagons are 1, 8, 15, 23, 32, 42, 53, 65, 78, 92, 107, and so on, the nonagons are 1, 9, 17, 26, 36, 47, 59, 72, 86, 101, 117, and so on, the decagons are 1, 10, 19, 29, 40, 52, 65, 79, 94, 110, 127, and so on, the hendecagons are 1, 11, 21, 32, 44, 57, 71, 86, 102, 119, 137, and so on, the dodecagons are 1, 12, 23, 35, 48, 62, 77, 93, 110, 128, 147, and so on, the tridecagons are 1, 13, 25, 38, 52, 67, 83, 100, 118, 137, 157, and so on, the tetradecagons are 1, 14, 27, 41, 56, 72, 89, 107, 126, 146, 167, and so on, the pentadecagons are 1, 15, 29, 44, 60, 77, 95, 114, 134, 155, 177, and so on, the hexadecagons are 1, 16, 31, 47, 64, 82, 101, 121, 142, 164, 187, and so on, the heptadecagons are 1, 17, 33, 50, 68, 87, 107, 128, 150, 173, 197, and so on, the octadecagons are 1, 18, 35, 53, 72, 92, 113, 135, 158, 182, 207, and so on, the enneadecagons are 1, 19, 37, 56, 76, 97, 119, 142, 166, 191, 217, and so on, the icosagons are 1, 20, 39, 59, 80, 102, 125, 149, 174, 200, 227, and so on, the hexasagons are 1, 21, 41, 62, 84, 107, 131, 156, 182, 209, 237, and so on, the heptasagons are 1, 22, 43, 65, 88, 112, 137, 163, 190, 218, 247, and so on, the octasagons are 1, 23, 45, 68, 92, 117, 143, 170, 198, 227, 257, and so on, the enneasagons are 1, 24, 47, 71, 96, 122, 149, 177, 206, 236, 267, and so on, the decasagons are 1, 25, 49, 74, 100, 127, 155, 184, 214, 245, 277, and so on, the hendecasagons are 1, 26, 51, 77, 104, 132, 161, 191, 222, 254, 287, and so on, the dodecasagons are 1, 27, 53, 81, 110, 140, 171, 203, 236, 270, 305, and so on, the tridecasagons are 1, 28, 55, 84, 114, 145, 177, 210, 244, 279, 315, and so on, the tetradecasagons are 1, 29, 57, 87, 118, 150, 183, 217, 252, 288, 325, and so on, the pentadecasagons are 1, 30, 59, 90, 122, 155, 189, 224, 260, 297, 335, and so on, the hexadecasagons are 1, 31, 61, 93, 126, 160, 195, 231, 268, 306, 345, and so on, the heptadecasagons are 1, 32, 63, 96, 130, 165, 201, 238, 276, 315, 355, and so on, the octadecasagons are 1, 33, 65, 99, 134, 170, 207, 245, 284, 324, 365, and so on, the enneadecasagons are 1, 34, 67, 102, 138, 175, 213, 252, 292, 333, 375, and so on, the icosasagons are 1, 35, 69, 105, 142, 180, 219, 259, 300, 342, 385, and so on, the hexasagons are 1, 36, 71, 108, 146, 185, 225, 266, 308, 351, 395, and so on, the heptasagons are 1, 37, 73, 111, 150, 190, 231, 273, 316, 360, 405, and so on, the octasagons are 1, 38, 75, 114, 154, 195, 237, 280, 324, 369, 415, and so on, the enneasagons are 1, 39, 77, 117, 158, 199, 242, 286, 331, 377, 423, and so on, the decasagons are 1, 40, 79, 120, 162, 203, 246, 291, 337, 384, 431, and so on, the hendecasagons are 1, 41, 81, 123, 166, 207, 251, 297, 343, 390, 437, and so on, the dodecasagons are 1, 42, 83, 126, 170, 211, 255, 302, 349, 396, 443, and so on, the tridecasagons are 1, 43, 85, 129, 174, 215, 259, 305, 352, 400, 447, and so on, the tetradecasagons are 1, 44, 87, 132, 178, 219, 263, 309, 356, 404, 451, and so on, the pentadecasagons are 1, 45, 89, 135, 182, 223, 267, 313, 360, 408, 455, and so on, the hexadecasagons are 1, 46, 91, 138, 186, 227, 271, 317, 364, 412, 459, and so on, the heptadecasagons are 1, 47, 93, 141, 190, 231, 275, 321, 368, 416, 463, and so on, the octadecasagons are 1, 48, 95, 144, 194, 235, 279, 325, 372, 420, 467, and so on, the enneadecasagons are 1, 49, 97, 147, 198, 239, 283, 329, 376, 424, 471, and so on, the icosasagons are 1, 50, 99, 150, 202, 243, 287, 333, 380, 428, 475, and so on, the hexasagons are 1, 51, 101, 153, 206, 247, 291, 337, 384, 432, 479, and so on, the heptasagons are 1, 52, 103, 156, 210, 251, 295, 341, 388, 436, 483, and so on, the octasagons are 1, 53, 105, 159, 214, 255, 299, 345, 392, 440, 487, and so on, the enneasagons are 1, 54, 107, 162, 218, 259, 303, 349, 396, 444, 491, and so on, the decasagons are 1, 55, 109, 165, 222, 263, 307, 353, 400, 448, 495, and so on, the hendecasagons are 1, 56, 111, 168, 226, 267, 311, 357, 404, 452, 499, and so on, the dodecasagons are 1, 57, 113, 171, 230, 271, 315, 361, 408, 456, 503, and so on, the tridecasagons are 1, 58, 115, 174, 234, 275, 319, 365, 412, 460, 507, and so on, the tetradecasagons are 1, 59, 117, 177, 238, 279, 323, 369, 416, 464, 511, and so on, the pentadecasagons are 1, 60, 119, 180, 242, 283, 327, 373, 420, 468, 515, and so on, the hexadecasagons are 1, 61, 121, 183, 246, 287, 331, 377, 424, 472, 519, and so on, the heptadecasagons are 1, 62, 123, 186, 250, 291, 335, 381, 428, 476, 523, and so on, the octadecasagons are 1, 63, 125, 189, 254, 295, 339, 385, 432, 480, 527, and so on, the enneadecasagons are 1, 64, 127, 192, 258, 299, 343, 389, 436, 484, 531, and so on, the icosasagons are 1, 65, 129, 195, 262, 303, 347, 393, 440, 488, 535, and so on, the hexasagons are 1, 66, 131, 198, 266, 307, 351, 397, 444, 492, 539, and so on, the heptasagons are 1, 67, 133, 201, 270, 311, 355, 401, 448, 496, 543, and so on, the octasagons are 1, 68, 135, 204, 274, 315, 359, 405, 452, 500, 547, and so on, the enneasagons are 1, 69, 137, 207, 278, 319, 363, 409, 456, 504, 549, and so on, the decasagons are 1, 70, 139, 210, 282, 323, 367, 413, 459, 507, 553, and so on, the hendecasagons are 1, 71, 141, 213, 286, 327, 371, 417, 463, 511, 557, and so on, the dodecasagons are 1, 72, 143, 216, 290, 331, 375, 421, 467, 515, 561, and so on, the tridecasagons are 1, 73, 145, 219, 294, 335, 379, 425, 471, 519, 565, and so on, the tetradecasagons are 1, 74, 147, 222, 298, 339, 383, 429, 475, 523, 569, and so on, the pentadecasagons are 1, 75, 149, 225, 302, 343, 387, 433, 479, 527, 573, and so on, the hexadecasagons are 1, 76, 151, 228, 306, 347, 391, 437, 483, 531, 577, and so on, the heptadecasagons are 1, 77, 153, 231, 310, 351, 395, 441, 487, 535, 581, and so on, the octadecasagons are 1, 78, 155, 234, 314, 355, 399, 445, 491, 539, 585, and so on, the enneadecasagons are 1, 79, 157, 237, 318, 359, 403, 449, 495, 543, 589, and so on, the icosasagons are 1, 80, 159, 240, 322, 363, 407, 453, 499, 547, 593, and so on, the hexasagons are 1, 81, 161, 243, 326, 367, 411, 457, 503, 551, 597, and so on, the heptasagons are 1, 82, 163, 246, 330, 371, 415, 461, 507, 555, 601, and so on, the octasagons are 1, 83, 165, 249, 334, 375, 419, 465, 511, 559, 605, and so on, the enneasagons are 1, 84, 167, 252, 338, 379, 423, 469, 515, 563, 609, and so on, the decasagons are 1, 85, 169, 255, 342, 383, 427, 473, 519, 567, 613, and so on, the hendecasagons are 1, 86, 171, 258, 346, 387, 431, 477, 523, 571, 617, and so on, the dodecasagons are 1, 87, 173, 261, 350, 391, 435, 481, 527, 575, 621, and so on, the tridecasagons are 1, 88, 175, 264, 354, 395, 439, 485, 531, 579, 625, and so on, the tetradecasagons are 1, 89, 177, 267, 358, 399, 443, 489, 535, 583, 629, and so on, the pentadecasagons are 1, 90, 179, 270, 362, 403, 447, 493, 539, 587, 633, and so on, the hexadecasagons are 1, 91, 181, 273, 366, 407, 451, 497, 543, 591, 637, and so on, the heptadecasagons are 1, 92, 183, 276, 370, 411, 455, 501, 547, 595, 641, and so on, the octadecasagons are 1, 93, 185, 279, 374, 415, 459, 505, 551, 599, 645, and so on, the enneadecasagons are 1, 94, 187, 282, 378, 419, 463, 509, 555, 603, 649, and so on, the icosasagons are 1, 95, 189, 285, 382, 423, 467, 513, 559, 607, 653, and so on, the hexasagons are 1, 96, 191, 288, 386, 427, 471, 517, 563, 611, 657, and so on, the heptasagons are 1, 97, 193, 291, 390, 431, 475, 521, 567, 615, 659, and so on, the octasagons are 1, 98, 195, 294, 394, 435, 479, 525, 571, 619, 663, and so on, the enneasagons are 1, 99, 197, 297, 398, 439, 483, 529, 575, 623, 667, and so on, the decasagons are 1, 100, 199, 300, 402, 443, 487, 533, 579, 627, 671, and so on, the hendecasagons are 1, 101, 201, 303, 406, 447, 491, 537, 583, 631, 675, and so on, the dodecasagons are 1, 102, 203, 306, 410, 451, 495, 541, 587, 635, 679, and so on, the tridecasagons are 1, 103, 205, 309, 414, 455, 499, 545, 591, 639, 683, and so on, the tetradecasagons are 1, 104, 207, 312, 418, 459, 503, 549, 595, 643, 687, and so on, the pentadecasagons are 1, 105, 209, 315, 422, 463, 507, 553, 599, 647, 691, and so on, the hexadecasagons are 1, 106, 211, 318, 426, 467, 511, 557, 603, 649, 693, and so on, the heptadecasagons are 1, 107, 213, 321, 430, 471, 515, 561, 607, 653, 697, and so on, the octadecasagons are 1, 108, 215, 324, 434, 475, 519, 565, 611, 657, 701, and so on, the enneadecasagons are 1, 109, 217, 327, 438, 479, 523, 569, 615, 661, 705, and so on, the icosasagons are 1, 110, 219, 330, 442, 483, 527, 573, 619, 665, 709, and so on, the hexasagons are 1, 111, 221, 333, 446, 487, 531, 577, 623, 669, 713, and so on, the heptasagons are 1, 112, 223, 336, 450, 491, 535, 581, 627, 673, 717, and so on, the octasagons are 1, 113, 225, 339, 454, 495, 539, 585, 631, 677, 721, and so on, the enneasagons are 1, 114, 227, 342, 458, 499, 543, 589, 635, 681, 725, and so on, the decasagons are 1, 115, 229, 345, 462, 503, 547, 593, 639, 685, 729, and so on, the hendecasagons are 1, 116, 231, 348, 466, 507, 551, 597, 643, 689, 733, and so on, the dodecasagons are 1, 117, 233, 351, 470, 511, 555, 601, 647, 693, 737, and so on, the tridecasagons are 1, 118, 235, 354, 474, 515, 559, 605, 651, 697, 741, and so on, the tetradecasagons are 1, 119, 237, 357, 478, 519, 563, 609, 655, 701, 745, and so on, the pentadecasagons are 1, 120, 239, 360, 482, 523, 567, 613, 659, 705, 749, and so on, the hexadecasagons are 1, 121, 241, 363, 486, 527, 571, 617, 663, 709, 753, and so on, the heptadecasagons are 1, 122, 243, 366, 490, 531, 575, 621, 667, 713, 757, and so on, the octadecasagons are 1, 123, 245, 369, 494, 535, 579, 625, 671, 717, 761, and so on, the enneadecasagons are 1, 124, 247, 372, 498, 539, 583, 629, 675, 721, 765, and so on, the icosasagons are 1, 125, 249, 375, 502, 543, 587, 633, 679, 725, 769, and so on, the hexasagons are 1, 126, 251, 378, 506, 547, 591, 637, 683, 729, 773, and so on, the heptasagons are 1, 127, 253, 381, 510, 551, 595, 641, 687, 733, 777, and so on, the octasagons are 1, 128, 255, 384, 514, 555, 599, 645, 691, 737, 781, and so on, the enneasagons are 1, 129, 257, 387, 518, 559, 603, 649, 695, 741, 785, and so on, the decasagons are 1, 130, 259, 390, 522, 563, 607, 653, 699, 745, 789, and so on, the hendecasagons are 1, 131, 261, 393, 526, 567, 611, 657, 703, 749, 793, and so on, the dodecasagons are 1, 132, 263, 396, 530, 571, 615, 661, 707, 753, 797, and so on, the tridecasagons are 1, 133, 265, 399, 534, 575, 619, 665, 711, 757, 801, and so on, the tetradecasagons are 1, 134, 267, 402, 538, 579, 623, 669, 715, 761, 805, and so on, the pentadecasagons are 1, 135, 269, 405, 542, 583, 627, 673, 719, 765, 809, and so on, the hexadecasagons are 1, 136, 271, 408, 546, 587, 631, 677, 723, 769, 813, and so on, the heptadecasagons are 1, 137, 273, 411, 550, 591, 635, 681, 727, 773, 817, and so on, the octadecasagons are 1, 138, 275, 414, 554, 595, 639, 685, 731, 777, 821, and so on, the enneadecasagons are 1, 139, 277, 417, 558, 599, 643, 689, 735, 781, 825, and so on, the icosasagons are 1, 140, 279, 420, 562, 603, 647, 693, 739, 785, 829, and so on, the hexasagons are 1, 141, 281, 423, 566, 607, 651, 697, 743, 789, 833, and so on, the heptasagons are 1, 142, 283, 426, 570, 611, 655, 701, 747, 793, 837, and so on, the octasagons are 1, 143, 285, 429, 574, 615, 659, 705, 751, 797, 841, and so on, the enneasagons are 1, 144, 287, 432, 578, 619, 663, 709, 755, 801, 845, and so on, the decasagons are 1, 145, 289, 435, 582, 623, 667, 713, 759, 805, 849, and so on, the hendecasagons are 1, 146, 291, 438, 586, 627, 671, 717, 763, 809, 853, and so on, the dodecasagons are 1, 147, 293, 441, 590, 631, 675, 721, 767, 813, 857, and so on, the tridecasagons are 1, 148, 295, 444, 594, 635, 679, 725, 771, 817, 861, and so on, the tetradecasagons are 1, 149, 297, 447, 598, 639, 683, 729, 775, 821, 865, and so on, the pentadecasagons are 1, 150, 299, 450, 602, 643, 687, 733, 779, 825, 869, and so on, the hexadecasagons are 1, 151, 301, 453, 606, 647, 691, 737, 783, 829, 873, and so on, the heptadecasagons are 1, 152, 303, 456, 610, 651, 695, 741, 787, 833, 877, and so on, the octadecasagons are 1, 153, 305, 459, 614, 655, 699, 745, 791, 837, 881, and so on, the enneadecasagons are 1, 154, 307, 462, 618, 659, 703, 749, 795, 841, 885, and so on, the icosasagons are 1, 155, 309, 465, 622, 663, 707, 753, 799, 845, 889, and so on, the hexasagons are 1, 156, 311, 468, 626, 667, 711, 757, 803, 849, 893, and so on, the heptasagons are 1, 157, 313, 471, 630, 671, 715, 761, 807, 853, 897, and so on, the octasagons are 1, 158, 315, 474, 634, 675, 719, 765, 811, 857, 901, and so on, the enneasagons are 1, 159, 317, 477, 638, 679, 723, 769, 815, 861, 905, and so on, the decasagons are 1, 160, 319, 480, 642, 683, 727, 773, 819, 865, 909, and so on, the hendecasagons are 1, 161, 321, 483, 646, 687, 731, 777, 823, 869, 913, and so on, the dodecasagons are 1, 162, 323, 486, 650, 691, 735, 781, 827, 873, 917, and so on, the tridecasagons are 1, 163, 325, 489, 654, 695, 739, 785, 831, 877, 921, and so on, the tetradecasagons are 1, 164, 327, 492, 658, 699, 743, 789, 835, 881, 925, and so on, the pentadecasagons are 1, 165, 329, 495, 662, 703, 747, 793, 839, 885, 929, and so on, the hexadecasagons are 1, 166, 331, 498, 666, 707, 751, 797, 843, 889, 933, and so on, the heptadecasagons are 1, 167, 333, 501, 670, 711, 755, 801, 847, 893, 937, and so on, the octadecasagons are 1, 168, 335, 504, 674, 715, 759, 805, 851, 897, 941, and so on, the enneadecasagons are 1, 169, 337, 507, 678, 719, 763, 809, 855, 901, 945, and so on, the icosasagons are 1, 170, 339, 510, 682, 723, 767, 813, 859, 905, 949, and so on, the hexasagons are 1, 171, 341, 513, 686, 727, 771, 817, 863, 909, 953, and so on, the heptasagons are 1, 172, 343, 516, 690, 731, 775, 821, 867, 913, 957, and so on, the octasagons are 1, 173, 345, 519, 694, 735, 779, 825, 871, 917, 961, and so on, the enneasagons are 1, 174, 347, 522, 698, 739, 783, 829, 875, 921, 965, and so on, the decasagons are 1, 175, 349, 525, 702, 743, 787, 833, 879, 925, 969, and so on, the hendecasagons are 1, 176, 351, 528, 706, 747, 791, 837, 883, 929, 973, and so on, the dodecasagons are 1, 177, 353, 531, 710, 751, 795, 841, 887, 933, 977, and so on, the tridecasagons are 1, 178, 355, 534, 714, 755, 799, 845, 891, 937, 981, and so on, the tetradecasagons are 1, 179, 357, 537, 718, 759, 803, 849, 895, 941, 985, and so on, the pentadecasagons are 1, 180, 359, 540, 722, 763, 807, 853, 899, 945, 989, and so on, the hexadecasagons are 1, 181, 361, 543, 726, 767, 811, 857, 903, 949, 993, and so on, the heptadecasagons are 1, 182, 363, 546, 730, 771, 815, 861, 907, 953, 997, and so on, the octadecasagons are 1, 183, 365, 549, 734, 775, 819, 865, 911, 957, 1001, and so on, the enneadecasagons are 1, 184, 367, 552, 738, 779, 823, 869, 915, 961, 1005, and so on, the icosasagons are 1, 185, 369, 555, 742, 783, 827, 873, 919, 965, 1009, and so on, the hexasagons are 1, 186, 371, 558, 746, 787, 831, 877, 923, 969, 1013, and so on, the heptasagons are 1, 187, 373, 561, 750, 791, 835, 881, 927, 973, 1017, and so on, the octasagons are 1, 188, 375, 564, 754, 795, 839, 885, 931, 977, 1021, and so on, the enneasagons are 1, 189, 377, 567, 758, 799, 843, 889, 935, 981, 1025, and so on, the decasagons are 1, 190, 379, 570, 762, 803, 847, 893, 939, 985, 1029, and so on, the hendecasagons are 1, 191, 381, 573, 766, 807, 851, 897, 943, 989, 1033, and so on, the dodecasagons are 1, 192, 383, 576, 770, 811, 855, 901, 947, 993, 1037, and so on, the tridecasagons are 1, 193, 385, 579, 774, 815, 859, 905, 951, 99



You can also set forth the succeeding polygonals in similar parallel lines

[5] In general, you will find that the squares are the sum of the triangles above those that occupy the same place in the series, plus the numbers of that same class in the next place back, for example, 4 equals 3 plus 1, 9 equals 6 plus 3, 16 equals 10 plus 6, 25 equals 15 plus 10, 36 equals 21 plus 15, and so on

The pentagons are the sum of the squares above them in the same place in the series, plus the elementary triangles that are one place further back in the series for example, 5 equals 4 plus 1, 12 equals 6 plus 3, 22 equals 16 plus 6, 35 equals 25 plus 10, and so on

[6] Again, the hexagonals are similarly the

12 plus 3, 28 equals 22 plus 6, 45 equals 35 plus 10, and as far as you like

[7] The same applies to the heptagonals, for 7 is the sum of 6 and 1, 18 equals 15 plus 3, 34 equals 28 plus 6 and so on Thus each polygonal number is the sum of the polygonal in the same place in the series with one less angle, plus the triangle, in the highest row, one place back in the series

[8] Naturally, then, the triangle is the element of the polygon both in figures and in numbers, and we say this because in the table, reading either up and down or across, the successive numbers in the rows are discovered to have as differences the triangles in regular order

### CHAPTER XIII

[1] From this it is easy to see what the solid number is and how its series advances with equal sides for the number which, in addition to the two dimensions contemplated in graphic representation in a plane length, and breadth, has a third dimension, which some call depth, others thickness, and some height, that number would be a solid number extended in three directions and having length, depth, and breadth

first after the triangular form from a triangular base, second after the form of the square from a square base, and succeeding these after the pentagonal form from a pentagonal base, then

figures, if one imagines three lines from the three angles of an equilateral triangle, equal

the sides of the square, each to each, and again converging in the dimension height to one and

[6] So likewise among numbers, each linear number increases from unity, as from a point as for example, 1, 2, 3, 4, 5 and successive numbers to infinity, and from these same numbers, which are linear and extended in one direction, combined in no random manner, the polygonal and plane numbers are fashioned—the trian-

angular pyramid from the triangles, the square pyramid from the squares, the pentagonal from the pentagons, the hexagonal from

piling up of the triangular numbers one upon the other, first 1, then 1, 3, then 1, 3, 6, then 10 in addition to these, and next 15 together with the foregoing, then 21 besides these, next 28, and so on to infinity

[2] It is clear that the greatest number is conceived of as being lowest for it is discovered to be the base the next succeeding one is on top of it, and the next on top of that, until unity appears at the apex and, so to speak tapers off the completed pyramid into a point

## CHAPTER XIV

[1] The next pyramids in order are those with a square base which rise in this shape to one and the same point. These are formed in

100, and again set the successive terms, as in a pile one upon the other in the dimension height when I put 1 on top of 4, the first actual pyramid with square base, 5 is produced, for here again unity is potentially the first [2] Once more, I put this same pyramid entire, composed of 5 units, just as it is, upon the square 9, and there is made up for me the pyramid 14 with square base and side 3—for the former pyramid had the side 2, and the one potentially first 1 as a side 1 or here too each

limit, or five times truncated at the next step, and so on as far as you care to carry the nomenclature

## CHAPTER XV

and going on to infinity in each case

[4] From this too it becomes evident that

truncated, and tri truncated pyramids the names of which we are sure to encounter in sci

so on with the succeeding squares throughout

[3] Here too, the sides will be composed of as many units as were in the sides of the squares from which they arose, in each case the sides of 8 will be 2, like those of 4 those of 27, 3, like those of 9 those of 64, 4, like those of 16, and so on so that likewise the side of unity, the potential cube, will be 1, which is the side of the potential square, 1

In general, each square is a single plane, and has four angles and four sides, while each several cube having increased out of some one square multiplied by its own side will have al-

becomes a plane figure with the same number of angles as the base. If, however, in addition to the failure to terminate in unity it does not even terminate in the polygon next to unity and

nal square

## CHAPTER XVI

[1] Now since the cube is a solid figure with equal sides in all dimensions, in length, depth,

... equally extended in all the  
... there is  
... ous in  
... th un-  
... al to  
... ies 4,  
or 2 times 4 times 2, or 12, or  
a figure which follows some other scheme of in-  
equality

[2] Such solid figures, in which the dimensions are everywhere unequal one to another, are called scalene in general. Some, however, using other names, call them "wedges," for carpenters', house-builders' and blacksmiths' wedges and those used in other crafts, having unequal sides in every direction, are fashioned so as to penetrate, they begin with a sharp end and continuously broaden out unequally in all the dimensions. Some also call them *sphekistos*, "wasps," because wasps' bodies also are very like them, compressed in the middle and showing the resemblance mentioned. From this also the *sphekoma*, "point of the helmet," must derive its name, for where it is compressed it imitates the waist of the wasp. Others call the same numbers "altars," using their own metaphor, for the altars of ancient style, particularly the *ionic*, do not have the breadth equal to the depth, nor either of these equal to the length, nor the base equal to the top, but are of varied dimensions everywhere.

[3] Now whereas the two kinds of numbers, cube and scalene, are extremes, the one equally extended in every dimension, the other unequally, the so-called parallelepipeds are solid numbers like means between them. The plane surfaces of these are heteromecic numbers,<sup>1</sup> just as in the case of the cubes the faces were squares, as has been shown.

## CHAPTER XVII

[1] Again, then, to take a fresh start, a number is called heteromecic if its representation, when graphically described in a plane, is quadrilateral and quadrangular, to be sure, but the sides are not equal one to another, nor is the length equal to the breadth, but they differ by 1. Examples are 2, 6, 12, 20, 30, 42, and so on, for if one represents them graphically he will always construct them thus: 1 times 2 equals 2, 2 times 3 equals 6, 3 times 4 equals 12, and the succeeding ones similarly, 4 times 5, 5 times 6, 6 times 7, 7 times 8, and thus indefinitely,

provided only that one side is greater than the other by 1 and by no other number. If, however, the sides differ otherwise than by 1, for instance, by 2, 3, 4 or succeeding numbers, as in 2 times 4, 3 times 6, 4 times 8, or however else they may differ, then no longer will such a number be properly called a heteromecic, but an oblong number. For the ancients of the school of Pythagoras and his successors saw "the other" and "otherness" primarily in 2, and "the same" and "sameness" in 1, as the two beginnings of all things, and these two are found to differ from each other only by 1. Thus "the other" is fundamentally "other" by 1, and by no other number, and for this reason customarily "other" is used, among those who speak correctly, of two things and not of more than two.

[2] Moreover, it was shown that all odd number is given its specific form<sup>2</sup> by unity, and all even number by 2. Hence we shall naturally say that the odd partakes of the nature of "the same," and the even of that of "the other," for indeed there are produced by the successive additions of each of these—naturally, and not by our decree—by the addition of the odd numbers from 1 to infinity the class of the squares, and by the addition of the evens from 2 to infinity, that of the heteromecic numbers.<sup>3</sup>

[3] There is, accordingly, every reason to think that the square once more shares in the nature of the same, for its sides display the same ratio, alike, unchanging and firmly fixed in equality, to themselves, while the heteromecic number partakes of the nature of the other, for just as 1 is differentiated from 2, differing by 1 alone, thus also the sides of every heteromecic number differ from one another, one differing from the other by 1 alone.

To illustrate, if I have set out before me the successive numbers in series beginning with 1, and select and arrange by themselves the odd numbers in the line and the even by themselves in another, there are obtained these two series

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27  
2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28

[4] Now, then, the beginning of the odd series is unity, which is of the same class as the series and possesses the nature of "the same," and so whether it multiplies itself in two dimensions

<sup>1</sup>Cf II 6 4

<sup>2</sup>See the following chapter.

<sup>3</sup>Cf Plato *Timaeus* 35 ff

<sup>4</sup>Cf I 7 2

<sup>5</sup>Cf II 18 2 and 20 3

or in three it is not made different, nor yet does

it make an other — — — — —

itself or another number, it causes a change,<sup>2</sup>

for example, 2 times 2, 2 times 3

[6] But in cases like 8 times 8 times 2, or 8 times 8 times 3, such solid forms are called "bricks," the product of a number by itself and then by a smaller — — — — —

provided only it be a greater number of times than the square itself, then the number is a "beam" — — — — —

that from which they began, these are called "beams" — — — — —

symmetrical and so — — — — —

if it were stronger — — — — —

Agassiz, 1, 1880, 1, 280

## CHAPTER XVIII

[1] Regarding the solid numbers this is for the present sufficient. The physical philosophers, however, and those that take their start with mathematics, call "the same" and "the other" — — — — —

— — — — —

1

1

— — — — — of the even numbers, because of the share of the original inequality<sup>3</sup> and "otherness" which they have in the difference between their sides. Therefore it is most necessary further to demonstrate how in these two, as in origins and — — — — —

[2] First, however, we must make the distinction whereby the oblong (promecic) number differs from the heteromecic. The heteromecic is, as was stated above,<sup>4</sup> the product of a number — — — — —

1

— — — — —

1

[3] Moreover, since squares are produced — — — — —

1

reason are limited and come to an end for "the equal" and "the same" are so in one definite way. But since the heteromecic — — — — —

1

1

1

<sup>2</sup>Cf II 6 3

<sup>3</sup>Cf Aristotle, *On the Soul* 406<sup>b</sup> 13

<sup>4</sup>Cf II 17 2

<sup>5</sup>Cf II 17 1

objects in the universe which have been created with reference to them are divided and classified and are seen to be opposite one to another, and well do the ancients at the very beginning of their account of Nature make the first subdivision in their cosmogony on this principle. Thus Plato<sup>1</sup> mentions the distinction between the natures of "the same" and "the other," and again, that between the essence which is indivisible and always the same and the one which is divided, and Philolaus says that existent things must all be either limitless or limited, or limited and limitless at the same time, by which it is generally agreed that he means that the universe is made up out of limited and limitless things at the same time, obviously after the image of number, for all number is composed of unity and the dyad, even and odd, and these in truth display equality and inequality, sameness and otherness, the bounded and the boundless, the defined and the undefined.

## CHAPTER XIX

[1] That we may be clearly persuaded of what is being said, namely, that things are made up of warring and opposite elements<sup>2</sup> and have in all likelihood taken on harmony—and harmony always arises from opposites, for harmony is the unification of the diverse and the reconciliation of the contrary minded—let us set forth in two parallel lines no longer, as just previously, the even numbers from 2 by themselves and the odd numbers from 1, but the numbers that are produced from these by adding them successively together, the squares from the odd numbers, and the heteromecic from the even. For if we give careful attention to their setting forth, we shall admire their mutual friendship and their coöperation to produce and perfect the remaining forms, to the end that we may with probability conceive that also in the nature of the universe from some such source as this a similar thing was brought about by universal providence.

[2] Let the two series then be as follows. That of the squares, from unity, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, and that of the heteromecic numbers, beginning

meic number, the second, compared to the second, is its sesquialter, the third, sesquitercian of the third, the fourth, sesquiquartan of the fourth, then sesquiquintan, sesquiseptan, and so on similarly *ad infinitum*. Their differences, too, will increase according to the successive numbers from 1, the difference of the first terms is 1, of the second 2, of the third 3, and so on. Next, if first the second term of the squares be compared with the first heteromecic number, the third with the second, the fourth with the third, and the rest similarly, they will keep unchanged the same ratios as before, but their differences will begin to progress no longer from 1, but from 2, remaining the same as before, and according to the advance observed in the former comparison, the first to the first will

[4] Furthermore, the squares among themselves will have only the odd numbers as differences the heteromecic, even numbers. And if we put the first heteromecic number as a mean term between the first two squares, the second between the next two, the third between the two following, and the fourth between the two next succeeding, therein will be seen still more regularly the numerical relations in groups of three terms. For as 4 is to 2, so is 2 to 1, and as 9 is sesquialter to 6, so is 6 to 4, and as 16 to 12, so is 12 to 9, and so on, with both numbers

ent one, by an increase. In all the groupings, too, the product of the extremes is equal to the square of the mean, and the extremes plus twice the mean, by exchange will always give a square. What is neatest of all, from the addition of both there comes about the production of the triangles in due order, showing that the nature of these is more ancient<sup>3</sup> than the origin of all things, thus 1 plus 2, 2 plus 4, 4 plus 6, 6 plus 9, 9 plus 12, 12 plus 16, 16 plus 20, and by this process the triangles, which give rise to the polygons, come forth in order.

## CHAPTER XX

[1] Still further, every square plus its own side becomes heteromecic, or by Zen<sup>4</sup>, if its side is subtracted from it. Thus, "the other" is con-

<sup>1</sup>Cf Plato, *Timaeus*, 35

<sup>2</sup>Plato, *Timaeus*, 30

<sup>3</sup>Cf II 17 3, 18 1 and II 7 4, cf II 8

may also be both greater and smaller than

8, 27, 64, 125, and 216, and those that go on

[2] This also is sufficient evidence that the two forms partake of sameness and otherness, of otherness in an indefinite fashion, but of sameness definitely, 1 and 2 generically, but the odd of sameness after the manner of a subordinate species because it belongs to the same class as 1, and the even of otherness because it is homogeneous with 2

[3] There is also a still clearer reason why the square, since it is the product of the addition of odd numbers, is akin to sameness, and the heteromec numbers to otherness because it is made up by adding even numbers, for as though they were friends of one another, these two forms share in their two rows the same differences when they do not have the same ratios, and conversely the same ratios when they do not have the same differences. For the difference between 4 and 2 in the double ratio is found between 6 and 4 as a superparticular, and again the difference between 9 and 6, as a sesquialter, is found between 12 and 9 as a sesquitercian, and so on. What is the same in quality is different in quantity, and just the opposite, what is the same in quantity is different in quality. [4] Again, it is clear that in all their relations the same difference between two terms will necessarily be called fractions with names that differ by 1, and be the half of one and the third of the other, or the third of one and the quarter of the other, or the fourth of one and the fifth of the other, and so on.

[5] But what will most of all confirm the fact that the odd, and never the even, is pre-eminently the cause of sameness, is to be demonstrated in every series beginning with 1 following some ratio, for example, the double ratio, 1, 2, 4, 8, 16, 32, 64, 128, 256, or the triple, 1, 3, 9, 27, 81, 243, 729, 2 187, and as far as you like. You will find that of necessity all the terms in the odd places in the series are squares, and no others by any device whatsoever, and that no square is to be found in an even place.

the next three, the third, the four next following, the fourth, the succeeding five, the fifth, the next six, the sixth, and so on.

## CHAPTER XXI

[1] After this it would be the proper time to incorporate the nature of proportions, a thing most essential for speculation about the nature of the universe and for the propositions of music, astronomy, and geometry, and not least for the study of the works of the ancients, and thus to bring the *Introduction to Arithmetic* to the end that is at once suitable and fitting.

[2] A proportion, then, is in the proper sense,

[3] Now a ratio<sup>1</sup> is the relation of two terms to one another, and the combination of such is a proportion, so that three is the smallest number of terms of which the latter is composed, although it can be a series of more, subject to the same ratio or the same difference. For example, 1, 2 is one ratio, where there are two terms, but 2, 4 is another similar ratio, hence 1, 2, 4 is a proportion, for it is a combination of ratios, or of three terms which are observed to be in the same ratio to one another. [4] The same thing may be observed also in greater numbers and longer series of terms, for let a fourth term, 8, be joined to the former after 4, again in a similar relation, the double, and then 16 after 8 and so on.

[5] Now if the same term, one and unchanging, is compared to those on either side of it, to the greater as consequent and to the lesser as antecedent, such a proportion is called continued, for example, 1, 2, 4 is a continued proportion as regards quality<sup>2</sup>, for 4 : 2 equals 2 : 1, and conversely 1 : 2 equals 2 : 4. In quantity, 1, 2, 3, for example, is a continued proportion, for as 3 exceeds 2, so 2 exceeds 1, and conversely, as 1 is less than 2, by so much 1 is less than 3.

[6] If, however, one term answers to the lesser term, and becomes its antecedent and a

<sup>1</sup>Cf Euclid, *Elements*, V, and

<sup>2</sup>Cf II 22, 23, 4 below

greater term, and another, not the same, takes the place of consequent and lesser term with reference to the greater, such a mean and such a proportion is called no longer continued, but disjunct for example, as regards quality, 1, 2, 4, 8, for 2 1 equals 8 4, and conversely 1 2 equals 4 8, and again 1 4 equals 2 8 or 4 1 equal in quantity, 1, 2, 3, 4, for as 1 is to 2, so 3 is to 4, and by interchange, as 3 exceeds 1, so 4 is exceeded by 2, by so much 2 is exceeded by 4

## CHAPTER XXII

[1] The first three proportions, then, which are acknowledged by all the ancients, Pythagoras, Plato, and Aristotle, are the arithmetic, geometric and harmonic and there are three others subcontrary to them, which do not have names of their own, but are called in more general terms the fourth, fifth and sixth forms of mean after which the moderns discover four others as well making up the number ten, which, according to the Pythagorean view, is the only possible one. It was in accordance with the former of the

illustrations, and forms of the exterior and interior, and feet and countless other things which we shall notice in the proper place

[2] Now however, we must treat from the beginning first that form of proportion which by quantity reconciles and binds together the comparison of the terms, which is a quantitative equality as regards the difference of the several terms to one another. This would be the arithmetic proportion, for it was previously reported that quantity is its peculiar belonging. [3] What then, is the reason that we shall treat of this first, and not another? Is it not clear that Nature shows it forth before the rest? For in the natural series of simple numbers, beginning with 1 with no term passed over or omitted, the definition of this proportion alone is preserved moreover, in our previous statement. It was demonstrated that the Arithmetic series, but is not abolished, and because it is implied by them but does not imply them. Thus it is natural that the mean which shares the name of arithmetic will not

unreasonably take precedence of the means which are named for the other sciences, the geometric and harmonic, for it is plain that all the more will it take precedence over the subcontraries, over which the first three hold the first and original, therefore

at our hands before

## CHAPTER XXIII

[1] It is an arithmetic proportion, then, whenever three or more terms are set forth in succession, or are so conceived, and the same quantitative difference is found to exist between the successive numbers, but not the same ratio among the terms, one to another. For example 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 for in this natural series of numbers, examined consecutively and without any omissions, every term whatsoever is discovered to be placed between two and to preserve the arithmetic proportion to them. For its differences as compared with those ranged on either side of it are equal the same ratio, however, is not preserved among them

[2] And we understand that in such a series there comes about both a continued and a disjunct proportion for if the same middle term answers to those on either side as both antecedent and consequent, it would be a continued proportion, but if there is another mean along with it a disjunct proportion comes about

[3] Now if we separate out of this series any three consecutive terms whatsoever, after the form of the continued proportion, or four or more terms after the disjunct form, and consider them the difference of them all would be 1, but their ratios would be different throughout. If however, again we select three or more terms not adjacent, but separated separated nevertheless by a constant interval, if one term was omitted in setting down each term, the difference in every case will be 2 and once more with three terms it will be a continued proportion with more disjunct. If two terms are omitted, the difference will always be 3 in all of them, continued or disjunct, if three, 4, if four, 5 and so on

[4] Such a proportion, therefore, partakes in equal quantity in its differences, but of unequal quality for this reason it is arithmetic. If on the contrary it partook of similar quality, but not quantity, it would be geometric instead of arithmetic

[5] A thing is peculiar to this proportion

that does not belong to any other, namely, the mean is either half of, or equal to, the sum of the extremes, whether the proportion be viewed as continuous or disjunct or by alternation, for either the mean term with itself, or the mean terms with one another, are equal to the sum of the extremes

[6] It has still another peculiarity, what ratio each term has to itself, thus the differences have to the differences, that is, they are equal

Again, the thing which is most exact, and which has escaped the notice of the majority, the product of the extremes when compared to the square of the mean is found to be smaller than it by the product of the differences, whether they be 1, 2, 3, 4, or any number whatever

In the fourth place, a thing which all previous writers also have noted, the ratios between the smaller terms are larger, as compared to those between the greater terms. It will be shown that in the harmonic proportion, on the contrary, the ratios between the greater terms are greater than the terms

as it were, between extremes, for this proportion has the ratios between the greater terms and those between the smaller equal, and we have seen that the equal is in the middle ground between the greater and the less. So much, then, about the arithmetic proportion

## CHAPTER XXIV

[1] The next proportion after this one, the geometric, is the only one in the strict sense of the word to be called a proportion, because its terms are seen to be in the same ratio. It exists whenever, of three or more terms, as the greatest is to the next greatest, so the latter is to the

same quantity but rather by the same quality of ratio, the opposite of what was seen to be the case with the arithmetic proportion

[2] For an example set forth the numbers beginning with 1 that advance by the double

or any number whatever that may be taken,

and conversely, they do not, however, have the same quantitative difference. Again, 2, 4, 8, 16, for not only does 16 have the same ratio to 2 as before, though not the same difference, but also by alternation it preserves a similar relation—as 16 is to 4, so 8 is to 2, and conversely, as 2 is to 8, so 4 is to 16, and disjunctly, as 2 is to 4, so 8 is to 16, and conversely and in disjunct form, as 16 is to 8 so 4 is to 2, for it has the double ratio

[3] The geometric proportion has a peculiar property shared by none of the rest, that the differences of the terms are in the same ratio to each other as the terms to those adjacent to them, the greater to the less and vice versa. Still another property is that the greater terms have as a difference, with respect to the lesser, the lesser terms themselves, and similarly difference differs from difference, by the smaller difference itself, if the terms are set forth in the double ratio, in the triple ratio both terms and

[4] Geometric proportions come about not only among the multiples, but also among all the superparticular, superpartient, and mixed forms, and the peculiar property of this proportion in all cases is preserved, that in the continued proportions the product of the extremes is equal to the square of the mean, but in disjunct proportions, or those with a greater number of terms, even if they are not continued, but with an even number of terms, that the product of the extremes equals that of the means

[5] As an illustration of the fact that in all the relations, all kinds of multiples, superparticulars, superpartients, and mixed ratios the peculiar property of this proportion is preserved, let that suffice and be sufficient for us wherein we fashioned, beginning with equality by the three rules all the kinds of inequality out of one another, when they were in both direct and reverse order, for each act of fashioning and each series set forth is a geometric proportion with all the aforesaid properties as well as a fourth namely, that they keep the



appear one after the other, the two, the  
quaternion, sesquiquartan, and so on

[6] It would be most reasonable, now that we have reached this point, to mention a corollary that is of use to us for a certain Platonic theorem<sup>1</sup> for plane numbers are bound together always by a single mean, solids by two, in the form of a proportion. For with two consecutive squares<sup>2</sup> only one mean term is discovered which preserves the geometric proportion, as antecedent to the smaller and consequent to the greater term, and never more than one. Hence we conceive of two intervals be-

dimensional and the plane or cubical. For example, 1 and 4 are planes, and 2 a middle term in proportion, or again 4 and 9, and 6 a middle term, held by

early to each, both together plane number 6. In cubes, however, for example 8 and 27, no longer one but two mean terms are found, 12 and 18, which just themselves and the terms in the same ratio as that which the differences bear to one another and the reason of this is that the two mean terms are the products of the sides of the cubes commingled, 2 times 2 times 3 and 3 times 3 times 2.

[10] In general, then, if a square takes a square, that is, multiplies it it always makes a square, but if a square multiplies a hetero-

with reference to the passage in the marriage number in the Republic<sup>4</sup> introduced in the person of the Muses. So then let us pass over to the third proportion, the so-called harmonic, and analyze it.

## CHAPTER XXV

[1] The proportion that is placed in the third order is one called the harmonic, which exists whenever among three terms the mean or ex-  
anumination is observed to be neither in the same ratio to the extremes, antecedent of one and consequent of the other, as in the geometric proportion, nor with equal intervals, but an inequality of ratios, as in the arithmetic, but on

the greatest term is to the  
antecedent  
mean  
4 6  
itself,  
ratio of 4

other, but in the second  
triple ratio

[2] It has a peculiar property, opposite, as

the terms are great  
smaller,  
mean  
in the  
middle

oppor-  
t and  
same  
ons of  
fraction of itself, the two

<sup>1</sup>Timaeus, 32

<sup>2</sup>Cf Euclid, Elements, VIII, 11

<sup>3</sup>Cf Euclid, Elements, VIII, 12

<sup>4</sup>Republic 546 ff

<sup>5</sup>Cf II 23 6

the terms that flank it, in the harmonic, however, it is the opposite, for the middle term is greater and less than the terms on either side by different fractions of itself, but always the

the mean, it makes twice the product of them-

multiplied by a ratio

term and another, but this form, with reference to relativity, appears now in one form,

the greatest term is to the smallest, so also is the difference between the greatest and the next greatest, or middle, term to the difference between the least term and the middle term, and vice versa

## CHAPTER XXVI

[1] In the classification of Being previously set forth we recognized the relative <sup>1</sup>as a thing peculiar to harmonic theory, but the musical ratios of the harmonic intervals are also rather to be found in this proportion. The most ele-

and sesquialter, is as 6 2, the ratio of term to term in the example in the triple ratio, and likewise of difference to difference in the same, and in the proportion with double ratio it is the ratio of the greatest term to the difference between that term and the mean term, or of the difference between the extremes to the difference between the smaller terms. The last and greatest interval, the so-called di-diapason, as it were twice the double, which is in the quadruple ratio, is as the middle term in the proportion in the double ratio to the difference between the lesser terms, or as the difference between the extremes, in the example in the triple ratio to the difference between the lesser terms

[2] Some, however, agreeing with Philolaus, believe that the proportion is called harmonic because it attends upon all geometric harmony, and they say that "geometric harmony" is the cube because it is harmonized in all three dimensions, being the product of a number thrice multiplied together. For in every cube this proportion is mirrored, there are in every cube 12 sides, 8 angles and 6 faces, hence 8, the mean between 6 and 12 is according to harmonic proportion for as the extremes are to each other, so is the difference between greatest and middle term to that between the middle and smallest terms, and, again, the middle term is greater than the smallest by one fraction of itself and by another is less than the greater term, but is greater and smaller by one and the same fraction of the extremes. And again, the sum of the extremes multiplied by the mean

ratio of term to term. Then the combination<sup>2</sup>

it been called harmonic

## CHAPTER XXVII

of the diapason and diapente together, which preserves the triple ratio of the two of them together, since it is the combination of double

[1] Just as in the division of the musical canon, when a single string is stretched or one length of a pipe is used, with immovable ends, and the mid point shifts in the pipe by means of the finger-holes, in the string by means of the bridge, and as in one way after another the aforesaid proportions, arithmetic, geometric,

<sup>1</sup>See I 3 1

<sup>2</sup>The examples referred to are the harmonic proportions cited in II 25 1

<sup>3</sup>Cf II 5 2.

## CHAPTER XXIX

[1] It remains for me to discuss briefly the most perfect proportion, that which is three-dimensional and embraces them all, and which is most useful for all progress in music and in the theory of the nature of the universe. This alone would properly and truly be called harmony<sup>1</sup> rather than the others, since it is not a plane, nor bound together by only one mean term, but with two, so as thus to be extended in three dimensions,<sup>2</sup> just as a while ago it was explained that the cube is harmony.

[2] When, therefore, there are two extreme terms, both of three dimensions, either numbers multiplied thrice by themselves so as to be a cube, or numbers multiplied twice by themselves and once by another number so as to be either "beams" or "bricks," or the products of three unequal numbers, so as to be scalene, and between them there are found two other terms which preserve the same ratios to the extremes alternately and together, in such a manner that, while one of them preserves the harmonic proportion, the other completes the arithmetic, it is necessary that in such a disposition of the four the geometric proportion appear, on examination, commingled with both mean terms—as the greatest is to the third removed from it, so is the second from it to the fourth, for such a situation makes the product of the means equal to the product of the extremes. And again, if the greatest term be

shown to differ from the one next beneath it by the amount whereby this latter differs from the least term, such an array becomes an arithmetic proportion and the sum of the extremes is twice the mean. But if the third term from the greatest exceeds and is exceeded by the same fraction of the extremes, it is harmonic and the product of the mean by the sum of the extremes is double the product of the extremes.

[3] Let this be an example of this proportion, 6, 8, 9, 12. 6 is a scalene number, derived from 1 times 2 times 3, and 12 comes from the successive multiplication of 2 times 2 times 3 of the mean terms the lesser is from 1 times 2 times 4, and the greater from 1 times 3 times 3. The extremes are both solid and three-dimensional, and the means are of the same class. According to the geometric proportion, as 12 is to 8, so 9 is to 6, according to the arithmetic, as 12 exceeds 9, by so much does 9 exceed 6, and by the harmonic, by the fraction by which 8 exceeds 6, viewed as a fraction of 6, 8 is also exceeded by 12, viewed as a fraction of 12.


[4] Moreover 6 or 12 is the diatessaron, in sesquitercian ratio, 9 or 8 is the diapente in the sesquialter. 12 or 6 is the diapason in the double. Finally, 9 or 8 is the interval of a tone, in the superoctave ratio, which is the common measure of all the ratios in music, since it is also the more familiar, because it is likewise the difference between the first and most elementary intervals.

[5] And let this be sufficient concerning the phenomena and properties of number, for a first Introduction.

<sup>1</sup>Cf II 26 2.

<sup>2</sup>Cf II 24 6.

## *THE GREAT IDEAS, Volumes 2 and 3*



MATTER	RELATION
MECHANICS	RELIGION
MEDICINE	REVOLUTION
MEMORY AND IMAGINATION	RHETORIC
METAPHYSICS	SAME AND OTHER
MIND	SCIENCE
MONARCHY	SENSE
NATURE	SIGN AND SYMBOL
NECESSITY AND CONTINGENCY	SIN
OLIGARCHY	SLAVERY
ONE AND MANY	SOUL
OPINION	SPACE
OPPOSITION	STATE
PHILOSOPHY	TEMPERANCE
PHYSICS	THEOLOGY
PLEASURE AND PAIN	TIME
POETRY	TRUTH
PRINCIPLE	TYRANNY
PROGRESS	UNIVERSAL AND PARTICULAR
PROPHECY	VIRTUE AND VICE
PRUDENCE	WAR AND PEACE
PUNISHMENT	WEALTH
QUALITY	WILL
QUANTITY	WISDOM
REASONING	WORLD